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CENTERS OF n-FOLD TENSOR PRODUCTS OF GRAPHS

Sarah Bendall

and

Richard Hammack

Department of Mathematics
Randolph-Macon College
Ashland, Virginia 23005-5505, USA

e-mail: sdw6z@cms.mail.virginia.edu
e-mail: rhammack@rmc.edu

Abstract

Formulas for vertex eccentricity and radius for the n-fold tensor product $G = \bigotimes_{i=1}^{n} G_i$ of n arbitrary simple graphs G_i are derived. The center of G is characterized as the union of $n+1$ vertex sets of form $V_1 \times V_2 \times \cdots \times V_n$, with $V_i \subseteq V(G_i)$.

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1. Introduction

The *tensor product* of two simple graphs $G_1 = (V(G_1), E(G_1))$ and $G_2 = (V(G_2), E(G_2))$ is the graph $G_1 \otimes G_2$ whose vertex set is $V(G_1) \times V(G_2)$, and whose edge set is $\{(x_1, x_2)(y_1, y_2) |\ x_1y_1 \in E(G_1) \text{ and } x_2y_2 \in E(G_2)\}$. The *n-fold tensor product* of simple graphs $G_1, G_2, \cdots, G_n$, denoted $\bigotimes_{i=1}^{n} G_i$, is the graph whose vertex set is $V(G_1) \times V(G_2) \times \cdots \times V(G_n)$, and whose edge set is $\{(x_1, x_2, \cdots, x_n)(y_1, y_2, \cdots, y_n) | x_iy_i \in E(G_i), 1 \leq i \leq n\}$. This is equivalent to the inductive definition $\bigotimes_{i=1}^{n} G_i = (\bigotimes_{i=1}^{n-1} G_i) \otimes G_n$.

In the literature, the tensor product is also called the *Kronecker product*, the *categorical product*, the *direct product*, or simply the *product*. See Section 5.3 of [3] for greater detail.

The *eccentricity* of a vertex x of a graph G is the maximum distance from x of a vertex y of G. The *radius* of G is the smallest eccentricity of the vertices of G. The *center* of G is the set of vertices whose eccentricity equals the radius of G. See [2] for a standard reference.

This article derives formulas which express vertex eccentricity and radius of an n-fold tensor product in terms of invariants of its factors. We also prove the center of such a graph is a union of $n+1$ vertex sets of the form $V_1 \times V_2 \times \cdots \times V_n$, with $V_i \subseteq V(G_i)$.

Previously, Suh-Ryung Kim [4] treated the case of the tensor product of two graphs, one of which is bipartite. More recently, Abay-Asmerom and Hammack [1] solved the case involving the tensor product of two arbitrary graphs, but the formulas did not generalize to products with more than two factors. The present article solves the problem in complete generality, and the results of [4] and [1] become corollaries and special cases. Moreover, the formulas from [1] are greatly simplified under our approach. The authors thank the referees for their valuable comments and suggestions.

2. Distance in a Tensor Product

This section reviews the notion of distance in a graph, and derives a few results concerning distance in a tensor product. The discussion is phrased in the language of walks.

Recall that a *walk* in G is a sequence of vertices $W = w_0w_1w_2\cdots w_m$, where any two consecutive vertices are adjacent, and form an *edge* of the walk. A walk is regarded as a traversal of its edges in a specified order. The *length* of W, denoted by $|W|$, is the number of edges in the walk (with the understanding that an edge may appear and be counted multiple times). A *trivial* walk consists of a single vertex, and has length 0. Two walks have the same *parity* if the difference of their lengths is even; otherwise they have *opposite parity*. A walk W and an integer q have the same (or opposite) parity if $|W| - q$ is even (or odd). An even (odd) walk is one whose length is even (odd). A walk that begins at vertex x and ends at vertex y is called an *x-y walk*.

The *distance* between two vertices x and y of a graph G, denoted by $d_G(x,y)$, is the length of the shortest x-y walk in G, or ∞ if no such walk exists. The *upper distance* between x and y, denoted $D_G(x, y)$, is the minimum length of an x-y walk whose parity differs from that of $d_G(x, y)$. If G is bipartite or trivial, then no such walk exists, and we say $D_G(x, y) = \infty$. Likewise, $D_G(x, y) = \infty$ if x and y happen to be in different components of G. Note that if G is connected and contains an odd cycle, then $D_G(x, y)$ must be finite. For example, in Figure 1, $d_G(a, d) = 2$, $D_G(a, d) = 3$, $d_G(a, a) = 0$, and $D_G(a, a) = 5$. Notice that D_G is *not* a distance function, as, in particular, $D_G(x, x) > 0$. The notion of upper distance, as well as the definitions in the next paragraph, first appeared in [1].

An x-y walk W in a graph G is called *minimal* if $|W| = d_G(x, y)$, and it is called *slack* if $d_G(x, y) < |W| < D_G(x, y)$. It is called *critical* if $|W| = D_G(x, y)$, and *ample* if $D_G(x, y) < |W|$. For example, if G is the 5-cycle *abcdea*, the walk *ab* is minimal, and *aedcb* is critical. The walk *abcb* is slack, and *abcbcb* is ample. Notice that any minimal walk is necessarily a path. Observe also that any walk in a bipartite graph is either minimal or slack — it can be neither critical nor ample. The following lemma will help prove our main results.

Lemma 1. *Any subwalk of a critical walk is either minimal or critical.*

Proof. Suppose the x-y walk X is a subwalk of a critical w-z walk W. Then $W = AXB$ for (possibly trivial) walks A and B. If X is minimal, there is nothing to prove, so suppose X is not minimal. Let Y be an x-y walk that is shorter than X. If we can show the parity of Y must differ from that of X, then (by the definition of a critical walk) X must be critical. But this is clear. For if Y had the same parity as X, then AYB would be a shorter w-z walk than W, yet it would have the same parity as W, contradicting the fact that W is critical. ■

If each factor G_i in $G = \bigotimes_{i=1}^n G_i$ has a walk $W_i = w_{i0}w_{i1}w_{i2}...w_{im}$ of length m, we denote by $W_1 \otimes W_2 \otimes \cdots \otimes W_n = \bigotimes_{i=1}^n W_i$ the walk $(w_{10}, w_{20}, \cdots, w_{n0})$ $(w_{11}, w_{21}, \cdots, w_{n1})$ $(w_{12}, w_{22}, \cdots, w_{n2}) \cdots (w_{1m}, w_{2m}, \cdots, w_{nm})$ in G. Notice that any walk of length m in G can be written uniquely as $\bigotimes_{i=1}^n W_i$, for appropriate walks W_i in G_i, all of length m.

Next, we present two lemmas concerning distance in an n-fold tensor product. These lemmas are generalizations to n factors of results that appeared in [1]. See [4] and [5] for another approach to distance in a tensor product.

Lemma 2. *Let $x = (x_1, x_2, \cdots, x_n)$ and $y = (y_1, y_2, \cdots, y_n)$ be two vertices of $G = \bigotimes_{i=1}^{n} G_i$ and suppose each factor G_i has a nontrivial x_i-y_i walk W_i. If all walks W_i have the same parity, then $d_G(x, y) \leq \max\{|W_i| \;\; |1 \leq i \leq n\}$.*

Proof. For each integer $1 \leq i \leq n$, denote the walk W_i as $x_i x_{i1} x_{i2} x_{i3} \cdots x_{im_i} y_i$. Choose an integer k, $1 \leq k \leq n$, for which $|W_k| = \max\{|W_i| \;\; |1 \leq i \leq n\}$. For each $i \neq k$, the x_i-y_i walk W_i can be extended to an x_i-y_i walk $\widetilde{W_i}$ of length $|W_k|$ by appending to its end the even walk $y_i x_{im_i} y_i x_{im_i} y_i \cdots x_{im_i} y_i$ of length $|W_k| - |W_i|$. Then $(\bigotimes_{i=1}^{k-1} \widetilde{W_i}) \otimes W_k \otimes (\bigotimes_{i=k+1}^{n} \widetilde{W_i})$ is an x-y walk of length $|W_k|$ in G. Hence $d_G(x, y) \leq |W_k| = \max\{|W_i| \;\; |1 \leq i \leq n\}$. ■

Lemma 3. *Let $x = (x_1, x_2, \cdots, x_n)$ and $y = (y_1, y_2, \cdots, y_n)$ be two vertices of $G = \bigotimes_{i=1}^{n} G_i$. If there is no integer m for which each G_i has an x_i-y_i walk of length m, then $d_G(x, y) = \infty$. Otherwise, $d_G(x, y) = \min\{m \in \mathbb{N} \mid$ each G_i has an x_i-y_i walk of length $m\}$.*

Proof. If there is no integer m for which each G_i has an x_i-y_i walk of length m, then there can be no x-y walk W in G, for such a walk would necessarily have the form $W = \bigotimes_{i=1}^{n} W_i$ with each W_i an x_i-y_i walk of length $m = |W|$. Hence $d_G(x, y) = \infty$.

Now suppose there is some integer m for which each G_i has an x_i-y_i walk W_i of length m. Set $M = \min\{m \in \mathbb{N}|$ each G_i has an x_i-y_i walk of length $m\}$. By Lemma 2, $d_G(x, y) \leq M$. Let $W = \bigotimes_{i=1}^{n} W_i$ be an x-y walk of length $d_G(x, y)$ in G. Then each W_i is an x_i-y_i walk of length $m = d_G(x, y)$, so $d_G(x, y) \geq M$. It follows $d_G(x, y) = M$. ■

The next result is our primary tool for constructing minimal walks in tensor products.

Proposition 1. *A walk $W = \bigotimes_{i=1}^{n} W_i$ in the graph $G = \bigotimes_{i=1}^{n} G_i$ is minimal if and only if one factor W_i is minimal, or one factor is slack and another factor is critical.*

Proof. Say W begins at $x = (x_1, x_2, \cdots, x_n)$ and ends at $y = (y_1, y_2, \cdots, y_n)$, so each W_i is an x_i-y_i walk in G_i.

Suppose W is minimal. First, suppose to the contrary that no factor of W is minimal and no factor is slack. Then each factor W_i is critical or ample, so for each $1 \leq i \leq n$ there is a shorter x_i-y_i walk than W_i. We can assume all these shorter walks have the same parity — the parity opposite to $|W|$ if W has any critical factors, or either parity if all factors are ample. But then Lemma 2 contradicts the minimality of W.

Now suppose that no factor of W is minimal and no factor is critical. Then each factor W_i is slack or ample, so for each $1 \leq i \leq n$ there is a shorter x_i-y_i walk than W_i. We can assume all these shorter walks have the same parity – the same parity as $|W|$ if W has any slack factors, or either parity if all factors are ample. But then Lemma 2 contradicts the minimality of W.

The previous two paragraphs show that if W is minimal, then one factor of W is minimal, or one factor is slack and another is critical.

Conversely, suppose that one factor of W is minimal, or one factor is slack and another factor is critical. If one factor is minimal, then W is minimal by Lemma 3. Next suppose that one factor W_k is slack and another factor W_l is critical. Then any x_k-y_k walk in G_k that is shorter than W_k has the same parity as W_k, while any x_l-y_l walk in G_l that is shorter than W_l has the opposite parity to W_l. As $|W_k| = |W_l|$, we conclude there is no integer $m < |W_k| = |W_l|$ for which there are x_k-y_k and x_l-y_l walks of length m. Then W is minimal by Lemma 3. ∎

3. Eccentricity and Centers

The *eccentricity* of $x \in V(G)$ is $e_G(x) = \max\{d_G(x,y)|y \in V(G)\}$. The *upper eccentricity* of x is $E_G(x) = \max\{D_G(x,y)|y \in V(G)\}$. Notice that $E_G(x) = \infty$ if and only if G is disconnected, bipartite, or trivial. As an illustration of these ideas, each vertex x of the graph G in Figure 1 is labeled with an ordered pair $(e_G(x), E_G(x))$.

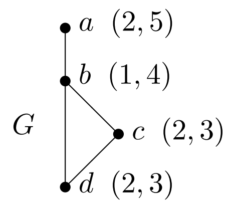

Figure 1

The *radius* of G is $r(G) = \min\{e_G(x)|x \in V(G)\}$, and the *upper radius* is $R(G) = \min\{E_G(x)|x \in V(G)\}$. For example, in Figure 1, $r(G) = 1$, and $R(G) = 3$.

Recall that the *center* of G is the subset of $V(G)$ consisting of all vertices x for which $e_G(x) = r(G)$. For example, the center of the graph G in Figure 1 consists of the single vertex b. Consideration of the upper eccentricity and radii in the factors of an n-fold tensor product will be instrumental in characterizing its center.

4. Results

Now we can compute the eccentricity of a vertex in an n-fold tensor product, and also find the radius and center of such a graph. This is done in Theorems 1, 2 and 3 below. These theorems involve a function μ, defined as follows. If X is a finite multiset with elements in $\mathbb{N} \cup \{\infty\}$, then

$$\mu(X) = \begin{cases} \max(X - \{\max(X)\}) & \text{if } \max(X) \text{ has multiplicity 1,} \\ \max(X) - 1 & \text{otherwise.} \end{cases}$$

In words, μ selects the second-largest element of X, unless X contains more than one largest element, in which case μ returns one less than the largest elements. As examples, $\mu(\{3, 6, 4, 9\}) = 6$, $\mu(\{2, 7, 7, \infty, \}) = 7$, $\mu(\{2, 4, 7, 7\}) = 6$, and $\mu(\{2, 4, \infty, \infty\}) = \infty$.

Theorem 1. *If no factor of $G = \bigotimes_{i=1}^{n} G_i$ is trivial, and $x = (x_1, x_2, \cdots, x_n) \in V(G)$, then $e_G(x_1, x_2, \cdots, x_n) = \mu(\{e_{G_i}(x_i), E_{G_i}(x_i) | 1 \leq i \leq n\})$.*

Proof. For brevity, set $M = \mu(\{e_{G_i}(x_i), E_{G_i}(x_i) | 1 \leq i \leq n\})$. First, we establish $e_G(x) \leq M$. For this it suffices to show any minimal walk $W = \bigotimes_{i=1}^{n} W_i$ in G, beginning at x, satisfies $|W| \leq M$. If W is such a walk, then, by Proposition 1, one factor of W is minimal or one is slack and another is critical. If some factor W_a is minimal, then, since it begins at x_a, we have $|W| = |W_a| \leq e_{G_a}(x_a) \leq M$, by definition of M. If W_a is slack and W_b is critical, then $|W| = |W_a| < E_{G_a}(x_a)$ and $|W| = |W_b| \leq E_{G_b}(x_b)$. It follows that $|W|$ is smaller than the largest element of $\{E_{G_1}(x_i) | 1 \leq i \leq n\}$, yet it is not larger than the second-largest element. Then $|W| \leq M$, by definition of M. This completes the proof that $e_G(x) \leq M$.

The rest of the proof is devoted to showing $e_G(x) \geq M$. Certainly this is true if G is disconnected, for then $e_G(x) = \infty \geq M$. So we may assume henceforward that G is connected. This means every factor G_i is connected and at most one factor is bipartite (c.f. Theorem 5.29 of [3]). To show

$e_G(x) \geq M$, it suffices to construct a minimal walk W in G beginning at x, and satisfying $|W| = M$. The rest of the proof is a construction of such a walk.

Choose indices $1 \leq a, b \leq n$ for which $E_{G_b}(x_b)$ is the largest element of the multiset $\{E_{G_i}(x_i) | 1 \leq i \leq n\}$, and $E_{G_a}(x_a)$ is the largest of the remaining elements once $E_{G_b}(x_b)$ has been removed. Thus $E_{G_a}(x_a) \leq E_{G_b}(x_b)$. Since $E_{G_i}(x_i) = \infty$ if and only if G_i is bipartite, it follows G_b is the only factor of G that can be bipartite, and, if it is, then $E_{G_b}(x_b) = \infty$ and $E_{G_a}(x_a)$ is finite.

For each $1 \leq i \leq n$, with $i \neq a, b$, let W_i be any walk in G_i that begins at x_i and has length M. (Such walks exist because each G_i is connected and nontrivial.) We are going to find walks W_a and W_b of length M for which Proposition 1 implies the walk $\bigotimes_{i=1}^n W_i$ of length M beginning at x is minimal. For the rest of the proof, let Z_a be a critical x_a-z_a walk of length $E_{G_a}(x_a)$ in G_a. Let Y_a be a minimal x_a-z_a walk in G_a. We consider three exhaustive cases.

Case 1. $E_{G_a}(x_a) < E_{G_b}(x_b) = \infty$. This is the case where G_b is bipartite. In the expression for M, the function μ disregards the largest value of $E_{G_b}(x_b) = \infty$ and selects the largest of the remaining values of $\{e_{G_i}(x_i), E_{G_i}(x_i) | 1 \leq i \leq n\}$. Since $e_{G_i}(x_i) < E_{G_i}(x_i)$ for each i, it follows that $M = \max\{e_{G_b}(x_b), E_{G_a}(x_a)\}$. We consider the cases $M = E_{G_a}(x_a)$ and $M = e_{G_b}(x_b)$ separately. First suppose $M = E_{G_a}(x_a)$. Let W_a be the critical walk Z_a of length $E_{G_a}(x_a) = M$, and let W_b be an arbitrary walk in G_b beginning at x_b and having length M. Then W_b is either minimal or slack because G_b is bipartite. The walk $\bigotimes_{i=1}^n W_i$ begins at x, has length M, and is minimal by Proposition 1. Next suppose $M = e_{G_b}(x_b)$. Let W_b be a minimal walk in G_b starting at x_b and having length $e_{G_b}(x_b) = M$, and let W_a be an arbitrary walk in G_a beginning at x_a and having length M. Then the walk $\bigotimes_{i=1}^n W_i$ begins at x, has length M, and is minimal by Proposition 1.

This takes care of the case where the factor G_b of G is bipartite, so $E_{G_b}(x_b)$ is finite for the rest of the proof. Let Z_b be a critical x_b-z_b walk of length $E_{G_b}(x_b)$ in G_b, and denote the last edge in Z_b as $y_b z_b$. Let Y_b be a minimal x_b-z_b walk in G_b.

Case 2. $E_{G_a}(x_a) < E_{G_b}(x_b) < \infty$. As in the previous case, $M = \max\{e_{G_b}(x_b), E_{G_a}(x_a)\}$, and, as in that case, if $M = e_{G_b}(x_b)$ there is a minimal walk in G, starting at x and having length M. Thus suppose

$M = E_{G_a}(x_a)$, so $e_{G_b}(x_b) \leq E_{G_a}(x_a) < E_{G_b}(x_b)$. Let W_a be the critical walk Z_a of length $E_{G_a}(x_a) = M$. Then since Y_b is minimal and begins at x_b, we have $|Y_b| \leq e_{G_b}(x_b) \leq E_{G_a}(x_a) = |W_a|$.

If the integer $k = |W_a| - |Y_b|$ is even, extend Y_b to a walk W_b of length M by appending to its end the even walk $z_b y_b z_b y_b \cdots y_b z_b$ of length k. Then the x_b-z_b walk W_b is minimal or slack because $|W_b| = |W_a| = E_{G_a}(x_a) < E_{G_b}(x_b) = D_{G_b}(x_b, z_b)$. Then the walk $\bigotimes_{i=1}^{n} W_i$ begins at x, has length M, and is minimal by Proposition 1.

On the other hand, if k is odd, extend Y_b to a x_b-y_b walk W_b of length M by appending to its end the odd walk $z_b y_b z_b y_b \cdots z_b y_b$ of length k. If we can show that W_b is minimal or slack, then $\bigotimes_{i=1}^{n} W_i$ will be the required minimal walk of length M beginning at x.

Now, the parity of Y_b is opposite to that of W_a (since k is odd) it is also opposite to Z_b by construction. Therefore $|W_a| = E_{G_a}(x_a)$ and $|Z_b| = E_{G_b}(x_b)$ have the same parity, and as the former is smaller that the latter we infer $E_{G_a}(x_a) < E_{G_b}(x_b) - 1$. And, since $e_{G_b}(x_b) \leq E_{G_a}(x_a)$, we get $e_{G_b}(x_b) < E_{G_b}(x_b) - 1$. Let X_b be the walk Z_b with its last edge $y_b z_b$ removed. By Lemma 1, X_b is either minimal or critical. But it cannot be minimal, for then $e_{G_b}(x_b) \geq d_{G_b}(x_b, y_b) = |X_b| = |Z_b| - 1 = E_{G_b}(x_b) - 1$, contradicting the above inequality. Therefore X_b is a critical x_b-y_b walk, so $|X_b| = D_{G_b}(x_b, y_b)$. But this means W_b is a minimal or slack x_b-y_b walk since $|W_b| = |W_a| = E_{G_a}(x_a) < E_{G_b}(x_b) - 1 = |X_b| = D_{G_b}(x_b, y_b)$.

Case 3. $E_{G_a}(x_a) = E_{G_b}(x_b) < \infty$. In this case, $M = E_{G_b}(x_b) - 1$ by definition of M. Let W_b be the walk Z_b with its last edge $y_b z_b$ removed, so $|W_b| = E_{G_b}(x_b) - 1 = M$, and W_b is either minimal or critical by Lemma 1. By construction, W_b and Y_a have the same parity (namely that opposite of $|Z_a| = |Z_b|$), and $|Y_a| \leq E_{G_a}(x_a) - 1 = |W_b|$. Extend Y_a to a x_a-z_a walk W_a of length M by appending to its end an even walk $z_a y_a z_a y_a \cdots y_a z_a$ of length $|W_b| - |Y_a|$. Then W_a is a minimal or slack x_a-z_a walk because $|W_a| = |W_b| = E_{G_b}(x_b) - 1 < E_{G_b}(x_b) = E_{G_a}(x_a) = D_{G_a}(x_a, z_a)$. The walk $\bigotimes_{i=1}^{n} W_i$ begins at x, has length M, and is minimal by Proposition 1. The proof is complete. ∎

Theorem 2. *If every factor of $G = \bigotimes_{i=1}^{n} G_i$ is nontrivial, then G has radius $r(G) = \mu(\{r(G_i), R(G_i) | 1 \leq i \leq n\})$.*

Proof. Choose a vertex $x = (x_1, x_2, \cdots, x_n)$ of G with the property that $r(G) = e_G(x)$. Using Theorem 1, $r(G) = e_G(x) = \mu(\{e_{G_i}(x_i), E_{G_i}(x_i) | 1 \leq i \leq n\}) \geq \mu(\{r(G_i), R(G_i) | 1 \leq i \leq n\})$.

To establish the reverse inequality, let $R(G_a) \leq R(G_b)$ be the two largest upper radii in the multiset $\{R(G_i)|1 \leq i \leq n\}$, and consider the following two cases.

If $R(G_a) = R(G_b)$, then for each $1 \leq i \leq n$, choose $x_i \in V(G_i)$ for which $E_{G_i}(x_i) = R(G_i)$. Then $E_{G_a}(x_a) = E_{G_b}(x_b)$ are the largest elements in the multiset $\{e_{G_i}(x_i), E_{G_i}(x_i)|1 \leq i \leq n\}$. Thus $r(G) \leq e_G(x_1, x_2, \cdots, x_n) = \mu(\{e_{G_i}(x_i), E_{G_i}(x_i)|1 \leq i \leq n\}) = E_{G_a}(x_a) - 1 = R(G_a) - 1 = \mu\{r(G_i), R(G_i)|1 \leq i \leq n\}$.

If $R(G_a) < R(G_b)$, choose $x_b \in V(G_b)$ for which $e_{G_b}(x_b) = r(G_b)$, and for $i \neq b$ take $x_i \in V(G_i)$ for which $E_{G_i}(x_i) = R(G_i)$. Then for $i \neq b$ we have $E_{G_i}(x_i) = R(G_i) \leq R(G_a) < R(G_b) \leq E_{G_b}(x_b)$. Thus $E_{G_b}(x_b)$ is the sole largest element of the multiset $\{e_{G_i}(x_i), E_{G_i}(x_i)|1 \leq i \leq n\}$, and the second-largest is either $e_{G_b}(x_b) = r(G_b)$ or $E_{G_a}(x_a) = R(G_a)$. Hence $R(G) \leq e_G(x_1, x_2, \cdots, x_n) = \mu(\{e_{G_i}(x_i), E_{G_i}(x_i)|1 \leq i \leq n\}) = \max\{e_{G_b}(x_b), E_{G_a}(x_a)\} = \max\{r(G_b), R(G_a)\} = \mu(\{r(G_i), R(G_i)|1 \leq i \leq n\})$. ■

The next theorem is an explicit description of the center of $G = \bigotimes_{i=1}^{n} G_i$. To set the stage, for each $1 \leq i \leq n$, define the following sets.

$$\begin{aligned}
X_i &= \{x \in V(G_i)|E_{G_i}(x) \leq r(G)\}, \\
\overline{X}_i &= \{x \in V(G_i)|E_{G_i}(x) \leq r(G) + 1\}, \\
\widetilde{X}_i &= \{x \in V(G_i)|e_{G_i}(x) \leq r(G)\}.
\end{aligned}$$

Observe that these sets are nested in the fashion $X_i \subseteq \overline{X}_i \subseteq \widetilde{X}_i$.

Theorem 3. *The center of $G = \bigotimes_{i=1}^{n} G_i$ is the following union of $n+1$ vertex sets:*
$(\overline{X}_1 \times \overline{X}_2 \times \cdots \times \overline{X}_n) \cup (\widetilde{X}_1 \times X_2 \times \cdots \times X_n) \cup (X_1 \times \widetilde{X}_2 \times \cdots \times X_n) \cup \cdots \cup (X_1 \times X_2 \times \cdots \times \widetilde{X}_n)$.

Proof. The theorem is obviously true if some factor of G is trivial or disconnected, or if more than one factor is bipartite. Thus we may assume each factor of G is connected and nontrivial, and at most one factor is bipartite.

We first verify that each set in the above union is in the center of G. For this it suffices to show that if vertex $x = (x_1, x_2, \cdots, x_n)$ is in one of these sets, then $e_G(x) \leq r(G)$.

If $x \in \overline{X}_1 \times \overline{X}_2 \times \cdots \times \overline{X}_n$, then $E_{G_i}(x_i) \leq r(G)+1$, for $1 \leq i \leq n$, and since $e_{G_i}(x_i) < E_{G_i}(x_i)$, it follows that $e_{G_i}(x_i) \leq r(G)$ for each i. Consequently, Theorem 1 gives $e_G(x) = \mu(\{e_{G_i}(x_i), E_{G_i}(x_i) | 1 \leq i \leq n\}) \leq r(G)$, since μ of a multiset is always less than its largest element, and in this case the largest element is at most $r(G)+1$.

If $x \in X_1 \times X_2 \times \cdots \times \widetilde{X}_k \times \cdots \times X_n$, then $e_{G_k}(x_k) \leq r(G)$ and $e_{G_i}(x_i) < E_{G_i}(x_i) \leq r(G)$ for $i \neq k$. Thus no element in the multiset $\{e_{G_i}(x_i), E_{G_i}(x_i) | 1 \leq i \leq n\}$ is greater than $r(G)$, with the possible exception of $E_{G_k}(x_k)$. By Theorem 1 and definition of M it follows that $e_G(x) = \mu(\{e_{G_i}(x_i), E_{G_i}(x_i) | 1 \leq i \leq n\}) \leq r(G)$.

Next, suppose x is in the center of G, so $e_G(x) = r(G)$. Let $E_{G_a}(x_a) \leq E_{G_b}(x_b)$ be the two largest upper eccentricities in the multiset $\{E_{G_i}(x_i) | 1 \leq i \leq n\}$.

If $E_{G_a}(x_a) = E_{G_b}(x_b)$, then $r(G) = e_G(x) = \mu\{e_{G_i}(x_i), E_{G_i}(x_i) | 1 \leq i \leq n\} = E_{G_a}(x_a) - 1$. This means $E_{G_i}(x_i) \leq E_{G_a}(x_a) \leq r(G)+1$ for $1 \leq i \leq n$, so $x \in \overline{X}_1 \times \overline{X}_2 \times \cdots \times \overline{X}_n$.

On the other hand, suppose $E_{G_a}(x_a) < E_{G_b}(x_b)$. Then using Theorem 1, $r(G) = e_G(x) = \mu(\{e_{G_i}(x_i), E_{G_i}(x_i) | 1 \leq i \leq n\})$. But μ will ignore the largest value of $E_{G_b}(x_b)$ and pick the largest of the remaining values. Hence

$$r(G) = \max\{E_{G_1}(x_1), E_{G_2}(x_2), \cdots, E_{G_{b-1}}(x_{b-1}), e_{G_b}(x_b),$$
$$E_{G_{b+1}}(x_{b+1}), \cdots, E_{G_n}(x_n)\}.$$

It follows that $e_{G_b}(x_b) \leq r(G)$, and $E_{G_i}(x_i) \leq r(G)$ for $i \neq b$. This means $x \in X_1 \times X_2 \times \cdots \times \widetilde{X}_b \times \cdots \times X_n$. ∎

Theorems 1, 2, and 3 simplify greatly if one or more factors of the tensor product is bipartite or disconnected. Of course, if one factor is disconnected or if more than one factor of is bipartite, then G is disconnected, and its radius and all its vertex eccentricities are infinite. Moreover, $X_i = \overline{X}_i = \widetilde{X}_i = V(G_i)$ in such cases, and Theorems 1, 2 and 3 give the expected result that the eccentricities and radius are infinite and every vertex of G is central. That is not particularly interesting. What *is* interesting is the case where *exactly one* of the factors, say G_1, is bipartite, while all other factors are connected and have odd cycles. In this situation $E_{G_1}(x_1) = \infty$, while $E_{G_i}(x_i)$ is finite when $1 < i \leq n$. In Theorem 1, μ disregards the largest value of $E_{G_1}(x_1) = \infty$ and selects the largest of the remaining finite values. Theorem 1 thus becomes

$e_G(x_1, x_2, \cdots, x_n) = \max\{e_{G_1}(x_1), E_{G_2}(x_2), E_{G_3}(x_3), \cdots, E_{G_n}(x_n)\}$. Theorem 2 reduces to $r(G) = \max\{r(G_1), R(G_2), R(G_3), \cdots, R(G_n)\}$, and in Theorem 3, $X_1 = \overline{X}_1 = \emptyset$. These observations prove the following.

Corollary 1. *Suppose every factor of $G = \bigotimes_{i=1}^n G_i$ is connected, and G_1 is bipartite, while all other factors have odd cycles. Then for any vertex $x = (x_1, x_2, \cdots, x_n)$ of G, $e_G(x) = \max\{e_{G_1}(x_1), E_{G_2}(x_2), E_{G_3}(x_3), \cdots, E_{G_n}(x_n)\}$. Also G has radius $r(G) = \max\{r(G_1), R(G_2), R(G_3), \cdots, R(G_n)\}$. Moreover, the center of G is the vertex set $\widetilde{X}_1 \times X_2 \times X_3 \times \cdots \times X_n$.*

For $n = 2$, this corollary reduces to Kim's Theorem 3 of [4]. Kim defines $d_e(a, b)$ and $d_o(a, b)$ to be the lengths of the shortest a-b walks of even and odd lengths, respectively, in a graph G. The *double eccentricity* of a vertex a of G is defined to be $de_G(a) = \max\{d_e(a, b), d_o(a, b) | b \in V(G)\}$, and the *double radius* is defined to be $dr(G) = \min\{de_G(a) | a \in V(G)\}$. Kim proves that if G is bipartite, then $e_{G \otimes H}(a, x) = \max\{e_G(a), de_H(x)\}$, and (a, x) is in the center of $G \otimes H$ if and only if $e_{G \otimes H}(a, x) = \max\{r(G), dr(H)\}$ (i.e., that $r(G \otimes H) = \max\{r(G), dr(H)\}$). Simply observe $de_G(a) = E_G(a)$, and $dr(G) = R(G)$, and these results are our Corollary 1 for the case $n = 2$.

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