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GRAPHS WITHOUT INDUCED P_5 AND C_5 *

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Abstract

Zverovich [Discuss. Math. Graph Theory **23** (2003), 159–162.] has proved that the domination number and connected domination number are equal on all connected graphs without induced P_5 and C_5. Here we show (with an independent proof) that the following stronger result is also valid: Every P_5-free and C_5-free connected graph contains a minimum-size dominating set that induces a complete subgraph.

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1. The Results

For a (simple, undirected) graph G with vertex set $V(G)$ and edge set $E(G)$, a subgraph $H \subseteq G$ is called a *dominating subgraph* if every $v \in V(G) \setminus V(H)$ is adjacent to some $w \in V(H)$. The *domination number* of G is the smallest number of vertices in a dominating subgraph H of G; and the *connected-domination number* is the minimum under the further requirement that the dominating subgraph $H \subseteq G$ be *connected*.

More than a decade ago, the present authors [1] and independently Cozzens and Kelleher [4] proved that a connected graph without induced paths and cycles on five vertices contains a dominating *complete subgraph.*

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Recently, Zverovich [6] showed that if such a graph has domination number k, then it also contains a *connected* dominating subgraph on k vertices. In this note we prove the following common generalization of these two results:

Theorem 1. *If a connected graph without induced P_5 and C_5 subgraphs has domination number k, then it is dominated by a complete subgraph on k vertices.*

This theorem will be proved in Section 2. Let us note that it has the following equivalent formulation, which is in fact a *necessary and sufficient condition:*

> *In each connected, induced subgraph of a graph G, the minimum number of vertices in a complete subgraph dominating G exists and is equal to the domination number, if and only if no induced subgraphs of G are isomorphic to P_5 and C_5.*

The reason is that P_5 and C_5 themselves do not have dominating cliques; i.e., if the property is required hereditarily, then P_5 and C_5 have to be forbidden induced subgraphs. Recalling from [6] that both P_5 and C_5 have domination number 2 and connected-domination number 3, also the following characterization theorem of [6] follows:

> *In each connected, induced subgraph of a graph G, the connected-domination number is equal to the domination number, if and only if G contains no induced subgraphs isomorphic to P_5 and C_5.*

In a more general setting, a theory has been developed for solving graph-class equations of the form $Dom\,(\mathcal{D}) = Forb\,(\mathcal{F})$ where $Dom\,(\mathcal{D})$ consists of the graphs in which every connected, induced subgraph is dominated by some induced subgraph $D \in \mathcal{D}$, and $Forb\,(\mathcal{F})$ means that no induced subgraph $F \in \mathcal{F}$ may occur. For recent results and more references of this kind, please see [2].

We also consider here the other extreme, i.e., domination with *largest* complete subgraphs. In this context, a subclass of P_5-free graphs has been studied in [3]. It is proved there that if a connected graph contains at least one triangle but it is $2K_2$-free (i.e., it does not contain any pair of disjoint edges as an induced subgraph), then it is dominated by some of its complete subgraphs of *maximum size.* As noted in [1], this result does not remain valid for P_5-free (and C_5-free) graphs. A simple counterexample is the graph, that we denote by P_5^+, obtained from P_5 by joining the middle vertex with one of the endpoints. In Section 3 we prove:

Theorem 2. *If a connected graph does not contain P_5, C_5, and P_5^+ as an induced subgraph, then it is dominated by a complete subgraph of maximum size.*

Similarly to Theorem 1, also this result can be transformed to a necessary and sufficient condition:

> *Every connected, induced subgraph H of a graph G is dominated by a maximum-size complete subgraph of H, if and only if G contains no induced subgraph isomorphic to P_5, C_5, and P_5^+.*

2. Small Complete Subgraphs

Before proving the main result of the paper, we state an auxiliary observation. In the proof we shall use the following terminology. If H is a dominating subgraph, a *private neighbor* of a vertex $w \in V(H)$ is a vertex z in $V(G) \setminus V(H)$ such that w is the unique neighbor of z inside $V(H)$.

Lemma 1. *Suppose that G contains no induced P_5 and C_5, and let H be a dominating induced subgraph of G. If H has a connected component H' which is not complete, then there exists a vertex $x \in V(H')$ such that*

1. *$H' - x$ is connected, and*
2. *$H - x$ is dominating in G.*

Proof. Let x and x' be two vertices at maximum distance apart in the induced subgraph H'. (It will be irrelevant whether or not their distance in G is smaller than in H'.) If H' is not complete, then x and x' are nonadjacent. Let us select a shortest x–x' path P in H'.

By the choice of x and x', both $H' - x$ and $H' - x'$ are connected. If some of them dominates G, then the lemma is proved. Otherwise, each of x and x' has at least one private neighbor, say y and y', respectively. In this case, however, the subgraph induced by $V(P) \cup \{y, y'\}$ either is a 5-cycle or contains an induced P_5, a contradiction. ∎

***Proof of Theorem* 1.** Given the graph G without induced P_5 and C_5, we select a dominating subgraph H such that

1. $|V(H)|$ is minimum, and

2. H has as few connected components as possible.

By Lemma 1, each component of H is a complete subgraph of G. We need to prove that H has just one component. Suppose, for a contradiction, that H is disconnected.

Case 1. Some vertex has neighbors in more than one component of H. Let w be such a vertex. Then the subgraph induced by $V(H) \cup \{w\}$ in G is dominating, and has fewer components than H. Since the component containing w is not complete, Lemma 1 allows us to remove a non-cutting vertex from this component, hence obtaining another dominating induced subgraph that has the same number of vertices as H, but with fewer components. This contradicts the choice of H.

Case 2. Each vertex has all of its neighbors in the same component of H. Since H is dominating and disconnected, but G is connected, there exists an edge $w'w''$ inside $G-H$, and two distinct components K', K'' of H, such that w' has neighbors in K' and w'' has neighbors in K''. Then the subgraph H^+ induced by $V(H) \cup \{w', w''\}$ is dominating, and one of its components, say H', is induced by $V(K') \cup V(K'') \cup \{w', w''\}$. This H' is not complete, and in order to keep it connected and make it complete, one needs to remove at least two vertices (one from $V(K') \cup \{w''\}$ and one from $V(K'') \cup \{w'\}$). On the other hand, H^+ contains just two more vertices than one of the *smallest* dominating subgraphs, H, therefore applying Lemma 1 twice we obtain a minimum-size dominating subgraph H^-. This H^- has fewer components than H, again a contradiction. ■

3. Large Complete Subgraphs

In the proof below, the notation $N(u)$ means the “open neighborhood” of vertex u (i.e., the set of vertices adjacent to u).

***Proof of Theorem* 2.** We prove the assertion by contradiction. Let G be a graph without induced P_5, C_5, and P_5^+, which is a minimal counterexample in the sense that, for every vertex v, if the subgraph $G - v$ is connected, then it is dominated by some of its maximum-size complete subgraphs. We denote by ω the maximum clique-size in G.

Choose any K_ω in G. Let v be a vertex at maximum distance from this K_ω. Since G is assumed to be a counterexample, this distance is at least 2.

Clearly, $G - v$ is connected and still has maximum clique size ω. Therefore, by the minimality of G, it is dominated by a subgraph $K \simeq K_\omega$. This K does not dominate v, but v has some neighbor, say w, dominated by K. Since K is of maximum size, $V(K) \setminus N(w) \neq \emptyset$.

Let us observe next that $V(K) \setminus N(w)$ consists of just one vertex. Indeed, otherwise we could select two vertices in $V(K) \setminus N(w)$ and one in $V(K) \cap N(w)$, hence obtaining an induced subgraph (the selected triangle with v and w) isomorphic to P_5^+, a contradiction.

We denote by y the single vertex of $V(K) \setminus N(w)$. It follows that the set $X = V(K - y) \cup \{w\}$ also induces K_ω in G. If it is not dominating, then some vertex z is adjacent to y and nonadjacent to the entire X. We finally choose a vertex $x \in V(K) \cap N(w)$. If vz is not an edge, then v, w, x, y, z induce P_5; and if vz is an edge, then they induce a C_5. In either case, we obtain a contradiction that proves the theorem. ∎

Note added in February 2004. One of the referees has informed us that Theorem 1 has been discovered independently by Goddard and Henning [5]; and another referee has observed that it can also be derived from the result of Zverovich by an argument avoiding Lemma 1.

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