Discussiones Mathematicae
Graph Theory 24 (2004) 485–490

DOMINANT-MATCHING GRAPHS

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Abstract

We introduce a new hereditary class of graphs, the dominant-matching graphs, and we characterize it in terms of forbidden induced subgraphs.

Keywords and phrases: domination number, matching number, hereditary class of graphs.

2000 Mathematics Subject Classsification: 05C69.

1. Dominant-Covering Graphs

Let G be a graph. The *neighborhood* of a vertex $x \in V(G)$ is the set $N_G(x) = N(x)$ of all vertices in G that adjacent to x. If vertices x and y of G are adjacent (respectively, non-adjacent), we shall use notation $x \sim y$ (respectively, $x \not\sim y$). For disjoint sets $X, Y \subseteq V(G)$, we write $X \sim Y$ (respectively, $X \not\sim Y$) to indicate that each vertex of X is adjacent to each vertex of Y (respectively, no vertex of X is adjacent to a vertex of Y).

A set $D \subseteq V(G)$ is called a *dominating set* in G if $V(G) = N[D] = \bigcup_{d \in D} N[d]$, where $N[d] = N(d) \cup \{d\}$ is the closed neighborhood of d. A *minimum* dominating set in G is a dominating set having the smallest cardinality. This cardinality is the *domination number* of G, denoted by $\gamma(G)$.

A set $C \subseteq V(G)$ is called a *vertex cover* in G if every edge of G is incident to at least one vertex in C. The minimum cardinality of a vertex cover in G is the *vertex covering number* of G, denoted by $\tau(G)$.

Definition 1. A graph G is called a *dominant-covering graph* if $\gamma(H) = \tau(H)$ for every isolate-free induced subgraph H of G.

Many similarly defined classes were characterized in terms of forbidden induced subgraphs by Zverovich [3], Zverovich [4], Zverovich and Zverovich [5], and Zverovich and Zverovich [6]. We give such a characterization for dominant-covering graphs, and then we extend it to dominant-matching graphs.

Theorem 1. *A graph G is a dominant-covering graph if and only if G does not contain any of $G_1, G_2, \ldots, G_{10}$ shown in Figure 1 as an induced subgraph.*

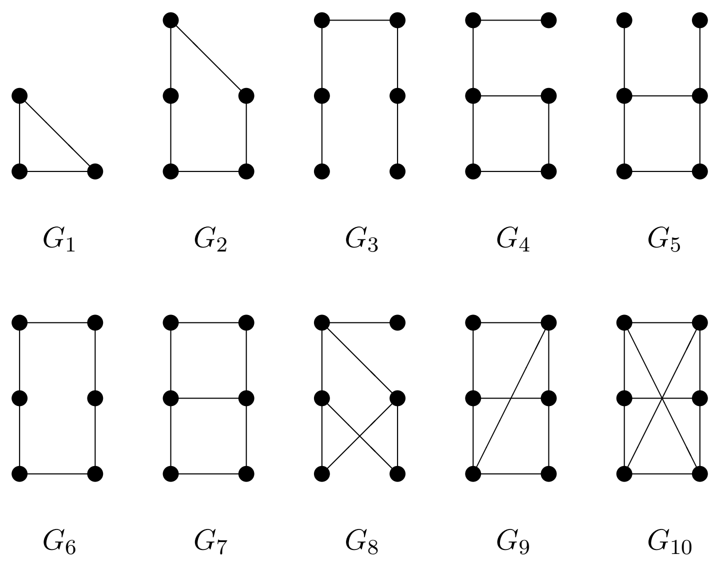

Figure 1. Forbidden induced subgraphs for dominant-covering graphs.

Proof. Necessity. It is easy to check that the graphs $G_i \in \{G_1, G_2, \ldots, G_{10}\}$ (Figure 1) satisfies $2 = \gamma(G_i) < \tau(G_i)$, and therefore they are not dominant-covering. It follows that no one of them can be an induced subgraph of a dominant-covering graph.

Sufficiency. Let G be a minimal forbidden induced subgraph for the class of all dominant-covering graphs. Suppose that $G \notin \{G_1, G_2, \ldots, G_{10}\}$. By minimality, G does not contain any of $G_1, G_2, \ldots, G_{10}$ as an induced subgraph. Also, each proper induced subgraph of G is a dominant-covering graph, therefore $\gamma(G) < \tau(G)$.

We consider a minimum dominating set D of G such that D covers the maximum possible number of edges of G [among all minimum dominating sets of G]. If D covers all edges of G, then $\gamma(G) = \tau(G)$, a contradiction. Thus, we may assume that an edge $e = uv$ is not covered by D.

Since D is a dominating set, there exist vertices w and x in D which are adjacent to u and v, respectively. If $w = x$ then $G(u, v, w) \cong G_1$, a contradiction. Therefore $w \neq x$. Moreover, u is non-adjacent to x, and v is non-adjacent to w.

Let $D_u = (D \setminus \{w\}) \cup \{u\}$. We have $|D_u| = |D|$, and D_u covers the edges uv, uw and vx.

Case 1. D_u is not a dominating set.
Suppose that D_u does not dominate a vertex y of G. Since D is a dominating set, y is adjacent to w. Thus, the edge $f = yw$ is covered by D, and it is not covered by D_u.

Case 2. D_u is a dominating set.
Clearly, D_u is a minimum dominating set. The choice of D implies that there exists an edge f which is covered by D and which is not covered by D_u. Obviously, f is incident to the vertex w, i.e., we may assume that $f = yw$ for some vertex $y \notin \{u, v, x\}$.

In both cases, we have obtained that there exists some edge yw covered by D and not covered by D_u. If y is adjacent to u or x, then G contains G_1 or G_2 as an induced subgraph, a contradiction. Hence edge-set of the induced subgraph $H = G(u, v, w, x, y)$ is one of the following:

Variant 1H: $E(H) = \{uv, uw, vx, wy\}$, or
Variant 2H: $E(H) = \{uv, uw, vx, wy, vy\}$, or
Variant 3H: $E(H) = \{uv, uw, vx, wy, wx\}$, or
Variant 4H: $E(H) = \{uv, uw, vx, wy, wx, vy\}$.

Now we consider the set $D_v = (D \setminus \{x\}) \cup \{v\}$. By symmetry, there exists an edge $g = zx$ which is covered by D and which is not covered by D_v. Again, we have four variants for the induced subgraph $F = G(u, v, w, x, z)$:

Variant 1F: $E(H) = \{uv, uw, vx, xz\}$, or
Variant 2F: $E(H) = \{uv, uw, vx, xz, uz\}$, or
Variant 3F: $E(H) = \{uv, uw, vx, xz, wx\}$, or
Variant 4F: $E(H) = \{uv, uw, vx, xz, wx, uz\}$.

Note that the vertices y and z may or may not be adjacent. Combinations of Variants 1H, 2H, 3H, 4H and Variants 1F, 2F, 3F, 4F shows that the set $\{u, v, w, x, y, z\}$ induces one of $G_3, G_4, \ldots, G_{10}$, a contradiction. ■

2. Dominant-Matching Graphs

The *matching number* of a graph G is denoted by $\mu(G)$, i.e., $\mu(G)$ is the maximum cardinality of a matching in G.

Proposition 1 (see Lovász and Plummer [1]). *$\mu(G) \leq \tau(G)$ for every graph G.*

Proposition 2 (Volkmann [2]). *$\gamma(G) \leq \mu(G)$ for every graph G without isolated vertices.*

Definition 2. A graph G is called a *dominant-matching graph* if $\gamma(H) = \mu(H)$ for every isolate-free induced subgraph H of G.

Note that the class of all graphs such that $\mu(H) = \tau(H)$ for every induced subgraph H of G coincides with the class of all bipartite graphs, see e.g. Minimax König's Theorem in Lovász and Plummer [1]. Now we extend Theorem 1 by characterization of the dominant-matching graphs in terms of forbidden induced subgraphs.

Theorem 2. *A graph G is a dominant-matching graph if and only if G does not contain any of $G_3, G_4, \ldots, G_{10}$ (Figure 1) and H_1, H_2, H_3, H_4, H_5 (Figure 2) as an induced subgraph.*

Proof. Necessity. It can be directly checked that

- $\gamma(H_i) = 1$ and $\mu(H_i) = 2$ for $i = 1, 2, 3$,
- $\gamma(H_j) = 2$ and $\mu(H_j) = 3$ for $j = 4, 5$, and
- $\gamma(G_k) = 2$ and $\mu(G_k) = 3$ for $k = 3, 4, \ldots, 10$.

Therefore none of $G_3, G_4, \ldots, G_{10}$ (Figure 1) and H_1, H_2, H_3, H_4, H_5 (Figure 2) can be an induced subgraph of a dominant-matching graph.

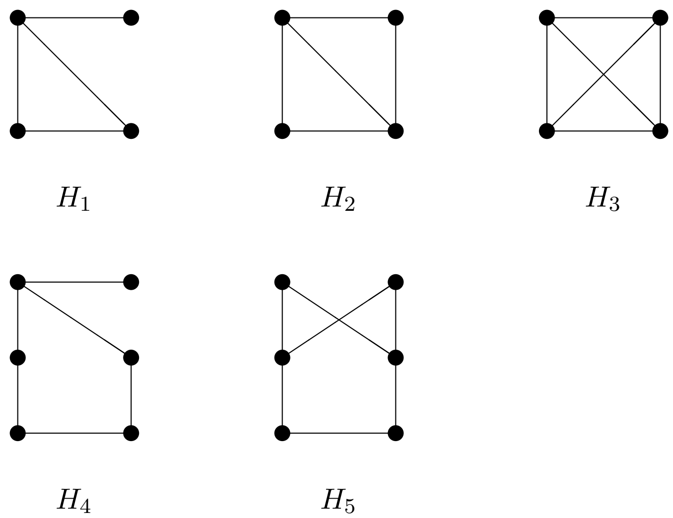

Figure 2. Some forbidden induced subgraphs for dominant-matching graphs.

Sufficiency. Suppose that the statement does not hold. We consider a minimal graph G such that

- G does not contain any of $G_3, G_4, \ldots, G_{10}$ (Figure 1) and H_1, H_2, H_3, H_4, H_5 (Figure 2) as an induced subgraph, and
- G is not a dominant-matching graph.

The minimality of G means that each proper induced subgraph of G is a dominant-matching graph. If G does not contain both G_1 and G_2 (Figure 1) induced subgraphs, then G is a dominant-covering graph by Theorem 1. Hence $\gamma(G) = \tau(G)$. Proposition 1 and Proposition 2 imply that $\gamma(G) = \mu(G)$, a contradiction to the choice of G.

Thus, it is sufficient to consider two cases where either G_1 or G_2 is an induced subgraph of G. By minimality of G, $\gamma(G) < \mu(G)$, and G is a connected graph.

Case 1. G_1 is an induced subgraph of G.
Since $\gamma(G) < \mu(G)$, $G \neq G_1$. By connectivity of G, there exists a vertex $u \in V(G) \setminus V(G_1)$ that is adjacent to at least one vertex of G_1. Clearly, the

set $V(G_1) \cup \{u\}$ induces one of H_1, H_2 or H_3 (Figure 2), a contradiction to the choice of G.

Case 2. G_2 is an induced subgraph of G.
As before, there exists a vertex $u \in V(G) \setminus V(G_2)$ that is adjacent to at least one vertex of G_2. We may assume that G has no induced G_1 [see Case 1]. Hence the set $V(G_2) \cup \{u\}$ induces either H_4 or H_5 (Figure 2), a contradiction to the choice of G. ■

Acknowledgement

We thank the referee for valuable suggestions and remarks.

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Received 14 July 2003
Revised 17 November 2003