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TOTAL DOMINATION SUBDIVISION NUMBERS OF GRAPHS

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Abstract

A set S of vertices in a graph $G = (V, E)$ is a total dominating set of G if every vertex of V is adjacent to a vertex in S. The total domination number of G is the minimum cardinality of a total dominating set of G. The total domination subdivision number of G is the minimum number of edges that must be subdivided (where each edge in G can be subdivided at most once) in order to increase the total domination number. First we establish bounds on the total domination subdivision number for some families of graphs. Then we show that the total domination subdivision number of a graph can be arbitrarily large.

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1. Introduction

In this paper we continue the study of the total domination subdivision numbers introduced in [9] and studied further in [6]. Total domination in graphs was introduced by Cockayne, Dawes, and Hedetniemi [2] and is now well studied in graph theory. The literature on domination and related parameters has been surveyed and detailed in the two books by Haynes, Hedetniemi, and Slater [7, 8].

Let G be a graph with no isolated vertex. If $S, T \subseteq V(G)$ and every vertex of T is adjacent to a vertex of S, then we say that S *totally dominates* T. In particular, if $T = V(G)$, then we call S a *total dominating set* (TDS) of G. The *total domination number* of G, denoted by $\gamma_t(G)$, is the minimum cardinality of a TDS. A TDS of G of cardinality $\gamma_t(G)$ is called a $\gamma_t(G)$*-set*.

Haynes, Hedetniemi, and van der Merwe [9] define the *total domination subdivision number* $\mathrm{sd}_{\gamma_t}(G)$ of a graph G to be the minimum number of edges that must be subdivided (where each edge in G can be subdivided at most once) in order to increase the total domination number. We assume that a component of G is of order at least three since the total domination number of the graph K_2 does not change when its only edge is subdivided. This concept for domination was defined by Arumugam [1] and studied in [3, 4, 5].

Constant upper bounds on the total domination subdivision number for several families of graphs were determined in [9]. For trees T, they [9] showed that $1 \le \mathrm{sd}_{\gamma_t}(T) \le 3$. The trees T having $\mathrm{sd}_{\gamma_t}(T) = 3$ are characterized in [6]. In [9], the evidence seemed to point to a constant upper bound on the total domination subdivision number. The largest value given in [9] for the total domination subdivision number is 4. Our aim in this paper is twofold. First we determine additional bounds on the total domination subdivision number improving the ones in [9] for some graphs. Then we show that there is no constant upper bound on $\mathrm{sd}_{\gamma_t}(G)$, that in fact, $\mathrm{sd}_{\gamma_t}(G)$ can be arbitrarily large.

1..1 Notation

For notation and graph theory terminology we in general follow [7]. For a graph $G = (V, E)$, the *open neighborhood* of a vertex $v \in V$ is $N(v) = \{u \in V \mid uv \in E\}$ and the *closed neighborhood* is $N[v] = N(v) \cup \{v\}$. For a set $S \subseteq V$, we denote the subgraph of G induced by S by $G[S]$. The minimum degree of G is denoted by $\delta(G)$.

A vertex u is a *triangular vertex* if every vertex in $N(u)$ is in a triangle with u. Stated equivalently, a vertex is triangular if the induced subgraph $G[N(u)]$ contains no isolated vertices. Notice if a vertex u is triangular, then $\deg(u) \geq 2$. We say that a graph G is *triangular* if it contains at least one triangular vertex, and is *completely triangular* if every vertex in G is triangular.

A vertex v in a graph G is called *simplicial* if the induced subgraph $G[N[v]]$ is a complete graph.

A *k-tree* is any graph which can be obtained from a complete graph on $k+1$ vertices, by repeatedly adding a new vertex and joining it to every vertex in a complete subgraph of the existing graph of order k.

A graph G is called *chordal* if every cycle of G of length greater than three has a *chord*, that is, an edge between two nonconsecutive vertices of the cycle.

A *maximal outerplanar graph* is a 2-tree which is obtained from a copy of K_3 by repeatedly adding a new vertex and joining it to two adjacent vertices on the exterior face of the existing graph.

A graph is *claw-free* if it does not contain $K_{1,3}$ as an induced subgraph.

2. Bounds on the Total Domination Subdivision Number

Our first bound is a generalization of the following result from [9].

Theorem 1 [9]**.** *For any connected graph G with adjacent vertices u and v, each of degree at least two,*

$$\mathrm{sd}_{\gamma_t}(G) \leq \deg(u) + \deg(v) - 1.$$

Theorem 2. *For any connected graph G with adjacent vertices u and v, each of degree at least two,*

$$\mathrm{sd}_{\gamma_t}(G) \leq \deg(u) + \deg(v) - |N(u) \cap N(v)| - 1.$$

Proof. We may assume that $N(u) \cap N(v) \neq \emptyset$ for otherwise our result follows from Theorem 1. Let $N(v) = \{v_1, v_2, \ldots, v_k\}$ where $u = v_1$ and if $N(u) - N[v] \neq \emptyset$, let $N(u) - N[v] = \{u_1, \ldots, u_t\}$. Let G' be the graph obtained by subdividing the edge vv_i with subdivision vertex x_i, for $1 \leq$

$i \leq k$, and the edge uu_j for $1 \leq j \leq t$. Let A be the set of the subdivision vertices and S' a $\gamma_t(G')$-set. Clearly, no vertex of G totally dominates v in G', and so $|S' \cap A| \geq 1$. We show that $\gamma_t(G') > \gamma_t(G)$. It suffices for us to show that $\gamma_t(G) \leq |S'| - 1$, since then $\gamma_t(G') = |S'| \geq \gamma_t(G) + 1$.

One of u or v must be in S' to totally dominate x_1. If both u and v are in S', then $S' - A$ is a TDS of G, and so $\gamma_t(G) \leq |S' - A| \leq |S'| - 1$.

Assume $u \in S'$ and $v \notin S'$. Then every neighbor of v in G is in S' to totally dominate $\{x_1, x_2, \ldots, x_k\}$ and some x_i is in S' to dominate v. If $|S' \cap A| \geq 2$, then $(S' - A) \cup \{v\}$ is a TDS of G, and so $\gamma_t(G) \leq |S' - A| + 1 \leq |S'| - 1$. On the other hand, if $|S' \cap A| = 1$, then since $S' - \{u\}$ totally dominates $N(u) - N[v]$, it follows that $(S' - A - \{u\}) \cup \{v\}$ is a TDS of G, and so $\gamma_t(G) \leq |S' - A| \leq |S'| - 1$.

Assume $v \in S'$ and $u \notin S'$. If $|S' \cap A| \geq 2$, then $(S' - A) \cup \{u\}$ is a TDS of G, and so $\gamma_t(G) \leq |S' - A| + 1 \leq |S'| - 1$. Therefore, assume that $|S' \cap A| = 1$. The vertex of A in S' is a neighbor of v in G'. If $x_1 \in S'$, then all neighbors of v in G, except for possibly u, are totally dominated by $S' - \{v, x_1\}$. Thus, $(S' - \{v, x_1\}) \cup \{v_i\}$ for some vertex $v_i \in N(u) \cap N(v)$ is a TDS of G, and so $\gamma_t(G) \leq |S'| - 1$. Therefore, we may assume that $x_1 \notin S'$. But then some $v_i \in N_G(v) \cap N_G(u)$ must be in S' to totally dominate u, whence $S' - A$ is a TDS of G and $\gamma_t(G) \leq |S'| - 1$. ■

Our next bound on the total domination subdivision number is for graphs with triangular vertices.

Theorem 3. *If a graph G contains a triangular vertex u, then* $\mathrm{sd}_{\gamma_t}(G) \leq \deg(u)$*, and this bound can be attained when* $deg(u) = 2$.

Proof. Let u be a triangular vertex in G, and let G' be the graph which results from subdividing every edge incident with u in G. Let A be the set of subdivision vertices, and let S' be any $\gamma_t(G')$-set. We show that $\gamma_t(G') > \gamma_t(G)$. It suffices for us to show that $\gamma_t(G) \leq |S'| - 1$.

Clearly, no vertex of G totally dominates u in G', and so S' must contain a subdivision vertex.

If $u \notin S'$, then every vertex of $N_G(u)$ is in S' to totally dominate the subdivision vertices. Since u is a triangular vertex, every vertex in $N_G(u)$ has a neighbor in $N_G(u)$, implying that $S' - A$ is a TDS of G, and so $\gamma_t(G) \leq |S' - A| \leq |S'| - 1$. Hence, we assume that $u \in S'$.

If $N_G(u) \cap S' \neq \emptyset$, then $S' - A$ is a TDS of G, and so $\gamma_t(G) \leq |S'| - 1$. If $|S' \cap A| \geq 2$, then $(S' - A) \cup \{x\}$, where $x \in N_G(u)$, is a TDS of G, and so $\gamma_t(G) \leq |S'| - 1$. Therefore, we assume that $N_G(u) \cap S' = \emptyset$ and $|S' \cap A| = 1$.

Let $S' \cap A = \{x'\}$ and let x be the neighbor of x' in $N_G(u)$. Now every vertex in $N_G(u) - \{x\}$ is totally dominated by vertices in $S' - \{u, x'\}$. Since u is a triangular vertex, x has a neighbor y in $N_G(u)$. Hence, $(S' - \{u, x'\}) \cup \{y\}$ is a TDS of G, and so $\gamma_t(G) \le |S'| - 1$. Hence, $\gamma_t(G') > \gamma_t(G)$.

To see that the bound is sharp, consider the family of graphs G that can be obtained from a complete graph K_t, $t \ge 3$, by adding a vertex u adjacent to exactly two vertices of K_t. It is easy to see that $\gamma_t(G) = 2$. We will show that $\mathrm{sd}_{\gamma_t}(G) = \deg(u) = 2$. Since u is a triangular vertex in G, by Theorem 3, we have $\mathrm{sd}_{\gamma_t}(G) \le \deg(u) = 2$. To see that $\mathrm{sd}_{\gamma_t}(G) \ge 2$, let $V(G) - \{u\} = \{v_1, v_2, \ldots, v_t\}$ and let $N(u) = \{v_1, v_2\}$. Subdivide an edge in G with a vertex w to form a new graph G' with minimum TDS S'. We will show that $\gamma_t(G') = \gamma_t(G) = 2$. If w subdivides an edge incident with u, without loss of generality, say uv_1, then $S' = \{v_1, v_2\}$ is a TDS of G'. If w subdivides an edge in $G - u$ incident to v_1 or v_2, without loss of generality, say $v_1 v_k$ where $k \ne 1$, then $S' = \{v_1, v_j\}$, where $j \ne k$, is a TDS of G'. If w subdivides an edge in G not incident to v_1 or v_2, say $v_i v_j$, then $S' = \{v_1, v_i\}$ is a TDS of G'. Hence, $\mathrm{sd}_{\gamma_t}(G) = \deg(u) = 2$. ■

Immediate consequences of Theorem 3 can be seen in the following six corollaries.

Corollary 4. *For every completely triangular graph G,* $\mathrm{sd}_{\gamma_t}(G) \le \delta(G)$.

Since every k-tree and every 2-connected chordal graph is completely triangular, we have the following two immediate corollaries.

Corollary 5. *For every k-tree G, $k \ge 2$,* $\mathrm{sd}_{\gamma_t}(G) \le k$.

Corollary 6. *For every* 2*-connected chordal graph G,* $\mathrm{sd}_{\gamma_t}(G) \le \delta(G)$.

Since any simplicial vertex with degree at least two is triangular, we have the following corollary.

Corollary 7. *If a graph G contains a simplicial vertex u of degree at least two, then* $\mathrm{sd}_{\gamma_t}(G) \le \deg(u)$.

It is easy to see that every maximal outerplanar graph G contains at least two vertices of degree two, which is the minimum degree of any vertex in G, and that each such vertex is a simplicial vertex. Notice that every maximal outerplanar graph is completely triangular.

Corollary 8. *For every maximal outerplanar graph G,* $\mathrm{sd}_{\gamma_t}(G) \le \delta(G) = 2$.

It is well known that every planar graph contains at least one vertex of degree at most five. One can also observe that every maximal planar graph is completely triangular.

Corollary 9. *For every maximal planar graph G,* $\mathrm{sd}_{\gamma_t}(G) \le \delta(G) \le 5$.

Our next bound is for graphs with simplicial vertices. To establish our first bound we shall use the fact that $\mathrm{sd}_{\gamma_t}(K_3) = 2$ and for complete graphs of order $n \ge 4$, $\mathrm{sd}_{\gamma_t}(K_n) = 3$ (a proof of this fact is straightforward and is therefore omitted).

Theorem 10. *If G is a graph having three or more pairwise-adjacent simplicial vertices, then* $\mathrm{sd}_{\gamma_t}(G) \le 3$.

Proof. If G has order 3, then $G = K_3$, and $\mathrm{sd}_{\gamma_t}(G) = 2$. Hence, we will assume that G contains a clique of order at least 4. Let u, v, and w be three pairwise-adjacent simplicial vertices in graph G. We will assume that these vertices are adjacent to at least one nonsimplicial vertex, else G is a complete graph of order $n \ge 4$ and $\mathrm{sd}_{\gamma_t}(G) = 3$.

Let C be the set of nonsimplicial vertices adjacent to u, v, and w, and let D be the set of simplicial vertices in the clique containing u, v, and w and the vertices in C. Let G' be the graph obtained from G by subdividing edges uv, vw, and uw with vertices a, b, and c, respectively. Let A be the set of subdivision vertices. We will show that $\gamma_t(G') > \gamma_t(G)$.

We show first that no $\gamma_t(G)$-set S is a TDS of G'. If $S \cap C \ne \emptyset$, then $|S \cap D| \le 1$, else S is not a $\gamma_t(G)$-set. But if at most one of u, v, and w is in S, then S does not dominate at least one of a, b, and c. If $S \cap C = \emptyset$, then $S \cap D \ne \emptyset$ which implies that $|S \cap D| = 2$. To totally dominate a, b, and c, it follows that $|S \cap \{u, v, w\}| = 2$. But then u, v, and w are not totally dominated in G'. Hence no $\gamma_t(G)$-set totally dominates G'.

It remains for us to show that no set in G' that contains a subdivision vertex and of cardinality $\gamma_t(G)$ is a TDS of G'. Suppose such a set S' exists. Then, $|S' \cap \{u, v, w\}| \ge 2$ to totally dominate A. But then $S' - A$ is a TDS of G with cardinality less than $\gamma_t(G)$, a contradiction. Hence no such set exists, as claimed. ■

We show next that the bound of Corollary 7 can be strengthened if the graph has a simplicial vertex of degree at least two.

Theorem 11. *Let u be a simplicial vertex of degree at least two in a graph G, and let v be a neighbor of u. Then,* $\mathrm{sd}_{\gamma_t}(G) \leq \min(\deg(u), \deg(v) - \deg(u) + 3)$.

Proof. The first bound has been established in Corollary 7. The proof of the second bound is as follows. Let $v, u_1, \ldots, u_r$, $r \geq 1$, be the neighbors of u and $v_1, \ldots, v_k$ the other neighbors of v (with $k = 0$ if v is also simplicial). We form a new graph G' by subdividing the edge uv with a vertex x, the edge u_1v with a vertex y, the edge uu_1 with a vertex z, and each edge vv_i, $1 \leq i \leq k$, with a vertex a_i. Let $A = \{x, y, z, a_1, \ldots, a_k\}$, and let S' be a minimum TDS of G' such that $|S' \cap A|$ is minimum. We show that $\gamma_t(G') > \gamma_t(G)$. It suffices for us to show that $\gamma_t(G) \leq |S'| - 1$.

Clearly, if u has only u_1 and v as neighbors, then the first bound holds, thus we will assume that $r \geq 2$. Since S' totally dominates $\{x, y, z\}$, we must have $|S' \cap \{u, u_1, v\}| \geq 2$. Since $|S' \cap A|$ is minimum, it follows that $u_i \in S'$ for some $i \neq 1$ to totally dominate u. If $u \in S'$, then $S' - \{u\}$ is a TDS of G, and so $\gamma_t(G) \leq |S'| - 1$. Hence we may assume $u \notin S'$, and so $\{u_1, v\} \subset S'$. Thus, $S' \cap \{x, y, z\} = \emptyset$. If $a_j \in S'$ for some j, then $S' - A$ is a TDS of G, and so $\gamma_t(G) \leq |S'| - 1$. On the other hand, if $S' \cap A = \emptyset$, then $S' - \{v\}$ is a TDS of G, and so $\gamma_t(G) \leq |S'| - 1$. ■

3. Claw-Free Graphs

In this section, we investigate upper bounds on the total domination subdivision number for claw-free graphs.

Theorem 12. *If G is a claw-free graph with minimum degree $\delta \geq 2$, then*

$$\mathrm{sd}_{\gamma_t}(G) \leq \max\{\delta + 1, 4\}.$$

Proof. Let u be a vertex of minimum degree δ in G, and let $\{u_1, u_2, \ldots, u_r\}$ be the neighbors of u.

If u is a triangular vertex, then by Theorem 3, $\mathrm{sd}_{\gamma_t}(G) \leq \deg(u) < \deg(u) + 1$ and the result holds. Hence we may assume that there is an isolate in $G[N(u)]$. Without loss of generality, let u_r be such an isolate. Since $\delta \geq 2$, u_r has at least one neighbor, say v, in $V - N[u]$. Since G is claw-free, it follows that $N_G[u] - \{u_r\}$ induces a complete graph as does $N_G[u_r] - \{u\}$. We now consider two cases depending on the value of the minimum degree.

Case 1. $\delta \geq 3$. Let G' be the graph which results from subdividing the edges uu_i for $1 \leq i \leq r$ with subdivision vertex a_i and the edge $u_r v$ with subdivision vertex a. Let A be the set of subdivision vertices. Let S' be any $\gamma_t(G')$-set. We show that $\gamma_t(G') > \gamma_t(G)$. It suffices for us to show that $\gamma_t(G) \leq |S'| - 1$. To totally dominate u in G', S' must contain a subdivision vertex a_i, $1 \leq i \leq r$, and to totally dominate the vertex a, at least one of u_r and v is in S'. Moreover if $u \notin S'$, then to totally dominate each subdivision vertex a_i for $1 \leq i \leq r$, $u_i \in S'$.

Assume that $u \in S'$. If $|S' \cap A| \geq 2$, then $(S' - A) \cup \{u_r\}$ is a TDS of G, and so $\gamma_t(G) \leq |S'| - 1$. Hence we may assume $|S' \cap A| = 1$ implying that $a_j \in S'$ for some j, $1 \leq j \leq r$, and $a \notin S'$. If $j < r$, then let u_k be on a triangle with u and u_j in G. Note that u_k is totally dominated by $S' - \{u, a_j\}$. Then $(S' - \{u, a_j\}) \cup \{u_k\}$ is a TDS of G, and once again $\gamma_t(G) \leq |S'| - 1$. Thus we may assume, $j = r$. If $u_r \in S'$, then $S' - \{a_r\}$ is a TDS of G, and so $\gamma_t(G) \leq |S'| - 1$. On the other hand, if $u_r \notin S'$, then $v \in S'$ and v is totally dominated by $S' - \{u, a_r\}$. Thus, $(S' - \{u, a_r\}) \cup \{u_r\}$ is a TDS of G, and again $\gamma_t(G) \leq |S'| - 1$.

Thus, assume that $u \notin S'$. Hence, $N_G(u) \subset S'$. If $|S' \cap A| \geq 2$, then $(S' - A) \cup \{u\}$ is a TDS of G, and so $\gamma_t(G) \leq |S'| - 1$. Hence we may assume $|S' \cap A| = 1$ implying that $a_j \in S'$ for some j, $1 \leq j \leq r$ to totally dominate u. If $j < r$, then $S' - \{a_j\}$ is a TDS of G, and so $\gamma_t(G) \leq |S'| - 1$. Thus we may assume, $j = r$. If $v \in S'$, then $S' - \{a_r\}$ is a TDS of G, and so $\gamma_t(G) \leq |S'| - 1$. Hence we may assume that $v \notin S'$. The vertex v is totally dominated by $S' - \{a_r, u_r\}$. Hence, $(S' - \{a_r, u_r\}) \cup \{v\}$ is a TDS of G, and so $\gamma_t(G) \leq |S'| - 1$.

Case 2. $\delta = 2$. If u has a neighbor with degree at most three, then the result follows from Theorem 1. Hence, we may assume that $\deg(u_i) > 3$ for $i = 1, 2$. Since G is claw-free, $N_G[u_i] - \{u\}$ induces a complete graph for $i = 1, 2$. Let $w \neq u$ be a neighbor of u_1 and $v \notin \{u, w\}$ be a neighbor of u_2. Form G' from G by subdividing the edges uu_1, uu_2, $u_1 w$, and $u_2 v$ with subdivision vertices a, b, c, and d, respectively. Let A be the set of subdivision vertices. Let S' be a $\gamma_t(G')$-set with a minimum number of subdivision vertices. We show that $\gamma_t(G') > \gamma_t(G)$. It suffices for us to show that $\gamma_t(G) \leq |S'| - 1$. To totally dominate u in G', S' must contain at least one of a and b, and so $|S' \cap A| \geq 1$. Moreover, if $u \notin S'$, then $\{u_1, u_2\} \subset S'$ to totally dominate $\{a, b\}$. To totally dominate c, $|S' \cap \{u_1, w\}| \geq 1$ while to totally dominate d, $|S' \cap \{u_2, v\}| \geq 1$.

Assume $\{a,b\} \subset S'$. If $u \notin S'$, then $\{u_1,u_2\} \subset S'$ and $(S' - \{a,b\}) \cup \{u\}$ is a TDS of G, and so $\gamma_t(G) \le |S'| - 1$. Hence we may assume $u \in S'$. If now $u_1 \in S'$, then $(S' - \{a,b\}) \cup \{u_2\}$ is a TDS of G, and so $\gamma_t(G) \le |S'| - 1$. Similarly, if $u_2 \in S'$, then $\gamma_t(G) \le |S'| - 1$. Thus we may assume $S' \cap \{u_1,u_2\} = \emptyset$, and therefore $\{v,w\} \subset S'$. But then $(S' - \{a,b,u\}) \cup \{u_1,u_2\}$ is a TDS of G, and so $\gamma_t(G) \le |S'| - 1$. Thus, without loss of generality, assume that $a \in S'$ and $b \notin S'$.

Assume that $u \in S'$. If both u_1 and u_2 are in S', then $S' - A$ is a TDS of G with cardinality less than $|S'|$. If $u_1 \in S'$ and $u_2 \notin S'$, then $v \in S'$ and u_2 is totally dominated by $S' - \{u,v\}$. In this case, $S' - A$ is again a TDS of G. If $u_1 \notin S'$ and $u_2 \in S'$, then $w \in S'$. If now $d \in S'$, then $|S' \cap A| \ge 2$ and $(S' - A - \{u\}) \cup \{u_1, v\}$ is a TDS of G, and so $\gamma_t(G) \le |S'| - 1$. On the other hand, if $d \notin S'$, then u_2 is totally dominated by $S' - A - \{u\}$ and $(S' - A - \{u\}) \cup \{u_1\}$ is a TDS of G, and so $\gamma_t(G) \le |S'| - 1$. Hence we may assume that neither u_1 nor u_2 is in S'. Thus, $\{v,w\} \subset S'$. If $d \in S'$, then we can exchange d for a vertex in $N_G(u_2) \cap N(v)$ to produce a new $\gamma_t(G')$-set that contains fewer subdivision vertices than does S', contrary to our choice of S'. Hence, $d \notin S'$, and so there exists a vertex $x \in N(u_2) \cap N(v)$ in S' to totally dominate u_2. Thus, $(S' - \{a,u\}) \cup \{u_1\}$ is a TDS of G, and so $\gamma_t(G) \le |S'| - 1$.

Hence we may assume that $u \notin S'$ implying that $\{u_1,u_2\} \subset S'$. If $d \in S'$, then we contradict our choice of S'. Thus, $d \notin S'$, and so there exists a vertex $x \in N(u_2) \cap N(v)$ in S' to totally dominate u_2. If $c \in S'$, then we contradict our choice of S' (simply replace c in S' with a vertex in $N_G(u_1) \cap N(w)$). Thus, $S' \cap A = \{a\}$. Now there must exist a vertex $z \in S'$ to totally dominate w. If $z \in N_G(u_1)$, then $S' - \{a\}$ is a TDS of G. If $z \notin N_G(u_1)$, then $(S' - \{a,u_1\}) \cup \{w\}$ is a TDS of G, and so again $\gamma_t(G) \le |S'| - 1$. ■

4. Graphs with Large Total Domination Subdivision Numbers

Our aim in this section is to show that the total domination subdivision number of a graph can be arbitrarily large.

Theorem 13. *For any integer $k \ge 2$, there exists a connected graph G with $\mathrm{sd}_{\gamma_t}(G) = k$.*

Proof. Let $X = \{1, 2, \ldots, 3(k-1)\}$ and let $\mathcal{Y} = \{Y \subset X \colon |Y| = k\}$. Thus, $\mathcal{Y}$ consists of all k-element subsets of X, and so $|\mathcal{Y}| = \binom{3(k-1)}{k}$. Let G be the graph with vertex set $X \cup \mathcal{Y}$ and with edge set constructed as follows: add an edge joining every two distinct vertices of X and for each $x \in X$ and $Y \in \mathcal{Y}$, add an edge joining x and Y if and only if $x \in Y$. Then, G is a connected graph of order $n = \binom{3(k-1)}{k} + 3(k-1)$. The set X induces a clique in G, while the set $\mathcal{Y}$ is an independent set each vertex of which has degree k in G.

To totally dominate $\mathcal{Y}$, any TDS of G must contain at least $2(k-1)$ vertices of X, and so $\gamma_t(G) \geq 2(k-1)$. On the other hand, any subset of X of cardinality $2(k-1)$ is a TDS of G, and so $\gamma_t(G) \leq 2(k-1)$. Consequently, $\gamma_t(G) = 2(k-1)$.

Let $F = \{e_1, \ldots, e_{k-1}\}$ be an arbitrary subset of $k-1$ edges of G. Let H be the graph obtained from G by subdividing each edge in F. We show that $\gamma_t(H) = \gamma_t(G)$. For $i = 1, \ldots, k-1$, let $e_i = u_i v_i$. Since every edge of G is incident with at least one vertex of X, we may assume $u_i \in X$ for each i. If $v_i \in \mathcal{Y}$, then since $\deg v_i = k$ and $|F| = k-1$, v_i is adjacent to a vertex w_i, say, in X such that $v_i w_i \notin F$. If $v_i \in X$, let w_i be any vertex of X that is incident with no edge of F. Let $D_F = \cup_{i=1}^{k-1} \{u_i, w_i\}$. Then, $|D_F| \leq 2(k-1)$. If $|D_F| < 2(k-1)$, let D be any subset of $2(k-1)$ vertices of X that contain D_F. If $|D_F| = 2(k-1)$, let $D = D_F$. Then, D is a TDS of H, and so $\gamma_t(H) \leq 2(k-1) = \gamma_t(G)$. On the other hand, since subdividing edges of G cannot decrease its total domination number, $\gamma_t(H) \geq \gamma_t(G)$. Consequently, $\gamma_t(H) = \gamma_t(G)$, whence $\mathrm{sd}_{\gamma_t}(G) \geq k$.

We show next that $\mathrm{sd}_{\gamma_t}(G) \leq k$. Let $y \in \mathcal{Y}$ and let G_y be the graph obtained from G by subdividing each edge incident with y. Let D be a $\gamma_t(G_y)$-set. To totally dominate $\mathcal{Y} - \{y\}$, D must contain at least $2k-3$ vertices of X. Further, if D contains exactly $2k-3$ vertices of X, then necessarily $X - D = N_G(y)$. But then in order to totally dominate y and the k new vertices of degree 2 in G_y, D must contain at least two additional vertices, and so $|D| \geq 2k-1$. On the other hand, if D contains at least $2k-2$ vertices of X, then in order to totally dominate y, D contains at least one additional vertex, and so once again $|D| \geq 2k-1$. Hence, $\gamma_t(G_y) = |D| \geq 2k-1 > \gamma_t(G)$, whence $\mathrm{sd}_{\gamma_t}(G) \leq k$. Consequently, $\mathrm{sd}_{\gamma_t}(G) = k$. ■

The number of vertices in the graph constructed in the proof of Theorem 13 is $n = \binom{3(k-1)}{k} + 3(k-1) < 2^{3(k-1)}$, and so $k > \frac{1}{3}\log_2 n + 1$. Hence we have the following corollary of Theorem 13.

Corollary 14. *There exist connected graphs G of arbitrarily large order n satisfying*

$$\mathrm{sd}_{\gamma_t}(G) > \frac{1}{3}\log_2 n + 1.$$

We close with the remark that for any $n \geq 6$, a connected graph H of order n satisfying $\mathrm{sd}_{\gamma_t}(H) > \frac{1}{3}\log_2 n+1$ can be constructed from a copy of the graph G constructed in the proof of Theorem 13 by adding, if necessary, a new set Z of vertices and adding edges so that the set $X \cup Z$ induces a clique in H.

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