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SHORT PATHS IN 3-UNIFORM QUASI-RANDOM HYPERGRAPHS

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Abstract

Frankl and Rödl [3] proved a strong regularity lemma for 3-uniform hypergraphs, based on the concept of δ-regularity with respect to an underlying 3-partite graph. In applications of that lemma it is often important to be able to "glue" together separate pieces of the desired subhypergraph. With this goal in mind, in this paper it is proved that every pair of typical edges of the underlying graph can be connected by a hyperpath of length at most seven. The typicality of edges is defined in terms of graph and hypergraph neighborhoods, and it is shown that all but a small fraction of edges are indeed typical.

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1. Introduction

Szemerédi's Regularity Lemma for graphs [10] has become one of the most important tools in modern graph theory. When solving some problems, it allows to concentrate on quasi-random subgraphs (called also ϵ-regular pairs) instead of considering the whole graph. Notable examples of such approach can be found in [5, 6].

Recently, Frankl and Rödl [3] have established an analogous, but much deeper regularity lemma for 3-uniform hypergraphs, with potentially even broader scope of applications. So far only few papers applied that lemma to solve some combinatorial problems (see [3, 9]). The general scheme is similar to that for graphs: the regularity lemma is applied to the entire hypergraph which is thus split into quasi-random blocks. Then a desired structure is built within some blocks and finally connected together by short paths constructed in other blocks.

While the existence of such paths in quasi-random graphs is almost trivial (see Section 2.1), it is not obvious at all for 3-uniform hypergraphs. One reason is the more complex structure of hypergraphs, and, in particular, the interaction between pairs (graph edges) and triplets (hypergraph edges).

This paper is a first step in the study of the short paths in quasi-random hypergraphs in the sense of the regularity lemma of Frankl and Rödl. We show that every pair of typical edges can be connected by a path of length at most seven (cf. Theorem 2.18 below).

We give relevant notation, background definitions and elementary facts in the forthcoming section. Section 3 provides the proof of our main result, while in the Appendix it is proved that almost all edges are, in a sense, typical.

2. Preliminaries

2..1 Graphs

Let $G = (V, E)$ be a given graph, where V and E are the vertex-set and the edge-set of G. We write $N_G(v)$ for the set of neighbors of $v \in V$ in the graph G. The size of $N_G(v)$ is $|N_G(v)| = \deg_G(v)$, the *degree* of v. We set $N_G(x, y) = N_G(x) \cap N_G(y)$ as the set of common neighbors of $x, y \in V$ in G. For a vertex $v \in V$ and a set $U \subset V - \{v\}$, we write $\deg_G(v, U)$ for the number of edges $\{v, u\}$ of G so that $u \in U$.

When U, W are subsets of V, we define

$$e_G(U, W) = |\{(x, y) : x \in U, y \in W, \{x, y\} \in E\}|\,.$$

For nonempty and disjoint U and W,

$$d_G(U, W) = \frac{e_G(U, W)}{|U||W|}$$

is the *density* of the graph G between U and W, or simply, the density of the pair (U, W).

Definition 2.1. Given $\epsilon_1, \epsilon_2 > 0$, a bipartite graph G with bipartition (V_1, V_2), where $|V_1| = n$ and $|V_2| = m$, is called (ϵ_1, ϵ_2)-*regular* if for every pair of subsets $U \subseteq V_1$ and $W \subseteq V_2$, $|U| > \epsilon_1 n, |W| > \epsilon_1 m$, the inequalities

$$d - \epsilon_2 < d_G(U, W) < d + \epsilon_2$$

hold for some real number $d > 0$. We may then also say that G, or the pair (V_1, V_2), is $(d; \epsilon_1, \epsilon_2)$-*regular.* Moreover, if $\epsilon_1 = \epsilon_2 = \epsilon$, we will use the names $(d; \epsilon)$-*regular* and ϵ-*regular.*

For example, according to the above definition, a complete bipartite graph has density equal to 1. Therefore it is ϵ-regular for all $\epsilon > 0$.

Remark 2.2. Note that each (ϵ_1, ϵ_2)-regular graph is ϵ-regular, where $\epsilon = \max\{\epsilon_1, \epsilon_2\}$. Note also that if G is $(d; \epsilon_1, \epsilon_2)$-regular, $|V_1| = |V_2| = n$, $U \subseteq V_1$, $W \subseteq V_2$, $|U| > \epsilon_1' n$ and $|W| > \epsilon_1' n$, where $\epsilon_1' > \epsilon_1$, then the induced subgraph $G[U, W]$ is $(d; \epsilon_1/\epsilon_1', \epsilon_2)$-regular.

The celebrated Szemerédi Regularity Lemma [10] asserts that the vertex set of every, sufficiently large graph can be decomposed into a bounded number t of parts so that all but at most $\epsilon\binom{t}{2}$ pairs of these parts induce ϵ-regular graphs.

Let $G = (V, E)$ be a $(d; \epsilon)$-regular bipartite graph with bipartition (V_1, V_2), where $|V_1| = |V_2| = n$ and $d > 2\epsilon$. In general, because G may contain isolated vertices, nothing can be said about the diameter of G. According to the definition of an ϵ-regular pair, we can control only pairs of sets of vertices each consisting of at least ϵn vertices. Therefore, it seems to be natural that the existence of a short path between two vertices $x, y \in V$ is guaranteed by the conditions:

$$\deg_G(x) > \epsilon n \quad \text{and} \quad \deg_G(y) > \epsilon n. \tag{1}$$

Indeed, in the case when $x \in V_1$ and $y \in V_2$, we set $A = N_G(x) \subseteq V_2$ and

$B = N_G(y) \subseteq V_1$ and observe that by (1)

$$d_G(A, B) > d - \epsilon > 0.$$

Thus there exists an edge $e = \{v, w\}$ with $v \in A$ and $w \in B$, and so the vertices x, v, w, y form in G a path from x to y of length three.

Slightly harder is the case when x and y belong to the same partition class, say V_1. To handle it, we need a simple property of ϵ-regular graphs. For a later application, we give it here in a more general form.

Fact 2.3. For all real $\epsilon_1, \epsilon_2 > 0$ and $d > 0$, and for all integers n and m, the following holds. Let G be a $(d; \epsilon_1, \epsilon_2)$-regular bipartite graph with a bipartition (V_1, V_2), where $|V_1| = n, |V_2| = m$. Further, let $A \subseteq V_2$, where $|A| > \epsilon_1 m$. Then all but at most $2\epsilon_1 n$ vertices $x \in V_1$ satisfy

$$(d - \epsilon_2)|A| < \deg_G(x, A) < (d + \epsilon_2)|A|.$$

Proof. We will show that no more than $2\epsilon_1 n$ vertices of V_1 violate either of the above inequalities. Suppose not. Then, without loss of generality, we may assume that there exists a subset B of V_1, $|B| > \epsilon_1 n$, such that for each vertex $x \in B$ we have $\deg_G(x, A) \geq |A|(d + \epsilon_2)$. Then

$$d_G(A, B) = \frac{e_G(A, B)}{|A||B|} \geq \frac{|B||A|(d + \epsilon_2)}{|A||B|} = d + \epsilon_2.$$

This contradiction with the $(d; \epsilon_1, \epsilon_2)$-regularity of the graph G concludes the proof. ∎

Returning to the question of the existence of a short path between x and y, as before we set $A = N_G(x) \subseteq V_2$ and $B = N_G(y) \subseteq V_2$. Assuming again that $d > 2\epsilon$, from (1) we conclude that there exists a vertex $z \in B$ such that $\deg_G(z) \geq (d - \epsilon)n > \epsilon n$. Otherwise $d_G(B, V_1) < d - \epsilon$ which contradicts the ϵ-regularity of G. We set $C = N_G(z) \subseteq V_1$, and, again by (1), we find an edge between C and A. Thus, there is a path of length four between x and y.

Remark 2.4. The $(d; \epsilon)$-regular graphs are often called quasi-random because they resemble random bipartite graphs $G(n, n, d)$ with edge probability

d (see [1] and [4]). Random graphs have indeed more uniform structure. E.g., it is well known that asymptotically almost surely (a.a.s.), the diameter of $G(n, d)$, d — constant, is three.

In this paper a similar problem is considered for quasi-random, 3-uniform, 3-partite hypergraphs. However, since the hypergraphs studied here will be "planted" on 3-partite, quasi-random graphs, before turning to hypergraphs, we will prove a few more simple facts about such graphs.

Definition 2.5. Let G be a 3-partite graph with a fixed 3-partition $V_1 \cup V_2 \cup V_3$. We shall write $G = \bigcup_{1 \leq i < j \leq 3} G^{ij}$, where $G^{ij} = G[V_i, V_j] = \{\{v_i, v_j\} \in E : v_i \in V_i, v_j \in V_j\}$. We call (G^{12}, G^{13}, G^{23}) a *triad.* Let $\mathbf{d} = (d_{12}, d_{13}, d_{23})$, where $d_{ij} > 0$, $1 \leq i < j \leq 3$, and $\epsilon > 0$ be given. We call G a $(\mathbf{d}, \epsilon)$-*triad* if each bipartite graph G^{ij}, $1 \leq i < j \leq 3$, is $(d_{ij}; \epsilon)$-regular. If all $d_{ij} = d$ then we call G a (d, ϵ)-*triad.*

Definition 2.6. Let G be a (d, ϵ)-triad, where $|V_1| = |V_2| = |V_3| = n$. We call a vertex $x \in V_i$ *proper* if the following holds:

$$n(d - \epsilon) < \deg_{G^{ij}}(x) < n(d + \epsilon),$$

$$n(d - \epsilon) < \deg_{G^{ik}}(x) < n(d + \epsilon),$$

where here and in similar statements throughout the paper we assume that $\{i, j, k\} = \{1, 2, 3\}$.

It follows immediately from Fact 2.3 that most vertices are proper.

Corollary 2.7. *For all real* $0 < \epsilon, d < 1$ *and for all integer* n*, the following holds. Let* G *be a* (d, ϵ)*-triad, where* $|V_1| = |V_2| = |V_3| = n$*. Then for each* $i = 1, 2, 3$*, all but at most* $4\epsilon|V_i|$ *vertices of* V_i *are proper.*

The next result says that also most pairs of vertices have typical joint neighborhoods.

Fact 2.8. For all $\epsilon > 0$ and $d > 2\epsilon$ and for all integer n, the following holds. Let G be a (d, ϵ)-triad, where $|V_1| = |V_2| = |V_3| = n$. Then for all but at most $4\epsilon n^2$ pairs of vertices $(x, y) \in V_i \times V_j$ we have

$$n(d - \epsilon)^2 < |N_G(x, y)| < n(d + \epsilon)^2.$$

Proof. Using Fact 2.3 (with $A = V_k$) we see that for all but at most $2\epsilon n$ vertices $x \in V_i$ the inequalities $n(d-\epsilon) < \deg_{G^{ik}}(x) < n(d+\epsilon)$ hold. We pick one such vertex x and note that $|N_{G^{ik}}(x)| > \epsilon n$. Now, using again Fact 2.3, this time to G^{jk} and with $A = N_{G^{ik}}(x)$, we conclude the proof. ∎

When we investigate 3-uniform hypergraphs, it will be important to be able to count triangles in quasi-random triads.

Definition 2.9. For a triad G, let $T(G)$ be the set of all (vertex sets of) triangles formed by the edges of $G = G^{12} \cup G^{23} \cup G^{13}$, and let $t(G) = |T(G)|$.

Fact 2.10. Let $G = (G^{12}, G^{13}, G^{23})$ be a $((d_{12}, d_{13}, d_{23}), \epsilon)$-triad, where $\min\{d_{12}, d_{13}\} > 2\epsilon$, then

$$(1-2\epsilon)(d_{12}-\epsilon)(d_{13}-\epsilon)(d_{23}-\epsilon) < \frac{t(G)}{|V_1||V_2||V_3|} < 4\epsilon+(d_{12}+\epsilon)(d_{13}+\epsilon)(d_{23}+\epsilon).$$

In particular, if $\epsilon < \frac{1}{10}d_{12}d_{13}d_{23}$ then $t(G) > \frac{1}{2}d_{12}d_{13}d_{23}|V_1||V_2||V_3|$.

Proof. Note that

$$t(G) = \sum_{x \in V_1} e_{G^{23}}(N_{G^{12}}(x), N_{G^{13}}(x)).$$

By the proof of Fact 2.3 and using the ϵ-regularity of G^{12} and G^{13} we see that each of the four inequalities

$$\epsilon|V_2| < |V_2|(d_{12}-\epsilon) < \deg_G(x, V_2) < (d_{12}+\epsilon)|V_2|,$$

$$\epsilon|V_3| < |V_3|(d_{13}-\epsilon) < \deg_G(x, V_3) < (d_{13}+\epsilon)|V_3|.$$

is violated by at most $\epsilon|V_1|$ vertices $x \in V_1$.

Observe also, this time from the ϵ-regularity of G^{23}, that the pair $(N_{G^{12}}(x), N_{G^{13}}(x))$ has density between $d_{23}-\epsilon$ and $d_{23}+\epsilon$. These inequalities together yield the desired estimates. ∎

2..2 Hypergraphs

A 3-uniform hypergraph $\mathcal{H}$ is a pair (V, E) of disjoint sets such that $E \subseteq [V]^3$, where $[V]^3$ is the set of all 3-element subsets of V. We call the elements of E *triplets* or *hyperedges*. We will often identify $\mathcal{H}$ with E.

The regularity lemma for 3-uniform hypergraphs established in [3] partitions a 3-uniform hypergraph into a bounded number of 3-partite 3-uniform hypergraphs, most of which are quasi-random in a well defined sense.

Definition 2.11. A 3-partite 3-uniform hypergraph $\mathcal{H}$ with a fixed 3-partition (V_1, V_2, V_3) will be called a 3-*graph.* If $G = G^{12} \cup G^{23} \cup G^{13}$ is a triad with the same vertex partition as $\mathcal{H}$, and, moreover, $\mathcal{H} \subseteq T(G)$, then we say that G *underlies* $\mathcal{H}$.

Definition 2.12. Let $\mathcal{H}$ be a 3-graph with an underlying triad $G = G^{12} \cup G^{23} \cup G^{13}$. The *density* of $\mathcal{H}$ with respect to G is defined by

$$d_{\mathcal{H}}(G) = \frac{|\mathcal{H} \cap T(G)|}{t(G)} \tag{2}$$

if $t(G) > 0$, and 0 otherwise.

In other words, the density counts the proportion of triangles of G which are triplets of $\mathcal{H}$.

Definition 2.13. Let $\alpha, \delta > 0$ be given. We say that a 3-graph $\mathcal{H}$ is $(\alpha; \delta)$-*regular* with respect to an underlying triad $G = G^{12} \cup G^{23} \cup G^{13}$ if for any subtriad $\mathcal{Q} = (\mathcal{Q}^{12}, \mathcal{Q}^{23}, \mathcal{Q}^{13})$ of G, where

$$\mathcal{Q}^{12} \subseteq G^{12}, \mathcal{Q}^{23} \subseteq G^{23}, \mathcal{Q}^{13} \subseteq G^{13},$$

the following holds: If $t(\mathcal{Q}) > \delta t(G)$, then

$$\alpha - \delta < d_{\mathcal{H}}(\mathcal{Q}) < \alpha + \delta.$$

We say that $\mathcal{H}$ is δ-*regular* with respect to G if it is $(\alpha; \delta)$-regular for some α. If the regularity condition fails for any α, we say that $\mathcal{H}$ is δ-*irregular* with respect to G, and if it fails with a particular α then we call $\mathcal{H}$ $(\alpha; \delta)$-*irregular.*

For instance, if $\mathcal{H}$ consists of all triangles of G, then $d_{\mathcal{H}}(G) = 1$. Consequently it is δ-regular for all $\delta > 0$.

The Hypergraph Regularity Lemma in [3] provides not only a partition of the vertex set $V(\mathcal{H})$ into a bounded number t of classes, but also, for any given pair (V_i, V_j) of partition classes, it provides a partition of the set

$V_i \times V_j$ into up to l graphs, in such a way that $\mathcal{H}$ is δ-regular with respect to all but at most $\delta\binom{t}{3}l^3$ triads thus formed.

Most of the partition graphs are $(1/l;\epsilon)$-regular and, therefore, a problem of significant interest in applications is to study the existence of short paths between typical edges of G in an $(\alpha;\delta)$-regular 3-graph $\mathcal{H}$ underlied by a $(1/l,\epsilon)$-triad $G = G^{12} \cup G^{23} \cup G^{13}$. It remains to define what we mean by a path and what we mean by a typical edge.

Definition 2.14. A *hyperpath* of length k in a 3-uniform hypergraph is a subhypergraph consisting of $k+2$ vertices and k hyperedges, whose vertices can be ordered, $v_1, \dots, v_{k+2}$, in such a way that for each $i = 1, 2, \dots, k$, we have $v_i v_{i+1} v_{i+2} \in \mathcal{H}$.

Sometimes such paths are called tight as opposed to loose ones, where $v_i v_{i+1} v_{i+2} \in \mathcal{H}$ holds only for odd i. In this paper we exclusively work with tight paths and therefore the word "tight" is omitted.

By analogy with the graph case, typicality of an edge is defined in terms of neighborhoods. It should not be surprising that there are two very different notions of a neighborhood in 3-uniform hypergraphs: the neighborhood of a pair of vertices, which is a subset of the vertex set, and the neighborhood of a vertex, which is a set of pairs of vertices, i.e., a graph.

Definition 2.15. Let $\mathcal{H}$ be a 3-uniform hypergraph and let $e = \{x, y\}$ be a pair of vertics in $V = V(\mathcal{H})$. We define the *hypergraph neighborhood* of e to be $\Gamma_e = \{z \in V : \{z, x, y\} \in \mathcal{H}\}$. The elements of Γ_e are called *neighbors* of e in the hypergraph $\mathcal{H}$.

Note that in a 3-graph, if $e \in G^{ij}$ then $\Gamma_e \subset V_k$.

Definition 2.16. Let $\mathcal{H}$ be a 3-graph with an underlying triad $G = G^{12} \cup G^{23} \cup G^{13}$ and let $x \in V_i$. We define the *link graph* $L(x) = (E_x, V_x)$ of x to be the graph on the vertex set $V_x = N_{G^{ij}}(x) \cup N_{G^{ik}}(x)$ and the edge set

$$E_x = \Big\{\{y, z\} \in G^{jk} : \{x, y, z\} \in \mathcal{H}\Big\}.$$

It is shown in the Appendix that almost all vertices x of an $(\alpha;\delta)$-regular 3-graph underlied by a $(1/l,\epsilon)$-triad have $(\alpha/l; \sqrt[4]{\delta}, \sqrt[4]{\delta}/l)$-regular links $L(x)$. It is also shown there that almost all edges e of the underlying triad have the neighborhood Γ_e of average size. Combining these two properties, together with that of having proper endpoints (see Definition 2.6) we arrive at the notion of a good edge.

Definition 2.17. Let for some real $\delta > 0, \alpha > 0$ and $\epsilon > 0$ and integer $l \geq 2$, a 3-graph $\mathcal{H}$ be $(\alpha; \delta)$-regular with respect to an underlying $(1/l, \epsilon)$-triad $G = G^{12} \cup G^{23} \cup G^{13}$, where $|V_1| = |V_2| = |V_3| = n$. We call an edge $e = \{x, y\} \in G^{ij}$, $1 \leq i < j \leq 3$, *good* if the following conditions are satisfied:

(i) x and y are proper,

(ii) $L(x)$ and $L(y)$ are $(\alpha/l; \sqrt[4]{\delta}, \sqrt[4]{\delta}/l)$-regular,

(iii) $(1/l - \epsilon)^2(\alpha - \sqrt[4]{\delta})n < |\Gamma_e| < (1/l + \epsilon)^2(\alpha + \sqrt[4]{\delta})n$.

Our aim in this paper is to prove the following theorem.

Theorem 2.18. *For all real $0 < \alpha < 1$ and all integer $l \geq 2$ there exist $\delta > 0$ and $\epsilon > 0$ such that whenever $\mathcal{H}$ is a 3-graph which is $(\alpha; \delta)$-regular with respect to an underlying $(1/l, \epsilon)$-triad $G = G^{12} \cup G^{13} \cup G^{23}$, where $|V_1| = |V_2| = |V_3| = n$ is sufficiently large, then every two good edges of G are connected in $\mathcal{H}$ by a hyperpath of length at most seven.*

Remark 2.19. It follows from Corollary 2.7, Lemma 4.1 and Lemma 4.2 that most of the edges of G are good (see Corollary 4.3 in Section 4). Also, it is possible to construct counterexamples showing that Theorem 2.18 ceases to be true if any one of the three conditions defining good edges is dropped.

Remark 2.20. The way Theorem 2.18 is quantified allows to choose δ much smaller than $1/l$. In applications of the Hypergraph Regularity Lemma, however, very often it is the other way around. A corresponding analog of Theorem 2.18 in that case seems to be much harder and is the subject of a forthcoming paper [7].

In the proof of Theorem 2.18 we have not attempted to optimize the dependence of δ and ϵ on α and l.

Remark 2.21. A random model corresponding to an $(\alpha; \delta)$-regular 3-graph with an underlying $(1/l, \epsilon)$-triad is a two-phase, random, binomial 3-graph $\mathcal{H}(\alpha, 1/l)$ in which, first each triplet is selected independently, with probability α, then a random graph $G(n, 1/l)$ is drawn, and only triplets coinciding with the triangles of $G(n, 1/l)$ remain. It is easy to show that in this model, for fixed $\alpha > 0$ and $l \geq 1$, a.a.s. for every choice of four vertices x, y, z, w there exists a vertex v such that xyv, yvz, vzw are all triplets, and so, every pair of edges of $G(n, 1/l)$ is joint in $\mathcal{H}(\alpha, 1/l)$ by a hyperpath of length at most three.

3. The Proof of Main Result

In the proof of Theorem 2.18 we will use twice the following fact.

Fact 3.1. For all real $0 < \alpha < 1$ and for all integer $l \geq 2$, there exist $\delta > 0$ and $\epsilon > 0$ such that whenever a 3-graph $\mathcal{H}$ is $(\alpha; \delta)$-regular with respect to an underlying $(1/l, \epsilon)$-triad $G = G^{12} \cup G^{23} \cup G^{13}$, where $|V_1| = |V_2| = |V_3| = n$, the following holds. If $e = \{x, y\} \in G^{12}$ is a good edge then for at least $4\epsilon n$ vertices $z \in \Gamma_e$ the edge $f = \{y, z\} \in G^{23}$ is also good.

Proof. Given α and l, choose δ and ϵ so that $4\epsilon < \sqrt{\delta}/2$ and

$$\frac{\alpha}{2l^2} - 5\sqrt[4]{\delta} > 4\epsilon. \tag{3}$$

and also so that Lemma 4.1 holds. Note that (3) implies that $\epsilon < 1/(16l)$ and $\sqrt[4]{\delta} < \alpha/10$.

Let $x \in V_1$, $y \in V_2$ and let $e = \{x, y\} \in G^{12}$ be a good edge. Let $B \subset N_{G^{23}}(y) \subset V_3$ be the set of all $z \in V_3$ for which

$$(1/l - \epsilon)^2(\alpha - \sqrt[4]{\delta})n < |\Gamma_{\{y,z\}}| < (1/l + \epsilon)^2(\alpha + \sqrt[4]{\delta})n.$$

Note that $\Gamma_{\{y,z\}} = N_{L(y)}(z)$ and that y is proper and $L(y)$ is $(\alpha/l; \sqrt[4]{\delta}, \sqrt[4]{\delta}/l)$-regular. We now apply Fact 2.3 to $L(y)$ with $d = \alpha/l$, $\epsilon_1 = \sqrt[4]{\delta}$, $\epsilon_2 = \sqrt[4]{\delta}/l$, $V_1 = N_{G^{23}}(y)$, and $V_2 = A = N_{G^{12}}(y)$. It then follows that $|B| > (1 - 2\sqrt[4]{\delta})|N_{G^{23}}(y)|$ and hence $|N_{G^{23}}(y) \setminus B| < 2\sqrt[4]{\delta}(1/l + \epsilon)n$.

Further, let $C = \{z \in V_3 : z$ is proper and has $(\alpha/l; \sqrt[4]{\delta}, \sqrt[4]{\delta}/l)$-regular link graph $L(z)\}$. By Corollary 2.7 and by Lemma 4.1 we have $|C| > (1 - 3\sqrt{\delta})n$. Note that $\{y, z\}$ is a good edge, if and only if $z \in B \cap C$. Thus, as e is good, $\Gamma_e \subset N_{G^{23}}(y)$, and $l \geq 2$, we have

$$\begin{aligned}
|\Gamma_e \cap B \cap C| &> |\Gamma_e| - |N_{G^{23}}(y) \setminus B| - |N_{G^{23}}(y) \setminus C| \\
&> (1/l - \epsilon)^2(\alpha - \sqrt[4]{\delta})n - 2\sqrt[4]{\delta}(1/l + \epsilon)n - 3\sqrt{\delta}n \\
&> \left(\frac{\alpha}{2l^2} - 5\sqrt[4]{\delta}\right)n > 4\epsilon n,
\end{aligned}$$

where the two last inequalities follow from (3). This ends the proof. ■

***Proof of Theorem* 2.18.** For a given real $\alpha > 0$ and integer $l \geq 2$, let $\delta > 0$ and $\epsilon > 0$ be such that the conclusion of Fact 3.1 holds, as well as the inequalities

$$\delta < \left(\frac{\alpha}{8l}\right)^{20} \tag{4}$$

and

$$\epsilon < \frac{\alpha^2}{16l^4} \tag{5}$$

are fulfilled. Note that the choice of such $\delta > 0$ and $\epsilon > 0$ is always possible.

Let $\mathcal{H}$ be a 3-graph which is $(\alpha;\delta)$-regular with respect to an underlying $(1/l,\epsilon)$-triad $G = G^{12} \cup G^{13} \cup G^{23}$, where $|V_1| = |V_2| = |V_3| = n$. Further, let e and f be any two good edges of G. Without loss of generality we may assume, that $e = \{v_e, w_e\}$ and $f = \{v_f, w_f\}$ are in the same graph, say G^{12}, $v_e, v_f \in V_1$. Otherwise, using Fact 3.1 (remember, that n is large), one can add one hyperedge of the form $\{v_f, w_f, z\}$ to achieve this.

Using again Fact 3.1, we see that for at least $4\epsilon n$ vertices $w \in V_3$ the edge $f_w = \{v_f, w\} \in G^{13}$ is good. We set $A = N_{G^{12}}(w_e) \subset V_1$ and apply Fact 2.3 to the graph G^{13} and the set A to get that there exists a vertex $w' \in V_3$ such that the edge $\{v_f, w'\}$ is good and

$$(1/l - \epsilon)^2 n < \deg_{G^{13}}(w', A) = |N_G(w_e, w')| < (1/l + \epsilon)^2 n.$$

Let us denote $f' = f_{w'} = \{v_f, w'\}$. Using the $(\alpha;\delta)$-regularity of $\mathcal{H}$ we will now show that there exists a path from e to f' of length five. Define a subtriad $Q = (Q^{12}, Q^{13}, Q^{23})$ of the triad $G = (G^{12}, G^{13}, G^{23})$ on the vertex sets $N_G(w_e, w') \subset V_1$, $\Gamma_{f'} \subset V_2$, $\Gamma_e \subset V_3$, where

$$\begin{aligned} Q^{12} &= L(w')[N_G(w_e, w'), \Gamma_{f'}], \\ Q^{13} &= L(w_e)[N_G(w_e, w'), \Gamma_e], \\ Q^{23} &= G^{23}[\Gamma_{f'}, \Gamma_e]. \end{aligned}$$

We will show that $t(Q) > \delta t(G)$, and consequently, by the $(\alpha;\delta)$-regularity of $\mathcal{H}$, that

$$d_{\mathcal{H}}(Q) = \frac{|\mathcal{H} \cap T(Q)|}{t(Q)} > \alpha - \delta > \frac{7}{8}\alpha,$$

where the last inequality follows from (4). This means by Fact 2.10 that there exist at least $\frac{7}{16}\alpha\delta(n/l)^3$ hyperedges $\{x, y, z\} \in \mathcal{H} \cap T(Q)$ with $x \in N_G(w_e, w')$, $y \in \Gamma_{f'}$ and $z \in \Gamma_e$. Therefore, one can choose such an hyperedge, so that $z \neq w'$, $x \neq v_e, v_f$ and $y \neq w_e, w_f$. Observe from the construction of Q that hyperedges $\{v_e, w_e, z\}$, $\{w_e, z, x\}$, $\{z, x, y\}$, $\{x, y, w'\}$, $\{y, w', v_f\}$ form a hyperpath Π of length five connecting the graph edges e and f'. Indeed, $\{v_e, w_e, z\}$ and $\{w_e, z, x\}$ are triplets of $\mathcal{H}$, because, respectively, $z \in \Gamma_e$, and $\{x, z\} \in L(w_e)$. Similarly, $\{x, y, w'\}$ and $\{y, w', v_f\}$ form hyperedges of $\mathcal{H}$.

The hyperpath Π, together with the hyperedge $\{w', v_f, w_f\}$ build a hyperpath of length six connecting e and f, provided they belong to the same graph G^{ij}, $1 \leq i < j \leq 3$. In case where e and f are in different graphs one has to add one more hyperedge to connect these edges by a hyperpath. Hence, Theorem 2.18 is proved if we can only show that $t(Q) > \delta t(G)$.

To this end we use Fact 2.10. By Remark 2.2, we see that the graphs Q^{12} and Q^{13} are $(\alpha/l; 8l\sqrt[4]{\delta}/\alpha, \sqrt[4]{\delta}/l)$-regular and Q^{23} is $(1/l; 4l^2\epsilon/\alpha)$-regular. For example, let us show how it follows for the graph Q^{12}. We know that $L(w')$ is $(\alpha/l; \sqrt[4]{\delta}, \sqrt[4]{\delta}/l)$-regular and that $|N_{G^{23}}(w')| < n(1/l + \epsilon) < 2n/l$ (since $\epsilon < 1/l$). Also, $|\Gamma_{f'}| > (1/l - \epsilon)^2(\alpha - \sqrt[4]{\delta})n > \alpha n/(4l^2)$ (by (5) and (4)) and, by the choice of w', $|N_G(w_e, w')| > n/(4l^2) > \alpha n/(4l^2)$ (since $\alpha < 1$). Thus, indeed, G^{12} is $(\alpha/l; 8l\sqrt[4]{\delta}/\alpha, \sqrt[4]{\delta}/l)$-regular.

We now use the fact that δ and ϵ are much smaller than α and $1/l$, and hence, in particular, by (4), Q^{12}, Q^{13} are $(\alpha/l; \delta^{1/5})$-regular while Q^{23}, by (5), is $(1/l; \epsilon^{1/2})$-regular. Thus, by Fact 2.10, (4) and (5),

$$t(Q) > (1 - 2\delta^{1/5})\left(\frac{\alpha}{l} - \delta^{1/5}\right)^2\left(\frac{1}{l} - \epsilon^{1/2}\right)\left(\frac{\alpha n}{4l^2}\right)^2\frac{n}{4l^2} > 2^{-10}\frac{\alpha^4 n^3}{l^9}.$$

On the other hand, again by Fact 2.10 and (5), we have

$$t(G) < \left(\frac{n}{l}\right)^3(4\epsilon l^3 + (1 + \epsilon l)^3) < 2\left(\frac{n}{l}\right)^3.$$

Consequently, by (4), $t(Q) > \delta t(G)$, which ends the proof. ∎

4. Appendix

In this additional section we show that almost all vertices and edges satisfy, respectively, parts (ii) and (iii) of Defintion 2.17, and consequently, taking into account Corollary 2.7, almost all edges are good. Both proofs are similar and rely on the δ-regularity of $\mathcal{H}$ with respect G. We begin by showing that link graphs $L(x)$ are $(\alpha/l; \sqrt[4]{\delta}, \sqrt[4]{\delta}/l)$-regular for almost all $x \in V_1$. The proof of this fact, in a slightly different form, has appeared in [3]. We include it here because our proof of Theorem 2.18 strongly depends on that fact.

Lemma 4.1. *For all real $0 < \alpha < 1$ and $0 < \delta < \alpha^4$, and for all integer $l \geq 2$, there exists $\epsilon > 0$ so that whenever a 3-graph $\mathcal{H}$ is $(\alpha;\delta)$-regular with respect to an underlying $(1/l,\epsilon)$-triad $G = G^{12} \cup G^{23} \cup G^{13}$, where $|V_1| = |V_2| = |V_3| = n$, then all but at most $\frac{5}{2}\sqrt{\delta}n$ proper vertices $x \in V_1$ have $(\alpha/l; \sqrt[4]{\delta}, \sqrt[4]{\delta}/l)$-regular link graphs $L(x)$.*

Proof. Given α, $\delta < \alpha^4$, and $l \geq 2$, let ϵ be such that

$$\sqrt[4]{\delta}\left(\frac{1}{l} - \epsilon\right) > \epsilon, \tag{6}$$

$$\epsilon < \frac{\sqrt[4]{\delta} - \delta}{(\alpha - \delta)l}, \tag{7}$$

and

$$\frac{5}{4}(1 - \epsilon l)^3 > 4\epsilon l^3 + (1 + \epsilon l)^3. \tag{8}$$

Note that by Corolloray 2.7 at least $(1 - 4\epsilon)n$ vertices $x \in V_1$ are proper, i.e., their degrees in both, V_2 and V_3 are squeezed between $(1/l \pm \epsilon)n$.

The method of the proof is by contradiction. Suppose that proper vertices of V_1, $x_1, \ldots, x_t$, $t > \frac{5}{2}\sqrt{\delta}n$, have $(\alpha/l; \sqrt[4]{\delta}, \sqrt[4]{\delta}/l)$-irregular link graphs $L(x_i)$. Without loss of generality, this means that for each $i = 1, \ldots, s$, $s > \frac{5}{4}\sqrt{\delta}n$, there exists a pair of subsets (U_i, V_i), $U_i \subseteq N_{G^{12}}(x_i)$, $V_i \subseteq N_{G^{13}}(x_i)$, satisfying

$$|U_i| > \sqrt[4]{\delta}|N_{G^{12}}(x_i)|, \quad |V_i| > \sqrt[4]{\delta}|N_{G^{13}}(x_i)|,$$

and

$$e_{L(x_i)}(U_i, V_i) \leq \frac{1}{l}(\alpha - \sqrt[4]{\delta})|U_i||V_i|. \tag{9}$$

Note that, by (6), $|U_i|, |V_i| > \epsilon n$, and as a consequence of the $(1/l;\epsilon)$-regularity of G^{23},

$$e_{G^{23}}(U_i, V_i) > |U_i||V_i|\left(\frac{1}{l} - \epsilon\right).$$

Set $Q^{12}(x_i) = \{\{x_i, y\} : y \in U_i\}$, $Q^{13}(x_i) = \{\{x_i, z\} : z \in V_i\}$. Consider the triad

$$Q = (G^{23}, \bigcup_{i=1}^{s} Q^{12}(x_i), \bigcup_{i=1}^{s} Q^{13}(x_i)).$$

Then, by summing up the number of triangles containing each x_i, $i = 1, 2, \ldots, s$,

$$t(Q) = \sum_{i=1}^{s} e_{G^{23}}(U_i, V_i) > \frac{5}{4}\delta(1 - \epsilon l)^3 \frac{n^3}{l^3}.$$

So, by Fact 2.10 and (8) we see that $t(Q) > \delta t(G)$.

Therefore, by the $(\alpha; \delta)$-regularity of $\mathcal{H}$, we have

$$|\mathcal{H} \cap T(Q)| > (\alpha - \delta)t(Q) > (\alpha - \delta) \sum_{i=1}^{s} |U_i||V_i| \left(\frac{1}{l} - \epsilon\right). \tag{10}$$

On the other hand, each x_i, $i=1, 2, \ldots, s$, is contained in at most $e_{L(x_i)}(U_i,V_i)$ triplets of $\mathcal{H} \cap T(Q)$ and, hence, by (7) and (9), we get

$$|\mathcal{H} \cap T(Q)| \leq \frac{1}{l}(\alpha - \sqrt[4]{\delta}) \sum_{i=1}^{s} |U_i||V_i| < (\alpha - \delta) \left(\frac{1}{l} - \epsilon\right) \sum_{i=1}^{s} |U_i||V_i|,$$

which contradicts (10). ∎

Call an edge of G *proper* if it satisfies the inequalities of Fact 2.8, and *typical* (in $\mathcal{H}$) if it satisfies the inequalities from Definition 2.17, part (iii). Our last result states that almost all proper edges of G are typical in $\mathcal{H}$, i.e., have typical neighborhoods in $\mathcal{H}$.

Lemma 4.2. *For all real $0 < \delta, \alpha < 1$, and for all integer $l \geq 2$, there exists $\epsilon > 0$ so that whenever a 3-graph $\mathcal{H}$ is $(\alpha; \delta)$-regular with respect to an underlying $(1/l, \epsilon)$-triad $G = G^{12} \cup G^{23} \cup G^{13}$, where $|V_1| = |V_2| = |V_3| = n$, then all but at most $6\sqrt{\delta}n^2/l$ proper edges of G^{12} are typical.*

Proof. Given α, δ, and l, let $0 < \epsilon < \sqrt{\delta}/(4l)$ be such that

$$\frac{3}{2}(1 - \epsilon l)^2 > 4\epsilon l^3 + (1 + \epsilon l)^3. \tag{11}$$

Let $F \subset G^{12}$ be the subgraph of all proper, but not typical edges of G^{12}, and let $L \subset V_1$ be the set of vertices whose degree in F is larger than $2\sqrt{\delta}n/l$. We will show that $|L| < 3\sqrt{\delta}n$, and thus, that $|E(F)| < 2\sqrt{\delta}n^2/l + 3\sqrt{\delta}(1/l+\epsilon)n^2 + \epsilon n^2 < 6\sqrt{\delta}n^2/l$.

Suppose that $|L| \geq 3\sqrt{\delta}n$. Then, there exists $B \subset V_1$, $|B| \geq \frac{3}{2}\sqrt{\delta}n$, such that for each $x \in B$ there exists a set $B_x \subset N_{G^{12}}(x) \subset V_2$, $|B_x| \geq \sqrt{\delta}n/l$, such that for each $y \in B_x$ we have $|\Gamma_{\{x,y\}}| \leq (1/l-\epsilon)^2(\alpha - \sqrt[4]{\delta})n$ (and the edge $\{x,y\}$ is proper).

Let $Q^{12} = \{\{x,y\} : x \in B, y \in B_x\}$. Consider the triad $Q = (Q^{12}, G^{13}, G^{23})$. Then,

$$t(Q) = \sum_{x \in B} \sum_{y \in B_x} |N_G(x,y)| > \frac{3}{2}\sqrt{\delta}n\sqrt{\delta}n\frac{1}{l}n\left(\frac{1}{l}-\epsilon\right)^2 > \delta t(G),$$

where the last inequality follows from Fact 2.10 and inequality (11). Therefore, by the $(\alpha;\delta)$-regularity of $\mathcal{H}$, we have

$$\begin{aligned}|\mathcal{H} \cap T(Q)| > (\alpha-\delta)t(Q) &> \sum_{x \in B} \sum_{y \in B_x} (\alpha-\delta)n\left(\frac{1}{l}-\epsilon\right)^2 \\ &> \sum_{x \in B} \sum_{y \in B_x} n\left(\frac{1}{l}-\epsilon\right)^2(\alpha-\sqrt[4]{\delta}).\end{aligned}$$

On the other hand,

$$|\mathcal{H} \cap T(Q)| = \sum_{x \in B} \sum_{y \in B_x} |\Gamma_{\{x,y\}}| \leq \sum_{x \in B} \sum_{y \in B_x} n\left(\frac{1}{l}-\epsilon\right)^2(\alpha-\sqrt[4]{\delta}),$$

which yields a contradiction. ■

Lemmas 4.1 and 4.2, Fact 2.8 and Corollary 2.7 imply that almost all edges of G are good and thus, by our Theorem 2.18, almost all pairs of edges of G can be connected in $\mathcal{H}$ by hyperpaths of length at most seven.

Corollary 4.3. *For all real $0 < \delta < \alpha^4 < 1$, and for all integer $l \geq 2$, there exists $\epsilon > 0$ so that whenever 3-graph $\mathcal{H}$ is $(\alpha;\delta)$-regular with respect to an underlying $(1/l,\epsilon)$-triad $G = G^{12} \cup G^{23} \cup G^{13}$, where $|V_1| = |V_2| = |V_3| = n$, then all but at most $50\sqrt{\delta}n^2/l$ edges of G are good.*

Proof. There are at most $12\epsilon n^2$ improper edges in G and at most $18\sqrt{\delta}n^2/l$ proper but untypical edges. Moreover, in each V_i there are at most $4\epsilon n$ improper vertices and at most $\frac{5}{2}\sqrt{\delta}n$ proper vertices but with irregular links. Each of this vertices may be incident to at most $2n(1/l+\epsilon)$ edges of G, except for at most $2\epsilon n$ vertices which may even have degree $2n$. Summing up, we get, roughly, at most $50\sqrt{\delta}n^2/l$ edges of G which are not good (provided, ϵ is small enough). ∎

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References

[1] B. Bollobás, Random Graphs (Academic Press, London, 1985).

[2] Y. Dementieva, Equivalent Conditions for Hypergraph Regularity (Ph.D. Thesis, Emory University, 2001).

[3] P. Frankl and V. Rödl, *Extremal problems on set systems*, Random Structures and Algorithms **20** (2002) 131–164.

[4] S. Janson, T. Łuczak and A. Ruciński, Random Graphs (Wiley, New York, 2000).

[5] J. Komlós, G.N. Sárközy and E. Szemerédi, *On the square of a Hamiltonian cycle in dense graphs*, Random Structures and Algorithms **9** (1996) 193–211.

[6] J. Komlós, G.N. Sárközy and E. Szemerédi, *On the Pósa-Seymour conjecture*, J. Graph Theory **29** (1998) 167–176.

[7] J. Polcyn, V. Rödl, A. Ruciński and E. Szemerédi, *Short paths in quasi-random triple systems with sparse underlying graphs*, in preparation.

[8] B. Nagle and V. Rödl, *Regularity properties for triple systems*, Random Structures and Algorithms **23** (2003) 264–332.

[9] V. Rödl, A. Ruciński and E. Szemerédi, *A Dirac-type theorem for 3-uniform hypergraphs*, submitted.

[10] E. Szemerédi, *Regular partitions of graphs*, in: Problèmes en Combinatoire et Théorie des Graphes, Proc. Colloque Inter. CNRS, (J.-C. Bermond, J.-C. Fournier, M. Las Vergnas, D. Sotteau, eds), (1978) 399–401.

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