

PACKING OF THREE COPIES OF A DIGRAPH INTO THE TRANSITIVE TOURNAMENT

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Abstract

In this paper, we show that if the number of arcs in an oriented graph \vec{G} (of order n) without directed cycles is sufficiently small (not greater than $\frac{2}{3}n - 1$), then there exist arc disjoint embeddings of three copies of \vec{G} into the transitive tournament TT_n . It is the best possible bound.

Keywords: packing of digraphs, transitive tournament.

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1. Introduction. Results

Let \vec{G} be a digraph of order n with the vertex set $V(\vec{G})$ and the arc set $E(\vec{G})$. A digraph \vec{G} is called *transitive* when it satisfies the condition of transitivity: if (u, v) and (v, w) are two arcs of \vec{G} then (u, w) is the arc, too. For any vertex $v \in V(\vec{G})$ let us denote by $d^+(v)$ the *outdegree* of v , i.e., the number of vertices of \vec{G} that are adjacent from v . By $d^-(v)$ we denote the *indegree* of v , i.e., the number of vertices adjacent to v . The *degree* of a vertex v , denoted by $d(v)$, is the sum $d(v) = d^-(v) + d^+(v)$. A digraph without directed cycles of length two is called an *oriented graph*. Replacing every arc (u, v) in an oriented graph \vec{G} by an edge uv yields its *underlying graph*.

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A *tournament* is an oriented graph such that its underlying graph is complete. A transitive tournament of order n will be denoted by TT_n . As it is unique up to isomorphism, throughout the paper, we will view TT_n as shown in Figure 1. And we can denote the vertices in TT_n by consecutive integers in such way that if $i < j$, then (i, j) is an arc of TT_n . The vertices 1, 2 and n will be called the *first*, the *second* and the *last vertex* of TT_n , respectively. We define the *length of an arc* (i, j) as the difference $j - i$.

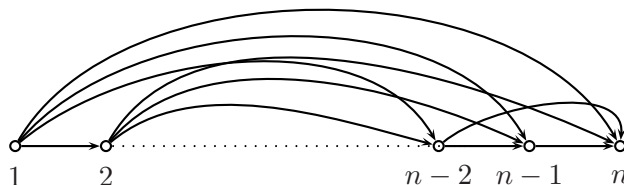


Figure 1: Transitive tournament TT_n

An (*oriented*) *path* between two distinct vertices u and v in an oriented graph \vec{G} is a finite sequence

$$u = v_0, v_1, \dots, v_{k-1}, v_k = v$$

of vertices, beginning with u and ending with v and edges $v_{i-1}v_i \in E(\vec{G})$ for $i \in \{1, \dots, k\}$. A *semipath* between two distinct vertices u and v is a path between u and v in the underlying graph G .

A vertex $x \in V(\vec{G})$ is an *end-vertex* if its degree $d(x) = 1$. An arc beginning or ending in x we call an *end-arc*.

Let u and v be end-vertices. The arcs $u'u$, $v'v$ (or uu' , vv') are called *independent* when $u' \neq v'$.

Let $\vec{G}(V, E)$ be an oriented graph of order n . An *embedding of \vec{G} into TT_n* is a couple (σ, σ') in which σ is a bijection $V \rightarrow \{1, \dots, n\} = V(TT_n)$ and σ' is an injection $E \rightarrow E(TT_n)$ induced by σ (i.e., for any edge $ij \in E$, $\sigma'(ij) = \sigma(i)\sigma(j)$). We will speak more simply of the embedding σ of \vec{G} . If $V(\vec{G}) = k < n$ we can also speak about an embedding of \vec{G} by adding $(n - k)$ isolated points to \vec{G} and we say that \vec{G} is embeddable into TT_n if $\vec{G}' := \vec{G} \cup \{\text{isolated vertices}\}$ is embeddable.

A *k-packing* of k oriented graphs $\vec{G}_1, \vec{G}_2, \dots, \vec{G}_k$ of order n into TT_n is a k -tuple $(\sigma_1, \dots, \sigma_k)$ in which σ_i is an embedding of \vec{G}_i for $1 \leq i \leq k$ such that the k sets $\sigma'_i(E_i)$ are disjoint.

We say that \vec{G} is k -packable into TT_n if a packing of k copies of \vec{G} into TT_n exists.

There are many results concerning packing of graphs. The basic result was proved, independently, in [2], [3] and [6].

Theorem 1. *Let G, H be graphs of order n . If $|E(G)| \leq n-2$ and $|E(H)| \leq n-2$ then G and H are packable into K_n .*

B. Bollobás and S.E. Eldridge made the following conjecture.

Conjecture 2. *Let G_1, G_2, \dots, G_k be k graphs of order n . If $|E(G_i)| \leq n-k$, $i = 1, \dots, k$, then G_1, G_2, \dots, G_k are packable into K_n .*

The case $k = 3$ of Conjecture 2 was proved by H. Kheddouci, S. Marshall, J.F. Saclé and M. Woźniak in [5].

If one restrains the study to the packing of three copies of the same graph, the hypothesis on size can slightly improved. The following theorem was proved in [7].

Theorem 3. *Let G be a graph of order n , $G \neq K_3 \cup 2K_1$, $G \neq K_4 \cup 4K_1$. If $|E(G)| \leq n-2$, then a 3-packing of G into K_n exists.*

The main result of this paper is similar to the basic result of Conjecture 2 for case $k = 3$ but for an acyclic digraph and its 3-packing into TT_n .

The motivation for us is the paper by A. Gölich, M. Pilśniak, M. Woźniak [4] where the existence of a 2-packing of \vec{G} into TT_n was shown. More precisely, the following result was proved therein.

Theorem 4. *Let \vec{G} be an acyclic digraph of order n such that $|E(\vec{G})| \leq \frac{3(n-1)}{4}$. Then \vec{G} is 2-packable into TT_n .*

The basic references of studies addressing packing problems can be found in [1, 8, 9, 10].

2. Some Lemmas

Before starting the proof of the main theorem we need some preliminary lemmas.

Lemma 5. *Let \vec{G} be a digraph isomorphic to a path of length k . If $k = \lfloor \frac{2}{3}n - 1 \rfloor$, then \vec{G} is 3-packable into TT_n .*

Proof. Notice that for $n \leq 3$ the length of a path is zero or one and it is clear that it is 3-packable into TT_n .

We use induction on the order of the transitive tournament. For $n = 4$ the length of a path \vec{P} is one, let $\vec{P} = v_0, v_1$. We can define its embedding $\sigma_1(v_0) = 1$ and $\sigma_1(v_1) = 4$ in TT_4 and the embeddings σ_2 and σ_3 as follows: $\sigma_2(v_0) = 2$, $\sigma_3(v_0) = 3$ and $\sigma_2(v_1) = 3$, $\sigma_3(v_1) = 4$.

For $n = 5$ the length of a path \vec{P} is two, let $\vec{P} = v_0, v_1, v_2$. We can define its embedding $\sigma_1(v_0) = 3$, $\sigma_1(v_1) = 4$ and $\sigma_1(v_2) = 5$ in TT_5 and the embeddings σ_2 and σ_3 as follows: $\sigma_2(v_0) = \sigma_3(v_0) = 1$, and $\sigma_2(v_1) = 2$, $\sigma_3(v_1) = 3$, and $\sigma_2(v_2) = 4$, $\sigma_3(v_2) = 5$.

For $n = 6$ the length of a path \vec{P} is three, let $\vec{P} = v_0, v_1, v_2, v_3$. We can define its embedding $\sigma_1(v_0) = 1$, $\sigma_1(v_1) = 4$, $\sigma_1(v_2) = 5$ and $\sigma_1(v_3) = 6$ in TT_6 and the embeddings σ_2 and σ_3 as follows: $\sigma_2(v_0) = \sigma_3(v_0) = 1$, $\sigma_2(v_1) = 2$, $\sigma_3(v_1) = 3$, and $\sigma_2(v_2) = 3$, $\sigma_3(v_2) = 4$, and $\sigma_2(v_3) = 5$, $\sigma_3(v_3) = 6$.

Now, let $n \geq 7$ and we assume that our result is true for all $n' < n$. Let v_0, \dots, v_k be the path \vec{P} of length $k = \lfloor \frac{2}{3}n - 1 \rfloor$ in TT_n . By induction, there exist the embeddings σ'_1 , σ'_2 and σ'_3 of path v_0, \dots, v_{k-2} into TT_{n-3} . Moreover, we can assume that vertices $\sigma'_1(v_{k-2}) = \sigma'_3(v_{k-2}) = n - 3$ and the number of $\sigma'_2(v_{k-2})$ in TT_{n-3} is less than $n - 3$. Now we add three vertices to TT_{n-3} at the end. Two vertices v_{k-1}, v_k of the path obtain the numbers: $n - 1$ and n , so $\sigma_1(v_{k-1}) = n - 1$, $\sigma_1(v_k) = n$. We define the embeddings σ_2 and σ_3 in TT_n as follows: $\sigma_2(v_{k-1}) = \sigma_3(v_{k-1}) = n - 2$ and $\sigma_2(v_k) = n - 1$, $\sigma_3(v_k) = n$, and $\sigma_1(v_i) = \sigma'_1(v_i)$, $\sigma_2(v_i) = \sigma'_2(v_i)$, $\sigma_3(v_i) = \sigma'_3(v_i)$ for $i \in \{0, \dots, k - 2\}$.

Thus, by induction, the proof is complete. ■

The following result may be proved in a similar way as Lemma 4.15 in [8].

Lemma 6. Let \vec{G} be an acyclic digraph of order n . Suppose that

- (a) $x'x, y'y, z'z$, or
- (b) xx', yy', zz'

are three independent end-arcs in $E(\vec{G})$. If $\vec{H} := \vec{G} - \{x, y, z\}$ is 3-packable into TT_{n-3} , then \vec{G} is 3-packable into TT_n .

Lemma 7. Let \vec{G} be an acyclic digraph of order n . Suppose that z is an isolated vertex and

(c) $x'x, y'y$, or

(d) xx', yy'

are two independent end-arcs in $E(\vec{G})$. If $\vec{H} := \vec{G} - \{x, y, z\}$ is 3-packable into TT_{n-3} , then \vec{G} is 3-packable into TT_n .

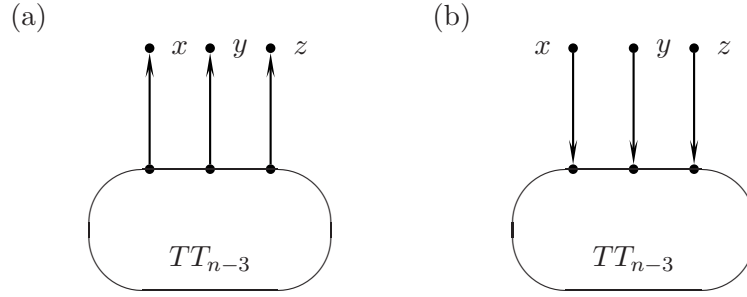


Figure 2. Two cases from Lemma 6

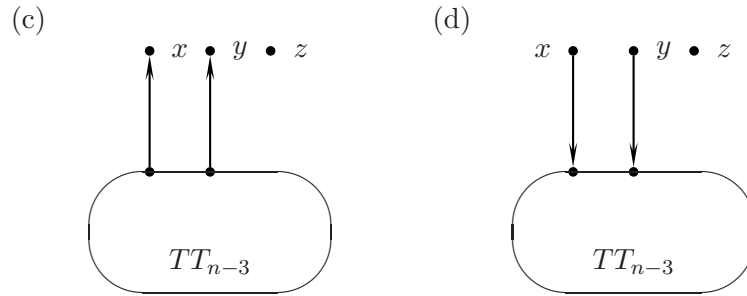


Figure 3. Two cases from Lemma 7

Proof. This lemma follows immediately from Lemma 6, (see Figure 2 and Figure 3). ■

Lemma 8. Let \vec{G} be an acyclic digraph of order n . Suppose that y, z are two isolated vertices and x is a vertex such that

(e) $d^-(x) \geq 2, d^+(x) = 0$, or

(f) $d^+(x) \geq 2, d^-(x) = 0$.

If $\vec{H} := \vec{G} - \{x, y, z\}$ is 3-packable into TT_{n-3} , then \vec{G} is 3-packable into TT_n .

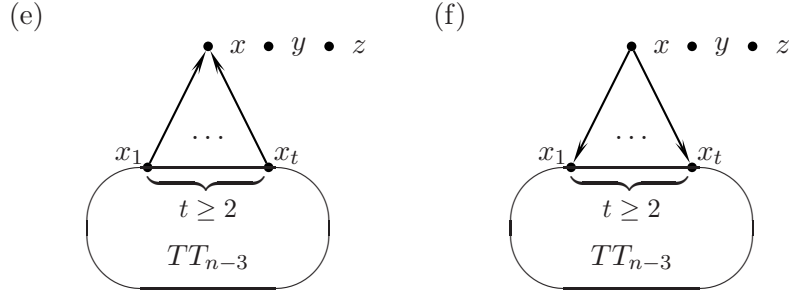


Figure 4. Two cases from Lemma 8

Proof. Without loss of generality we can consider only the case (e). By assumption there exist arc disjoint embeddings σ'_1 , σ'_2 and σ'_3 of \vec{H} into TT_{n-3} . Add three vertices to TT_{n-3} at the end and we obtain the transitive tournament TT_n .

Now, we define the embeddings of \vec{G} : $\sigma_1(v) = \sigma'_1(v)$, $\sigma_2(v) = \sigma'_2(v)$, $\sigma_3(v) = \sigma'_3(v)$ for all vertices of \vec{H} , and $\sigma_1(x) = n - 2$, $\sigma_2(x) = n - 1$, $\sigma_3(x) = n$. This is the correct 3-packing of \vec{G} into TT_n , which completes the proof. ■

Lemma 9. Let \vec{G} be an acyclic digraph of order n . Suppose that y, z are two isolated vertices in \vec{G} , the end-vertices x_1, \dots, x_k are adjacent to a vertex x , which is such that $d^+(x) = t \geq 1$, $d^-(x) = k \geq 2$ and $k + t \geq 4$.

If $\vec{H} := \vec{G} - \{x, y, z, x_1, \dots, x_k\}$ is 3-packable into TT_{n-3-k} , then \vec{G} is 3-packable into TT_n .

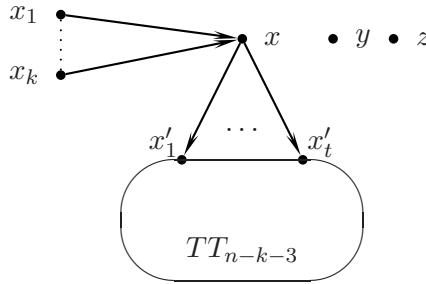


Figure 5. The case from Lemma 9

Proof. Let us imagine a transitive tournament TT_{n-3-k} with the vertices numbered from $k + 4$ to n . Let us assume that embeddings σ'_1 , σ'_2 and σ'_3 of

\vec{H} exist in TT_{n-3-k} . Let us add $k+3$ vertices to TT_{n-3-k} at the beginning and we obtain the transitive tournament TT_n .

Now, we define the embeddings σ_1, σ_2 and σ_3 of \vec{G} into TT_n as follows: $\sigma_1(x_i) = \sigma_2(x_i) = \sigma_3(x_i) = i$ for $i \in \{1, \dots, k\}$, $\sigma_1(x) = k+1$, $\sigma_2(x) = k+2$, $\sigma_3(x) = k+3$, and $\sigma_1(v) = \sigma'_1(v)$, $\sigma_2(v) = \sigma'_2(v)$, $\sigma_3(v) = \sigma'_3(v)$ for all the remaining vertices. We obtain a 3-packing of \vec{G} . ■

Lemma 10. *Let \vec{G} be an acyclic digraph of order n . Suppose that x, y are two isolated vertices in \vec{G} , a, b are two end-vertices adjacent to a vertex c . Let d be a vertex adjacent from c such that $d^-(c) = 2$, $d^+(c) = 1$, $d^-(d) = 1$, $d^+(d) \geq 1$. If $\vec{H} := \vec{G} - \{x, y, a, b, c, d\}$ is 3-packable into TT_{n-6} , then \vec{G} is 3-packable into TT_n .*

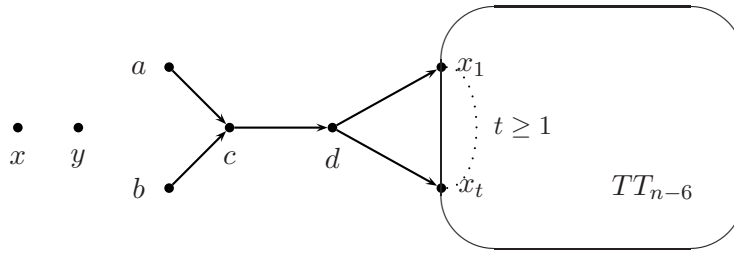


Figure 6. The case from Lemma 10

Proof. Let us imagine a transitive tournament TT_{n-6} with the vertices numbered from 7 to n . Let us assume that embeddings σ'_1, σ'_2 and σ'_3 of \vec{H} exist in TT_{n-6} . Let us add the vertices a, b, c, d, x, y to TT_{n-6} at the beginning and we obtain a transitive tournament TT_n .

We can define the embedding σ_1 of \vec{G} into TT_n as follows: $\sigma_1(a) = 1$, $\sigma_1(b) = 2$, $\sigma_1(c) = 3$, $\sigma_1(d) = 4$, $\sigma_1(x) = 5$, $\sigma_1(y) = 6$ and $\sigma_1(v) = \sigma'_1(v)$ for all the remaining vertices. Now, we define the embeddings σ_2 and σ_3 of \vec{G} into TT_n as follows: $\sigma_2(a) = \sigma_3(a) = 1$, $\sigma_2(b) = \sigma_3(b) = 2$, $\sigma_2(c) = 4$ and $\sigma_3(c) = 5$, $\sigma_2(d) = 5$ and $\sigma_3(d) = 6$ and $\sigma_2(v) = \sigma'_2(v)$, $\sigma_3(v) = \sigma'_3(v)$ for all the remaining vertices. So a 3-packing of \vec{G} into TT_n exists. ■

Lemma 11. *Let \vec{G} be an acyclic digraph of order n . Suppose that a_k ($k > 1$) is a vertex in \vec{G} such that a path of length $k-1$ from a_1 to a_k exists and $d^+(a_k) \geq 2$. Moreover, suppose that $y_1, \dots, y_{k'}$ ($k' = \lfloor \frac{k+3}{2} \rfloor$) are isolated vertices in \vec{G} .*

If $\vec{H} := \vec{G} - \{y_1, \dots, y_{k'}, a_1, \dots, a_k\}$ is 3-packable into $TT_{n-k-k'}$, then \vec{G} is 3-packable into TT_n .

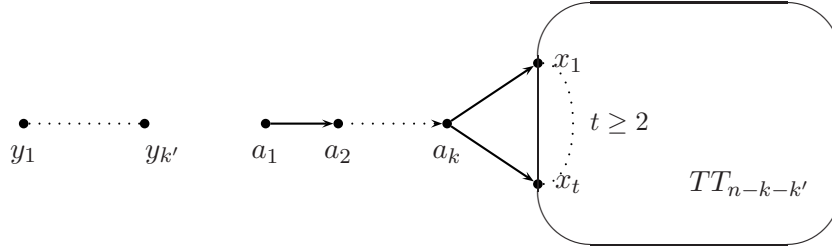


Figure 7. The case from Lemma 11

Proof. Let us imagine a transitive tournament $TT_{n-k-k'}$ with the vertices numbered from $k+k'+1$ to n . Let us assume that there are embeddings σ'_1 , σ'_2 and σ'_3 of \vec{H} into $TT_{n-k-k'}$. Let us add $k+k'$ vertices to $TT_{n-k-k'}$ at the beginning and we obtain the transitive tournament TT_n .

In Lemma 5 we show that the path of length $k-1$ is 3-packable into $TT_{\lfloor \frac{3}{2}k + \frac{1}{2} \rfloor}$. So there are embeddings σ''_1 , σ''_2 and σ''_3 of this path into $TT_{k+k'-1}$. Now we extend the embeddings σ''_1 , σ''_2 and σ''_3 to embeddings σ^*_1 , σ^*_2 and σ^*_3 into $TT_{k+k'}$ with the last isolated vertex added. We will modify these embeddings if necessary so that $\sigma^*_1(a_k) \neq \sigma^*_2(a_k) \neq \sigma^*_3(a_k)$.

We consider three cases:

1. In the case of two embeddings of a path, the vertex a_k is embedded in the same vertex of $TT_{k+k'-1}$, for example $\sigma''_1(a_k) \neq \sigma''_2(a_k) = \sigma''_3(a_k)$,
2. In the case of three embeddings of a path, the vertex a_k is embedded in the same vertex of $TT_{k+k'-1}$ but not in the last, say $\sigma''_1(a_k) = \sigma''_2(a_k) = \sigma''_3(a_k) = i < k+k'-1$,
3. In the case of three embeddings of a path, the vertex a_k is embedded in the last vertex of $TT_{k+k'-1}$.

In the first case we may choose for $\sigma^*_2(a_k)$ the last vertex of $TT_{k+k'}$.

In the second case we may choose $\sigma^*_2(a_k) = k+k'-1$ and $\sigma^*_3(a_k) = k+k'$.

In the third case we must have $\sigma''_1(a_{k-1}) > \sigma''_2(a_{k-1}) > \sigma''_3(a_{k-1})$. If $\sigma''_1(a_k) - \sigma''_1(a_{k-1}) > 1$, then we may assume $\sigma^*_1(a_k)$ is in the $k+k'-2$ vertex, and $\sigma^*_2(a_k) = k+k'$. If $\sigma''_1(a_k) - \sigma''_1(a_{k-1}) = 1$ (in $TT_{k+k'-1}$), then either we may assume $\sigma^*_2(a_k)$ is in the $k+k'-2$ vertex or we may assume $\sigma^*_3(a_k)$ is in the $k+k'-2$ vertex and the other one in $k+k'$ vertex.

Now $\sigma_1^*(a_k) \neq \sigma_2^*(a_k) \neq \sigma_3^*(a_k)$ and we can define the embeddings σ_1 , σ_2 and σ_3 of \vec{G} into TT_n as follows: $\sigma_1(a_i) = \sigma_1^*(a_i)$, $\sigma_2(a_i) = \sigma_2^*(a_i)$, $\sigma_3(a_i) = \sigma_3^*(a_i)$ for all $i \in \{1, \dots, k\}$, $\sigma_1(y_j) = \sigma_1^*(y_j)$, $\sigma_2(y_j) = \sigma_2^*(y_j)$, $\sigma_3(y_j) = \sigma_3^*(y_j)$ for all $j \in \{1, \dots, k'\}$ and $\sigma_1(v) = \sigma_1'(v)$, $\sigma_2(v) = \sigma_2'(v)$, $\sigma_3(v) = \sigma_3'(v)$ for all the remaining vertices. ■

3. The Main Result

In this section, we consider the existence of a 3-packing of \vec{G} into TT_n and we prove the following theorem.

Theorem 12. *Let \vec{G} be an acyclic digraph of order n such that $|E(\vec{G})| \leq \frac{2}{3}n - 1$. Then \vec{G} is 3-packable into TT_n .*

3.1 The bound in Theorem 12 is the best possible

First, we show that the size condition in Theorem 12 cannot be weakened.

Let us consider a path of length k and suppose that a 3-packing of such a path into TT_n exists, where $n > k$. It means that \vec{G} , \vec{G}' and \vec{G}'' are three arc disjoint subgraphs of the transitive tournament TT_n isomorphic to such a path. Let k_1 , k'_1 and k''_1 denote the number of arcs of length one in \vec{G} , \vec{G}' and \vec{G}'' , k_2 , k'_2 and k''_2 denote the number of arcs of length two and k_3 , k'_3 and k''_3 denote the number of arcs of length greater than two, respectively. Thus

$$(*) \quad \left. \begin{aligned} k_1 + k_2 + k_3 &= k, \\ k'_1 + k'_2 + k'_3 &= k, \\ k''_1 + k''_2 + k''_3 &= k. \end{aligned} \right\}$$

Since \vec{G} , \vec{G}' and \vec{G}'' are subgraphs of TT_n , we have

$$k_1 + 2k_2 + 3k_3 \leq n - 1,$$

$$k'_1 + 2k'_2 + 3k'_3 \leq n - 1,$$

$$k''_1 + 2k''_2 + 3k''_3 \leq n - 1.$$

By adding the last three inequalities we get

$$k_1 + k'_1 + k''_1 + 2k_2 + 2k'_2 + 2k''_2 + 3k_3 + 3k'_3 + 3k''_3 \leq 3n - 3.$$

But on the other hand, since \vec{G} , \vec{G}' and \vec{G}'' are arc disjoint and the total number of arcs of length 1 in TT_n is equal to $(n-1)$, we have:

$$2(k_1 + k'_1 + k''_1) \leq 2(n-1)$$

and since the total number of arcs of length 2 in TT_n is equal to $(n-2)$, we have:

$$k_2 + k'_2 + k''_2 \leq n-2.$$

By adding these three inequalities and using (*) we get

$$9k \leq 6n-7.$$

Finally, we obtain

$$k \leq \frac{2}{3}n - 1.$$

3..2 Proof of Theorem 12

At the beginning, we can notice that for $n \leq 4$ an oriented graph satisfying the assumption of Theorem 12 has zero or one arc and, obviously, is 3-packable into TT_n . For $n = 5$ an oriented graph satisfying the assumption of Theorem 12 has at most two arcs and it is also easily seen that it is 3-packable.

Now, let us assume that \vec{G} is a counterexample of Theorem 12 for minimum possible $n \geq 6$.

Let us notice that for $6 \leq n \leq 9$, if \vec{G} does not have any isolated vertex and has, of course, at most $\frac{2}{3}n - 1$ edges, then \vec{G} has only tree-components and at least three of them are isolated arcs. So by Lemma 6, we get a contradiction with the minimality of \vec{G} .

As above, if \vec{G} (for $6 \leq n \leq 9$) has only one isolated vertex, then \vec{G} has at least two isolated arcs (for $7 \leq n \leq 9$) or one isolated arc and one end-arc ($n = 6$). So by Lemma 7, we get a contradiction with the minimality of \vec{G} . Hence in the next part of the proof we can assume that for $n \leq 9$ \vec{G} has at least two isolated vertices.

It is obvious that every oriented graph \vec{G} , for $n \geq 10$ which satisfies the conditions of Theorem 12 is not connected and at least $\lceil \frac{n}{3} + \frac{7}{9} \rceil$ of its components are oriented trees (including, the isolated points as trivial oriented trees). If in \vec{G} there are more than four non-trivial oriented trees as its components, then \vec{G} has at least five independent end-vertices. So three

of them have to be such as in case (a) or (b) in Lemma 6. We get a contradiction with the minimality of \vec{G} . Hence \vec{G} has at most four components being non-trivial oriented trees and at least $\lceil \frac{n}{3} + \frac{7}{9} \rceil$ of its components are oriented trees. For order $n \geq 10$ we obtain an isolated point in \vec{G} .

Now, if in \vec{G} there are more than two non-trivial oriented trees as its components, then \vec{G} has at least three independent end-vertices. So two of them have to be such as in case (c) or (d) in Lemma 7 and since in \vec{G} there is an isolated vertex, we get a contradiction with the minimality of \vec{G} .

Hence from this moment in the proof (for order $n \geq 6$) \vec{G} has at most two components being non-trivial oriented trees and at least $\max\{2, \lceil \frac{n}{3} - \frac{11}{9} \rceil\}$ of its components are isolated vertices.

Let \vec{H} be a non-trivial connected component of \vec{G} of the greatest order. Let a vertex $x \in V(\vec{H})$ be such that $d^-(x) = 0$. It is easily seen that there is not more than one vertex adjacent from x , since if there is more than one, then \vec{G} satisfies the assumptions of Lemma 8 and it leads to a contradiction with the minimality of \vec{G} .

It means that $d^+(x) = 1$. If y is a neighbour of x , \vec{G} satisfies one of the following properties:

1. $d^-(y) \geq 3$;
2. $d^-(y) = 2$ and $d^+(y) \geq 2$;
3. $d^-(y) = 2$ and $d^+(y) \leq 1$;
4. there is a path $(a_1 = x, a_2 = y, \dots, a_k)$, $k \geq 2$ and $d^+(a_k) \geq 2$;
5. \vec{G} is an oriented path.

It is easily seen that in the first, the second and the third case we may assume that all vertices adjacent to y are end-vertices. If not, in the graph \vec{G} either there are two end-vertices like in Lemma 7 or there is a vertex with indegree zero and outdegree greater than or equal to 2, hence it satisfies the assumptions of Lemma 8. In both the cases we obtain a contradiction with the minimality of \vec{G} .

Case 1. It is obvious that in this case such a graph is 3-packable since either $d^+(y) = 0$ and it satisfies the assumptions of Lemma 8 or $d^+(y) > 0$ and the assumptions of Lemma 9.

Case 2. Such a graph is 3-packable since it satisfies the assumptions of Lemma 9.

Case 3. As in the first case, if $d^+(y) = 0$, it satisfies the assumptions of Lemma 8.

Let $d^+(y) = 1$ and z be a vertex adjacent from y . If $d(z) = 1$, assume first that \vec{H} is a not unique non-trivial component of \vec{G} . In the second non-trivial component \vec{K} of \vec{G} there is a vertex $v \in V(\vec{K})$ such that $d^-(v) = 0$. For the same reason as before the outdegree of v must be equal to 1. And then there are two end-arcs: one ending in x and the other ending in v , so by Lemma 7 \vec{G} is 3-packable, which contradicts the minimality of \vec{G} . Hence in this case \vec{G} has a unique non-trivial component \vec{H} . So \vec{H} has three arcs and in \vec{G} , which satisfies the assumption of Theorem 12, there are two isolated vertices. Three copies of such a graph can be packed in the same way as in the proof of Lemma 10, but \vec{G} is not 3-packable, so $d^-(z) > 1$.

If $d^-(z) > 1$, then two end-vertices, like in Lemma 7, exist in the graph \vec{G} and \vec{G} is 3-packable. If $d^-(z) = 1$ and $d^+(z) \geq 1$ such a graph is 3-packable since it satisfies the assumptions of Lemma 10.

Case 4. We may observe that if $d^-(a_i) > 1$, for any $i > 2$, in the graph \vec{G} either there are two end-vertices like in Lemma 7 or there is a vertex with indegree zero and outdegree greater than or equal to 2, hence it satisfies the assumptions of Lemma 8. In both the cases we obtain a contradiction with the minimality of \vec{G} .

It is obvious that in the fourth case such a graph is 3-packable since it satisfies the assumptions of Lemma 11.

Case 5. Such a graph is 3-packable since it satisfies the assumptions of Lemma 5.

Therefore the set of counterexamples is empty and the proof of Theorem 12 is complete. ■

4. A Conjecture — m -Packable into TT_n

Finally we can make a general conjecture.

Conjecture 13. *Let \vec{G} be an acyclic digraph of order n such that $|E(\vec{G})| \leq \frac{m+1}{2m}n - \frac{m^2+5}{6m}$. Then \vec{G} is m -packable into TT_n .*

We show only that the size condition in Theorem 13 cannot be weakened. Let us consider a path of length k . Then we suppose that there is an

m -embedding of such a path into TT_n , where $n > k$. It means that $\vec{G}_1, \vec{G}_2, \dots, \vec{G}_m$ are m arc disjoint subgraphs of the transitive tournament TT_n isomorphic to such a path. Let for $1 \leq i \leq m-1$, $k_1^i, k_2^i, \dots, k_m^i$ denote the numbers of arcs in $\vec{G}_1, \vec{G}_2, \dots, \vec{G}_m$ of length i in TT_n and $k_1^m, k_2^m, \dots, k_m^m$ denote the number of arcs in $\vec{G}_1, \vec{G}_2, \dots, \vec{G}_m$ of length greater than $m-1$, respectively. Thus

$$(*) \quad \left. \begin{aligned} k_1^1 + k_1^2 + \dots + k_1^m &= k, \\ k_2^1 + k_2^2 + \dots + k_2^m &= k, \\ &\dots \\ k_m^1 + k_m^2 + \dots + k_m^m &= k. \end{aligned} \right\}$$

Since $\vec{G}_1, \vec{G}_2, \dots, \vec{G}_m$ are subgraphs of TT_n we have for each \vec{G}_m

$$k_i^1 + 2k_i^2 + \dots + mk_i^m \leq n-1.$$

By adding those inequalities we get

$$\sum_{i=1}^m k_i^1 + 2 \sum_{i=1}^m k_i^2 + \dots + m \sum_{i=1}^m k_i^m \leq mn - m.$$

But on the other hand, since $\vec{G}_1, \vec{G}_2, \dots, \vec{G}_m$ are arc disjoint and the total number of arcs of length 1 is equal to $n-1$ we have:

$$(m-1) \sum_{i=1}^m k_i^1 \leq (m-1)(n-1),$$

Since the total number of arcs of length 2 is equal to $n-2$ we have:

$$(m-2) \sum_{i=1}^m k_i^2 \leq (m-2)(n-2)$$

and similar inequalities, up to

...

$$\sum_{i=1}^m k_i^{m-1} \leq (n-m+1).$$

By adding these inequalities and using $(*)$ we obtain

$$m^2k \leq (m+m-1+m-2+\dots+1)n - (m+1)(m-1) + 2(m-2) + \dots + (m-1)1$$

hence finally

$$k \leq \frac{m+1}{2m}n - \frac{m^2+5}{6m}.$$

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