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SOME REMARKS ON α-DOMINATION

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Abstract

Let $\alpha \in (0,1)$ and let $G = (V_G, E_G)$ be a graph. According to Dunbar, Hoffman, Laskar and Markus [3] a set $D \subseteq V_G$ is called an α-dominating set of G, if $|N_G(u) \cap D| \geq \alpha d_G(u)$ for all $u \in V_G \setminus D$. We prove a series of upper bounds on the α-domination number of a graph G defined as the minimum cardinality of an α-dominating set of G.

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1. Introduction

Let $\alpha \in (0,1)$ and let $G = (V_G, E_G)$ be a simple and finite graph. The neighbourhood and the degree of a vertex $u \in V_G$ in the graph G are denoted by $N_G(u)$ and $d_G(u)$, respectively. For all further notation and terminology we refer the reader to [5].

In [3] Dunbar, Hoffman, Laskar and Markus introduce the concept of an *α-dominating set*. They define a set $D \subseteq V_G$ to be an α-dominating set of G, if $|N_G(u) \cap D| \geq \alpha d_G(u)$ for all $u \in V_G \setminus D$, i.e., for every vertex that does not belong to D at least the α-fraction of its neighbours belongs to D.

In the present paper we prove a series of upper bounds on the α-domination number $\gamma_\alpha(G)$ of a graph G defined as the minimum cardinality of an α-dominating set of G. For further results on this parameter we refer the reader to [3] and [6].

2. Results

Our first bound follows easily from a colouring result of Cowen and Emerson (cf. Theorem 5.1 in [9]). In order to make our presentation self-contained we include a short proof using a classical Erdős-type exchange argument.

Theorem 2.1. *If $\alpha \in (0,1)$ and G is a graph of order n, then $\gamma_\alpha(G) \leq (1 - \frac{1}{\lceil \frac{1}{1-\alpha} \rceil})n$.*

Proof. Let $r = \lceil \frac{1}{1-\alpha} \rceil$ and let $V_G = V_1 \cup V_2 \cup \ldots \cup V_r$ be a partition of V_G such that $\sum_{i=1}^{r} \sum_{u \in V_i} |N_G(u) \cap V_i|$ is minimum. Note that $r(1-\alpha) \geq 1$.

If $|N_G(u_0) \cap V_{i_0}| > (1-\alpha) d_G(u_0)$ for some $i_0 \in \{1, 2, \ldots, r\}$ and some $u_0 \in V_{i_0}$, then there is some index i_1 with $1 \leq i_1 \leq r$ such that $|N_G(u_0) \cap V_{i_1}| < (1-\alpha) d_G(u_0)$. If $V'_{i_0} = V_{i_0} \setminus \{u_0\}$, $V'_{i_1} = V_{i_1} \cup \{u_0\}$ and $V'_l = V_l$ for $1 \leq l \leq r$ with $l \notin \{i_0, i_1\}$, then

$$\begin{aligned}\sum_{i=1}^{r} \sum_{u \in V'_i} |N_G(u) \cap V'_i| &= \sum_{i=1}^{r} \sum_{u \in V_i} |N_G(u) \cap V_i| - 2|N_G(u_0) \cap V_{i_0}| + 2|N_G(u_0) \cap V_{i_1}| \\ &< \sum_{i=1}^{r} \sum_{u \in V_i} |N_G(u) \cap V_i|\end{aligned}$$

which is a contradiction.

For each $i \in \{1, 2, \ldots, r\}$ we therefore have $|N_G(u) \cap V_i| \leq (1 - \alpha)d_G(u)$ for all $u \in V_i$, which implies that $V_G \setminus V_i$ is an α-dominating set of G. Since $\max\{|V_i| \mid 1 \leq i \leq r\} \geq \frac{n}{r}$, the desired bound follows. ∎

The following corollary is actually equivalent to Theorem 2.1.

Corollary 2.2. *Let $i \in \mathbf{N}_0 = \{0, 1, 2, \ldots\}$ and let $\alpha \in \mathbf{R}$ be such that $\frac{i}{i+1} < \alpha \leq \frac{i+1}{i+2}$. If G is a graph of order n, then $\gamma_\alpha(G) \leq \frac{i+1}{i+2}n < \frac{n}{2-\alpha}$.*

Proof. If $\frac{i}{i+1} < \alpha \leq \frac{i+1}{i+2}$ for some $i \in \mathbf{N}_0$, then $i + 1 < \frac{1}{1-\alpha} \leq i + 2$ and $\frac{i+1}{i+2} < \frac{1}{2-\alpha}$. These inequalities easily imply the desired result. ∎

Corollary 2.3 (Corollary 10 in [3])**.** *If $\alpha \in (0, \frac{1}{2}]$ and G is a graph of order n, then $\gamma_\alpha(G) \leq \frac{n}{2}$.*

The following observation shows that the bound given in Theorem 2.1 and Corollary 2.2 is essentially best-possible.

Observation 2.4. *Let $i \in \mathbf{N}_0 = \{0, 1, 2, \ldots\}$ and let $\alpha \in \mathbf{R}$ be such that $\frac{i}{i+1} < \alpha \leq \frac{i+1}{i+2}$. For every $\epsilon > 0$, there is a connected graph G of order n such that $\gamma_\alpha(G) \geq (\frac{i+1}{i+2} - \epsilon)n$.*

Proof. For $l \geq 1$ let the connected graph $G(l)$ arise from the disjoint union of l cliques H_1, H_2 ,$\ldots$, H_l of order $i+2$ and one isolated vertex v by joining v and one vertex u_j of H_j by a new edge for $1 \leq j \leq l$.

Let D be a minimum α-dominating set of $G(l)$. If $V_{H_j} \setminus \{u_j\} \subseteq D$ for some $1 \leq j \leq l$, then $|D \cap V_{H_j}| \geq i + 1$. If $V_{H_j} \setminus \{u_j\} \not\subseteq D$ for some $1 \leq j \leq l$, then there is a vertex $u'_i \in V_{H_j} \setminus (\{u_j\} \cup D)$ with $d_{G(l)}(u'_j) = i + 1$ and thus $|D \cap V_{H_j}| \geq \lceil \alpha d_{G(l)}(u_j) \rceil = \lceil \alpha(i+1) \rceil = i+1$. Altogether, we obtain $\gamma_\alpha(G(l)) \geq l(i+1)$ and thus $\frac{\gamma_\alpha(G(l))}{n_{G(l)}} \geq \frac{l(i+1)}{l(i+2)+1}$. Since $\lim_{l\to\infty} \frac{l(i+1)}{l(i+2)+1} = \frac{i+1}{i+2}$, the desired result follows. ∎

Note that the graphs we considered for the proof of Observation 2.4 were connected but not 2-connected. In fact, we believe the following.

Conjecture 2.5. *If $\alpha \in (0, 1)$ and G is a 2-(vertex-)connected graph of order n, then $\gamma_\alpha(G) \leq \lceil \alpha(n - 1) \rceil$.*

Using a probabilistic argument, our next result shows that the bound in Conjecture 2.5, i.e., $\approx \alpha n$, is essentially the right bound for graphs with a sufficiently large minimum degree.

Theorem 2.6. *For $\alpha \in (0,1)$ and $\epsilon > 0$ there is some constant $\delta_{\alpha,\epsilon}$ depending only on α and ϵ such that $\gamma_\alpha(G) \leq (\alpha+\epsilon)n$ for all graphs G of order n with minimum degree at least $\delta_{\alpha,\epsilon}$.*

Proof. Let G be a graph of order n and minimum degree δ. Without loss of generality, we may assume that $\alpha + \frac{\epsilon}{2} \leq 1$. Independently for each vertex $u \in V_G$ we introduce a random indicator variable Y_u such that $\Pr[Y_u = 1] = \alpha + \frac{\epsilon}{2}$ and $\Pr[Y_u = 0] = 1 - (\alpha + \frac{\epsilon}{2})$. Let $D_0 = \{u \in V_G \mid Y_u = 1\}$ and

$$D_1 = \{u \in V_G \mid Y_u = 0 \text{ and } |N_G(u) \cap D_0| < \alpha d_G(u)\}.$$

Clearly, $D_0 \cup D_1$ is an α-dominating set of G. The expected value of $|D_0|$ is $\mathrm{E}[|D_0|] = (\alpha + \frac{\epsilon}{2})n$. It remains to estimate the expected value $\mathrm{E}[|D_1|]$.

For $u \in V_G$ let $Z_u = |N_G(u) \cap D_0| = \sum_{v \in N_G(u)} Y_v$, i.e., Z_u is the sum of mutually independent indicator variables. Clearly, $\mathrm{E}[Z_u] = (\alpha + \frac{\epsilon}{2})d_G(u)$. Using Corollary A.1.14 in [1] – a well-known bound of large deviation whose origin goes back to Chernoff [2] – we obtain that there is some constant $c(\alpha,\epsilon) > 0$ depending only on α and ϵ such that

$$\begin{aligned}
\Pr[|N_G(u) \cap D_0| < \alpha d_G(u)] &= \Pr[Z_u < \alpha d_G(u)] \\
&= \Pr\left[\left(\alpha + \frac{\epsilon}{2}\right) d_G(u) - Z_u > \frac{\epsilon}{2} d_G(u)\right] \\
&\leq \Pr\left[\left|Z_u - \left(\alpha + \frac{\epsilon}{2}\right) d_G(u)\right| > \frac{\epsilon}{2} d_G(u)\right] \\
&= \Pr\left[|Z_u - \mathrm{E}[Z_u]| > \frac{\epsilon}{2\alpha+\epsilon}\mathrm{E}[Z_u]\right] \\
&\leq 2e^{-c(\alpha,\epsilon)\mathrm{E}[Z_u]} \\
&= 2e^{-c(\alpha,\epsilon)(\alpha+\frac{\epsilon}{2})d_G(u)}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\mathrm{E}[|D_1|] &= \sum_{u\in V_G} \Pr[u\in D_1]\\
&= \sum_{u\in V_G} \Pr[Y_u=0]\cdot \Pr[|N_G(u)\cap D_0|<\alpha d_G(u)]\\
&\le \sum_{u\in V_G} \left(1-\left(\alpha+\frac{\epsilon}{2}\right)\right) 2e^{-c(\alpha,\epsilon)(\alpha+\frac{\epsilon}{2})d_G(u)}\\
&\le n\left(1-\left(\alpha+\frac{\epsilon}{2}\right)\right) 2e^{-c(\alpha,\epsilon)(\alpha+\frac{\epsilon}{2})\delta}.
\end{aligned}$$

Since (for fixed α and ϵ and hence also fixed $c(\alpha,\epsilon)$)

$$\lim_{\delta\to\infty}\left(1-\left(\alpha+\frac{\epsilon}{2}\right)\right) 2e^{-c(\alpha,\epsilon)(\alpha+\frac{\epsilon}{2})\delta}=0,$$

the desired result follows. ■

We proceed to some best-possible bounds for bipartite graphs and for cacti, i.e., graphs all cycles of which are edge-disjoint.

Proposition 2.7. *Let $\alpha\in(0,1)$.*

(i) *If G is a bipartite graph of order n, then $\gamma_\alpha(G)\le\frac{1}{2}n$.*

(ii) *If G is a non-bipartite cactus of order n and the length of a shortest odd cycle in G is at least g_{odd}, then $\gamma_\alpha(G)\le\frac{1}{2}(1+\frac{1}{g_{\mathrm{odd}}})n$.*

Proof. (i) Since the smaller partite set of G is an α-dominating set of G, the result is obvious.

(ii) Let G be a non-bipartite cactus of order $n\ge 2$ such that the length of a shortest odd cycle in G is at least g_{odd}. We may assume, without loss of generality, that G is connected. We prove the bound by induction on the number b of blocks of G.

First we assume that $b=1$, i.e., G is an odd cycle of length at least g_{odd}. This implies

$$\gamma_\alpha(G)=\left\lceil\frac{n}{2}\right\rceil=\frac{n+1}{2}\le\frac{1}{2}\left(1+\frac{1}{g_{\mathrm{odd}}}\right)n.$$

Now let $b > 1$ and let $B = (V_B, E_B)$ be an endblock of G, i.e. G has exactly one cutvertex v. As above, we see that there is an α-dominating set D_B of B such that $|D_B| \leq \frac{1}{2}(1 + \frac{1}{g_{\text{odd}}})|V_B|$ and $v \in D_B$.

Let $H_1,\ H_2, \ldots, H_l$ be the components of $G - V_B$. Clearly, H_i is a cactus having less blocks than G such that H_i is either bipartite or the length of a shortest odd cycle in H_i is at least g_{odd} for $1 \leq i \leq l$. By (i) and the induction hypothesis, there is an α-dominating set D_i of H_i such that $|D_i| \leq \frac{1}{2}(1 + \frac{1}{g_{\text{odd}}})|V_{H_i}|$ for $1 \leq i \leq l$.

Since $v \in D_B$, it follows that $D = D_B \cup D_1 \cup D_2 \cup \ldots \cup D_l$ is an α-dominating set of G such that $|D| \leq \frac{1}{2}(1 + \frac{1}{g_{\text{odd}}})(|V_B| + |V_{H_1}| + |V_{H_2}| + \ldots + |V_{H_l}|) = \frac{1}{2}(1 + \frac{1}{g_{\text{odd}}})n$ and the proof is complete. ■

The term $\frac{n}{2}$ in Corollary 2.3 and Proposition 2.7 is a well-known upper bound for the (ordinary) domination number $\gamma(G)$ of a graph G of order n without isolated vertices [7].

The graphs for which the equality $\gamma(G) = \frac{n}{2}$ holds have been well-studied and characterized in [4] and [8]. The main obstacle for proving an analogous characterization for the α-domination number is the fact that — in contrast to the (ordinary) domination number — this parameter has no monotonicity with respect to spanning subgraphs, i.e., if $G = (V_G, E_G)$ and $H = (V_H, E_H)$ are graphs such that $V_G = V_H$ and $E_H \subseteq E_G$, then $\gamma(H) \geq \gamma(G)$ but no such estimate holds for γ_α. As an example consider the star $K_{1,5}$ of order 6, the complete graph K_6 of order 6 and the tree E that arises by subdividing two edges of $K_{1,3}$. Clearly, $K_{1,5}$ and E are two spanning subgraphs of K_6 but $1 = \gamma_{\frac{2}{5}}(K_{1,5}) < 2 = \gamma_{\frac{2}{5}}(K_6) < 3 = \gamma_{\frac{2}{5}}(E)$.

We close with a corresponding characterization for trees.

Theorem 2.8. *If $\alpha \in (0, 1)$ and T is a tree of order n, then $\gamma_\alpha(T) = \frac{n}{2}$ if and only if T has a perfect matching M such that $\min\{d_T(u), d_T(v)\} < \lceil \frac{1}{1-\alpha} \rceil$ for all $uv \in M$.*

Proof. We first prove the 'if'-part. Let T be a tree that has a perfect matching M such that $\min\{d_T(u), d_T(v)\} < \lceil \frac{1}{1-\alpha} \rceil$ for all $uv \in M$. Let D be a minimum α-dominating set of T. If there is some edge $u_0v_0 \in M$ such that $u_0, v_0 \notin D$, then $d_T(u_0) - 1 \geq |N_G(u_0) \cap D| \geq \alpha d_T(u_0)$ and $d_T(v_0) - 1 \geq |N_G(v_0) \cap D| \geq \alpha d_T(v_0)$. This implies the contradiction $\min\{d_T(u_0), d_T(v_0)\} \geq \lceil \frac{1}{1-\alpha} \rceil$. Therefore, $|\{u, v\} \cap D| \geq 1$ for all $uv \in M$.

In view of Proposition 2.7, this implies

$$\frac{n}{2} = |M| \leq \sum_{uv \in M} |\{u, v\} \cap D| = |D| = \gamma_\alpha(T) \leq \frac{n}{2}$$

and the proof of the 'if'-part is complete.

Now we prove the 'only if'-part by induction on the order n of T. If $n \leq 2$, then the result is immediate. Now let T be a tree of order $n \geq 3$ such that $\gamma_\alpha(T) = \frac{n}{2}$. Let u be an endvertex of T and let v be the unique neighbour of u in T.

If v is adjacent to an endvertex u' of T different from u, then let $T' = T - \{u\}$. It is obvious that there is a minimum α-dominating set D' of T' that contains v. Therefore, D' is also an α-dominating set of T and we obtain the contradiction $\gamma_\alpha(T) \leq \gamma_\alpha(T') \leq \frac{n-1}{2}$. Hence u is the unique endvertex of T adjacent to v and all components $T_1, T_2, \ldots, T_l$ of $T - \{u, v\}$ are trees of order at least two. For $1 \leq i \leq l$ let D_i be a minimum α-dominating set of T_i. Since the set $\{v\} \cup D_1 \cup D_2 \cup \ldots \cup D_l$ is an α-dominating set of T, we obtain

$$\frac{n-2}{2} = \gamma_\alpha(T) - 1 \leq \sum_{i=1}^{l} |D_i| = \sum_{i=1}^{l} \gamma_\alpha(T_i) \leq \sum_{i=1}^{l} \frac{|V_{T_i}|}{2} = \frac{n-2}{2}.$$

This easily implies that $\gamma_\alpha(T_i) = \frac{|V_{T_i}|}{2}$ and, by induction, the tree T_i has a perfect matching M_i for $1 \leq i \leq l$. Clearly, $M = \{uv\} \cup M_1 \cup M_2 \cup \ldots \cup M_l$ is a perfect matching of T.

For contradiction we assume that $\min\{d_T(u_0), d_T(v_0)\} \geq \lceil \frac{1}{1-\alpha} \rceil$ for some edge $u_0v_0 \in M$. Since T is a tree, it is straightforward to verify that the set

$$\begin{aligned} D = {} & \{w \in V_T \mid \mathrm{dist}_G(w, u_0) \text{ is odd and } \mathrm{dist}_G(w, u_0) < \mathrm{dist}_G(w, v_0)\} \\ & \cup \{w \in V_T \mid \mathrm{dist}_G(w, v_0) \text{ is odd and } \mathrm{dist}_G(w, v_0) < \mathrm{dist}_G(w, u_0)\} \end{aligned}$$

is an α-dominating set of T such that $u_0, v_0 \notin D$ and $|\{u, v\} \cap D| = 1$ for all $uv \in M \setminus \{u_0v_0\}$. This leads to the contradiction $\gamma_\alpha(T) \leq |D| = \frac{n-2}{2}$ and the proof is complete. ■

Note that $\lceil \frac{1}{1-\alpha} \rceil = 2$ for $\alpha \in (0, \frac{1}{2}]$. Thus for these values of α the trees described by the condition given in Theorem 2.8 correspond exactly to the trees T of order n that satisfy $\gamma(T) = \frac{n}{2}$ (every non-endvertex of these trees is adjacent to a unique endvertex).

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