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# ON THE STRUCTURE OF PLANE GRAPHS OF MINIMUM FACE SIZE 5

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**Abstract**

A subgraph of a plane graph is light if the sum of the degrees of the vertices of the subgraph in the graph is small. It is known that a plane graph of minimum face size 5 contains light paths and a light pentagon. In this paper we show that every plane graph of minimum face size 5 contains also a light star $K_{1,3}$ and we present a structural result concerning the existence of a pair of adjacent faces with degree–bounded vertices.

**Keywords:** plane graph, light graph, face size.

**2000 Mathematics Subject Classification:** 05C10.

## 1. Introduction

Throughout this paper we consider connected plane graphs without loops or multiple edges. For a plane graph $G$, $V(G)$ and $F(G)$ denotes the set of its vertices and faces. A *k-vertex* (*k-face*) will stand for a vertex (a face) of degree (size) $k$, a $\geq k$*-vertex*/$\leq k$*-vertex* ($\geq k$*-face*/$\leq k$*-face*) for those of degree (size) at least $k$/at most $k$. Let $P_r$ and $C_r$ denote a path and a cycle, respectively, on $r$ vertices (an *r-path* and an *r-cycle* in the sequel). Let an *r-star* be a graph $S_r = K_{1,r} = [x; a_1, \ldots, a_r]$.

Let $\mathcal{G}$ be a family of graphs and let $\mathcal{H}$ be a finite family of graphs with the property that each graph of $\mathcal{G}$ contains a proper subgraph isomorphic

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to at least one member of $\mathcal{H}$. Let $\tau(\mathcal{H},\mathcal{G})$ be the smallest integer with the property that every graph $G \in \mathcal{G}$ contains a subgraph $K$ which is isomorphic to one of the elements in $\mathcal{H}$ such that, for every vertex $v \in V(K)$,

$$\deg_G(v) \le \tau(\mathcal{H},\mathcal{G}).$$

If such a finite $\tau(\mathcal{H},\mathcal{G})$ does not exist we write $\tau(\mathcal{H},\mathcal{G}) = +\infty$. We shall say that the family $\mathcal{H}$ is *light in the family* $\mathcal{G}$ if $\tau(\mathcal{H},\mathcal{G}) < +\infty$.

Similarly, let $f(\mathcal{H},\mathcal{G})$ be the smallest integer with the property that every graph $G \in \mathcal{G}$ contains a subgraph $K$ which is isomorphic to one of the elements of $\mathcal{H}$ such that

$$\sum_{v\in V(K)} \deg_G(v) \le f(\mathcal{H},\mathcal{G}).$$

Obviously, $\mathcal{H}$ is light in the family $\mathcal{G}$ if and only if $f(\mathcal{H},\mathcal{G}) < +\infty$.

Note that in the case $|\mathcal{H}| = 1$ we obtain the definition of *light graph* $H$ *in the family* $\mathcal{G}$, see [8], and we will use notation $\varphi(H,\mathcal{G})$ and $w(H,\mathcal{G})$ instead of $\tau(\mathcal{H},\mathcal{G})$, $f(\mathcal{H},\mathcal{G})$.

Let $\mathcal{G}_c(\delta,\rho)$ be the family of all $c$-connected plane graphs with minimum vertex degree at least $\delta$ and minimum face size at least $\rho$; we will use $\mathcal{G}(\delta,\rho)$ instead of $\mathcal{G}_3(\delta,\rho)$ and $\varphi(H;\delta,\rho)$ instead of $\varphi(H,\mathcal{G}(\delta,\rho))$ (similarly for $w,\tau,f$).

It is well known that every plane graph contains a vertex of degree at most 5. The excellent Kotzig's theorem [10] shows that each graph $G \in \mathcal{G}(3,3)$ contains a light edge $e = \{u,v\}$ such that $\deg_G(u)+\deg_G(v) \le 13$, the bound being precise; thus, $w(e;3,3) = 13$ and $\varphi(e;3,3) = 10$. These results were generalized in several directions and led to study of the existence of light graphs and families of graphs for various values of parameters $c,\delta,\rho$. For the light graph problems, Fabrici and Jendrol' [5] proved $\varphi(P_k;3,3) = 5k$ and $\varphi(H;3,3) = +\infty$ for $H \not\cong P_k$. The situation is similar in families $\mathcal{G}(4,3)$ and $\mathcal{G}(3,4)$ – the only light graphs are paths and $\varphi(P_k;4,3) = 5k-7$ for $k \ge 8$ ([4]), $5\lfloor\frac{k}{2}\rfloor \le \varphi(P_k;3,4) \le \frac{5}{2}k$ ([7]). For the values of $w$, it is known that $k\log_2 k \le w(P_k;3,3) \le k^2 + 13k$ ([5, 3]). In [6] and [1] there is proved that the family $\mathcal{T}_k$ of all trees on $k$ vertices is light in $\mathcal{G}(3,3)$ and $\tau(\mathcal{T}_k;3,3) = 4k+3$ for $k \ge 3$, $8k-5 \le f(\mathcal{T}_k;3,3) \le 8k-1$ for $k \ge 5$; in [2] it is shown that $\tau(\mathcal{T}_k;4,3) = 4k-1$ for $k \ge 4$.

The aim of this paper is to extend the family of light graphs in the class $\mathcal{G}(3,5)$. From the result of Jendrol' and Fabrici [5] it follows that a $k$-path

$P_k$, $k \geq 1$ is light in $\mathcal{G}(3,5)$, and in [9] it is shown that $\varphi(P_k;3,5) \leq \frac{5}{3}k$. The classical result of Lebesgue [11] implies that the 5-face $C_5$ is light in $\mathcal{G}(3,5)$ and $\varphi(C_5;3,5) = 5$. Jendrol' and Owens [9] also showed that no $k$-cycle $C_k$, $k > 5$, except, possibly, $C_{14}$, is light in $\mathcal{G}(3,5)$.

We prove

**Theorem 1.1.** *Each $G \in \mathcal{G}(3,5)$ contains a 3-star $S_3 = [x;a,b,c]$ such that $deg_G(x) = 3, deg_G(a) = deg_G(b) = 3, deg_G(c) \leq 4$. Moreover, the bounds 3 and 4 are best possible.*

**Corollary 1.2.** *$S_3$ is light in $\mathcal{G}(3,5)$, $\varphi(S_3;3,5) = 4$, $w(S_3;3,5) = 13$.*

**Theorem 1.3.** *Each $G \in \mathcal{G}(3,5)$ contains a 5-face adjacent to a $\leq 6$-face such that every their vertex is of degree at most 9 in $G$.*

**Corollary 1.4.** *The family $\{C_8, C_9\}$ is light in $\mathcal{G}(3,5)$ and $6 \leq \tau(\{C_8, C_9\}; 3,5) \leq 9$.*

## 2. Proof of Theorem 1.1

The proof is by contradiction. Suppose that there exists a 3-connected plane graph $G$ of minimum face size 5 such that for $S_3 \subseteq G$, $S_3 = [x;a,b,c]$ with $\deg_G(x) = 3$, at least one of $a,b,c$ is a $\geq 5$-vertex, or at least two of them are $\geq 4$-vertices.

It is a consequence of Euler's theorem that

$$\sum_{v \in V(G)} (\deg_G(v) - 4) + \sum_{f \in F(G)} (\deg_G(f) - 4) = -8.$$

According to this formula, assign to each vertex $v \in V(G)$ the initial charge $\mu(v) = \deg_G(v) - 4$ and to each face $f \in F(G)$ the initial charge $\mu(f) = \deg_G(f) - 4$. Thus $\sum_{x \in V(G) \cup F(G)} \mu(x) = -8$. Then, we define the local redistribution of charges such that the sum of charges remains the same and we show that the new charge function $\overline{\mu} : V(G) \cup F(G) \to \mathbb{Q}$ is nonnegative. This will contradict the fact that $\sum_{x \in V(G) \cup F(G)} \overline{\mu}(x) = \sum_{x \in V(G) \cup F(G)} \mu(x) = -8$.

The local redistribution of charges is performed by the following rules:

**Rule 1.** Each $k$-vertex $v, k \geq 4$ sends $\frac{k-4}{m(v)}$ to each adjacent 3-vertex; $m(v)$ is the number of 3-vertices adjacent to $x$. If $m(v) = 0$, no charge is transferred.

**Rule 2.** Each $k$-face $f$ sends $\frac{k-4}{m(f)}$ to each incident 3-vertex; $m(f)$ is the number of 3-vertices incident to $f$. If $m(f) = 0$, no charge is transferred.

Let $\widetilde{\mu} : V(G) \cup F(G) \to \mathbb{Q}$ be the charge of vertices and faces of $G$ after applying Rules 1 and 2. A vertex $x$ is called *overcharged* if $\widetilde{\mu}(x) > 0$, and it is called *undercharged* if $\widetilde{\mu}(x) < 0$.

**Rule 3.** Each overcharged 3-vertex $x$ sends $\frac{\widetilde{\mu}(x)}{\widetilde{m}(x)}$ to each adjacent undercharged 3-vertex; $\widetilde{m}(x)$ is the number of undercharged 3-vertices adjacent to $x$. If $\widetilde{m}(x) = 0$, no charge is transferred.

Let $\overline{\mu}$ be the charge of vertices and faces of $G$ after applying Rules 1,2 and 3. Note that for faces and vertices of degree at least 4, $\widetilde{\mu}$ and $\overline{\mu}$ is the same, and for overcharged 3-vertex $x$, $\overline{\mu}(x) \geq 0$. To show that $\overline{\mu}$ is a nonnegative function, several cases are considered.

**(1)** Let $f$ be a $k$-face of $G$. If $f$ is not incident with a 3-vertex, then $\widetilde{\mu}(f) = k - 4 > 0$; otherwise $\widetilde{\mu}(f) = k - 4 - m(f) \cdot \frac{k-4}{m(f)} = 0$.

**(2)** Let $x$ be a $k$-vertex of $G, k \geq 4$. If $x$ is not adjacent with a 3-vertex, then $\widetilde{\mu}(x) = k - 4 \geq 0$; otherwise $\widetilde{\mu}(x) = k - 4 - m(x) \cdot \frac{k-4}{m(x)} = 0$.

**(3)** Let $x$ be a 3-vertex; denote $u, v, w$ the neighbours of $x$.

**(3.1)** Let $x$ be incident with three 5-faces; denote $i, j$ and $k, l$ ($j$ being adjacent to $k$) the remaining neighbours of $u$ and $v$, respectively. If $u, v, w$ are $\geq$ 4-vertices, then $\widetilde{\mu}(x) \geq -1 + 3 \cdot \frac{5-4}{3} = 0$. Suppose that $u, v$ are 3-vertices. Then $w$ is a $\geq$ 5-vertex and at least one vertex from each pair $i, j$ and $k, l$ is a $\geq$ 4-vertex. Hence, either $\widetilde{\mu}(x) \geq -1 + \frac{1}{5} + 2 \cdot \frac{1}{4} + \frac{1}{3} = \frac{1}{30} > 0$ or $\widetilde{\mu}(x) \geq -1 + \frac{1}{5} + 2 \cdot \frac{1}{3} + \frac{1}{5} = \frac{1}{15} > 0$.

Suppose that $u$ is a 3-vertex and $v, w$ are $\geq$ 4-vertices. If at least one of them is a $\geq$ 5-vertex, $\widetilde{\mu}(x) \geq -1 + \frac{1}{5} + \frac{1}{3} + 2 \cdot \frac{1}{4} = \frac{1}{30} > 0$; so we can assume that $v, w$ are 4-vertices. If $i, j$ are $\geq$ 4-vertices, then $\widetilde{\mu}(x) \geq -1 + 3 \cdot \frac{1}{3} = 0$, so, without loss of generality suppose that $i$ is a 3-vertex; then $j$ is a $\geq$ 5-vertex. Moreover, we can suppose that $w$ is the only $\geq$ 4-vertex sharing the common face $\omega$ with $x, u, i$ (otherwise $\widetilde{\mu}(x) \geq -1 + 3 \cdot \frac{1}{3} = 0$). Denote by $z$ the fifth vertex of $\omega$. Then the remaining neighbour of $i$ is a $\geq$ 5-vertex, the remaining neighbour (distinct from $i, w$) of $z$ is a $\geq$ 4-vertex and $\widetilde{\mu}(u) \geq -1 + \frac{1}{5} + 2 \cdot \frac{1}{3} + \frac{1}{4} = \frac{7}{60} > 0$, $\widetilde{\mu}(i) \geq -1 + \frac{1}{5} + 2 \cdot \frac{1}{3} + \frac{1}{4} = \frac{7}{60} > 0$. Thus $i, u$ are overcharged and $u$ has only one undercharged 3-neighbour; hence, $\overline{\mu}(x) \geq -1 + 2 \cdot \frac{1}{3} + \frac{1}{4} + \frac{7}{60} = \frac{1}{30} > 0$.

**(3.2)** Let $x$ be incident with two 5-faces and one $\geq$ 6-face $\alpha$. If all neighbours of $x$ are $\geq$ 4-vertices, then $\widetilde{\mu}(x) \geq -1 + 3 \cdot \frac{5-4}{3} = 0$. Suppose that two of neighbours of $x$ are 3-vertices. Then the third one is a $\geq$ 5-vertex and we have $\widetilde{\mu}(x) \geq -1 + \frac{1}{5} + 2 \cdot \frac{1}{4} + \frac{1}{3} = \frac{1}{30} > 0$ (if both 3-vertices are adjacent to $\alpha$) or $\widetilde{\mu}(x) \geq -1 + \frac{1}{5} + \frac{1}{5} + \frac{1}{4} + \frac{6-4}{5} = \frac{1}{20} > 0$.

Let $x$ be adjacent with exactly one 3-vertex $u$ and remaining neighbours be $\geq$ 4-vertices. If $\geq$ 4-vertices are adjacent to $\alpha$, then $\widetilde{\mu}(x) \geq -1 + 2 \cdot \frac{1}{4} + \frac{6-4}{4} = 0$; if at least one of them is a $\geq$ 5-vertex, $\widetilde{\mu}(x) \geq -1 + \frac{1}{3} + \frac{1}{4} + \frac{6-4}{5} + \frac{1}{5} = \frac{11}{60} > 0$. In the remaining case at least one neighbour of $u$ is a $\geq$ 4-vertex, hence, $\widetilde{\mu}(x) \geq -1 + 2 \cdot \frac{1}{3} + \frac{6-4}{5} = \frac{1}{15} > 0$ or $\widetilde{\mu}(x) \geq -1 + \frac{1}{3} + \frac{1}{4} + \frac{6-4}{4} = \frac{1}{12} > 0$.

**(3.3)** Let $x$ be incident with one 5-face and two $\geq$ 6-faces. If $x$ is adjacent to a $\geq$ 5-vertex, we have $\widetilde{\mu}(x) \geq -1 + \frac{1}{5} + 2 \cdot \frac{6-4}{5} + \frac{1}{5} > 0$ or $\widetilde{\mu}(x) \geq -1 + \frac{1}{5} + \frac{1}{3} + \frac{6-4}{5} + \frac{1}{4} > 0$. Otherwise $x$ is adjacent to at least two $\geq$ 4-vertices, so $\widetilde{\mu}(x) \geq -1 + \frac{1}{3} + 2 \cdot \frac{6-4}{5} > 0$ or $\widetilde{\mu}(x) \geq -1 + \frac{6-4}{4} + \frac{6-4}{5} + \frac{1}{4} > 0$.

**(3.4)** Let $x$ be incident to $\geq$ 6-faces. Then $\widetilde{\mu}(x) \geq -1 + 3 \cdot \frac{6-4}{6} > 0$.
To show that the bounds 3 and 4 are best possible, consider the graph of the dodecahedron and subdivide each its edge by a new vertex. We obtain a plane graph $G'$ consisting only of 10-faces. For $j = 1, \ldots, 12$ let $\alpha^j = x_1^j x_2^j \ldots x_{10}^j$ be an arbitrary face of $G'$ such that $x_i^j$ is of degree 2 for $i$ even. Into $\alpha^j$ insert a 5-cycle $C^j = y_1^j y_2^j \ldots y_5^j$ and add new edges $y_i^j x_{2i}^j$ for $i = 1, \ldots 5$. It is easy to see that, in the graph so obtained, each 3-vertex is adjacent to at least one 4-vertex. ∎

## 3. Proof of Theorem 1.3

The proof is by contradiction. Suppose that there exists a 3-connected plane graph $G$ with minimum face size 5 such that each its 5-face is adjacent only with $\geq$ 7-faces, or if it is adjacent with a $\leq$ 6-face, then at least one of their vertices is of degree $\geq$ 10.

We use again the Discharging method; the charge $\nu : V(G) \cup F(G) \to \mathbb{Q}$ of vertices and faces of $G$ is assigned according to the formula (which is another consequence of the Euler theorem)

$$\sum_{v \in V(G)} (2 \cdot \deg_G(v) - 6) + \sum_{f \in F(G)} (\deg_G(f) - 6) = -12,$$

so, $\nu(v) = 2\deg_G(v) - 6$, $\nu(f) = \deg_G(f) - 6$. The local redistribution of charges is performed according to the following rules:

**Rule 1.** Each $k$-vertex $x$ sends $\frac{2k-6}{m(x)}$ to each incident $\leq 6$-face; $m(x)$ is the number of $\leq 6$-faces incident to $x$. If $m(x) = 0$, no charge is transferred.

Let $\widetilde{\nu}$ be the charge of vertices and faces after application of Rule 1; like in Section 2, we will use the notion of overcharged and undercharged faces.

**Rule 2.** Each face $\alpha$ with $\widetilde{\nu}(\alpha) > 0$ sends $\frac{\widetilde{\nu}(\alpha)}{\widetilde{m}(\alpha)}$ to each adjacent undercharged 5-face; $\widetilde{m}(\alpha)$ is the number of undercharged 5-faces incident to $\alpha$. If $\widetilde{m}(\alpha) = 0$, no charge is transferred.

Let $\overline{\nu}$ be the charge of vertices and faces after application of Rules 1 and 2. To show that $\overline{\nu}$ is a nonnegative function, it is enough to count it only for 5-faces, since, due to the definition of discharging rules, vertices are either completely discharged (here $\overline{\nu}(x) = \widetilde{\nu}(x) = 0$) or they keep their charge (in this case, $\overline{\nu}(x) = \widetilde{\nu}(x) = 2k - 6 \geq 0$); the same holds for $\geq 7$-faces and $\leq 6$-faces overcharged after Rule 1.

If a 5-face $\alpha$ is incident with a $\geq 6$-vertex, then $\widetilde{\nu}(\alpha) \geq -1 + \frac{2\cdot 6-6}{6} = 0$, and if $\alpha$ is incident to at least two $\geq 4$-vertices, then $\widetilde{\nu}(\alpha) \geq -1 + 2\cdot\frac{2\cdot 4-6}{4} = 0$; so we can suppose that $\alpha$ is incident with four 3-vertices and one $\leq 5$-vertex.

**(1)** Let $\alpha$ be incident with five 3-vertices. Then each face $\beta$ adjacent to $\alpha$ is either a $\geq 7$-face or a $\leq 6$-face containing a $\geq 10$-vertex. If $\beta$ is a $\geq 7$-face and $\gamma,\ \omega$ are faces incident both with $\beta,\ \alpha$, then $\beta$ does not send a charge to $\gamma,\ \omega$ by Rule 2 ($\widetilde{\nu}(\gamma) > 0,\ \widetilde{\nu}(\omega) > 0$), thus $\widetilde{m}(\beta) \leq \deg_G(\beta) - 2$ and $\beta$ sends at least $\frac{7-6}{7-2} = \frac{1}{5}$ to $\alpha$ by Rule 2.

If $\beta$ is a 6-face, then it is incident with a $\geq 10$-vertex, thus, $\widetilde{\nu}(\beta) \geq 0 + \frac{2\cdot 10-6}{10} = \frac{7}{5}$ and $\beta$ sends at least $\frac{\frac{7}{5}}{6} > \frac{1}{5}$ to $\alpha$ by Rule 2.

If $\beta$ is a 5-face, then, again, it is incident with a $\geq 10$-vertex, thus, $\widetilde{\nu}(\beta) \geq -1 + \frac{2\cdot 10-6}{10} = \frac{2}{5}$. Since $\beta$ is incident with at most two undercharged 5-faces, it sends at least $\frac{1}{5}$ to $\alpha$.

Therefore, $\alpha$ receives at least $\frac{1}{5}$ from each of its neighbouring faces and $\overline{\nu}(\alpha) \geq -1 + 5\cdot\frac{1}{5} = 0$.

**(2)** Let $\alpha$ be incident with four 3-vertices and a vertex $y$ of degree 4 or 5. Consider three neighbouring faces of $\alpha$ which are not incident with edges having $y$ as endvertex. They are either $\geq 7$-faces or $\leq 6$-faces containing $\geq 10$-vertices. Using the same arguments as above, these faces send at least $\frac{1}{5}$ to $\alpha$, thus, $\overline{\nu}(\alpha) \geq -1 + \frac{2\cdot 4-6}{4} + 3\cdot\frac{1}{5} > 0$. ■

In [9] there is an example of a pentagonal plane graph with the property that each pair of adjacent 5-faces contains a 6-vertex. This implies the lower bound for the number $\tau(\{C_8, C_9\}; 3, 5)$.

## 4. Concluding Remarks

**1.** Theorem 1.3 implies that the graphs of Figure 1 are light in $\mathcal{G}(3,5)$ (since each of them is the subgraph of the pair of adjacent 5-faces or of the 5-face which is adjacent with a 6-face).

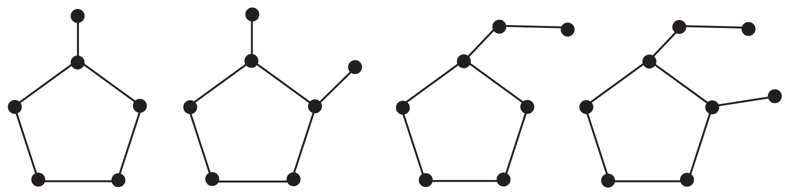

Figure 1: Some light graphs in $\mathcal{G}(3,5)$

**2.** Corollary 1.4 also gives an example of two cycles such that each of them is not light in $\mathcal{G}(3,5)$, but the family comprised of them is light. This leads to the following

**Problem 4.1.** Determine all pairs $\{C_i, C_j\}$ of cycles such that neither $C_i$ nor $C_j$ is light in $\mathcal{G}(3,5)$, but the family $\{C_i, C_j\}$ is light. In general, for given $n$, determine all families $\{C_{i_1}, \ldots, C_{i_n}\}$ such that no subfamily is light in $\mathcal{G}(3,5)$, but the whole family is light.

The condition on minimum face size 5 cannot be omitted: for the family $\mathcal{G}(3,3)$, the graph of $n$-wheel, $n$ being large shows that no family of cycles is light (since each of them contains the $n$-vertex); similarly, for the family $\mathcal{G}(3,4)$, the dual of $n$-sided antiprism (i.e., the dual of the unique 4-regular plane graph consisting of $2n$ 3-faces and two nonadjacent $m$-faces) shows that no family of even cycles is light.

**3.** Providing the additional information on degrees of face vertices, Theorem 1.3 can be also viewed as a strengthening of the theorem of Wernicke [12] on existence of 5-face adjacent to a $\leq$ 6-face in every plane graph of minimum face size 5. We believe that it is possible, in the similar way, also to strengthen Franklin's theorem about the triple of neighbouring faces. According to this, we formulate the following

**Problem 4.2.** Find the maximum number $n_0$ and the minimum numbers $k(n_0)$, $t(n_0)$ with the property that each plane graph of minimum face size 5 contains a chain (that is, the path in the dual graph) of $n_0$ faces of size $\leq k(n_0)$ such that each of their vertices is of degree at most $t(n_0)$. Determine the corresponding values $k(n)$, $t(n)$ for each integer $n \leq n_0$.

From Theorem 1.3 we obtain $k(2) = 6$, $6 \leq t(2) \leq 9$. In order to be $t(n) < +\infty$, $n \leq 6$ is necessary, as seen from the following example: Choose $m$ arbitrarily large and consider the plane graph $K_{2,m}$. Let $\alpha^i = ux_ivy_i, i = 1, \ldots, m$ be a 4-face of $K_{2,m}$ such that $u, v$ are its $m$-vertices. Insert into $\alpha^i$ a copy $C^i$ of the configuration of Figure 2 (arisen from the dodecahedron by splitting its one edge to half-edges); let $a^i, b^i, e^i$ and $f^i$ be counterparts of $a, b, e$ and $f$ in $C^i$. Next, for each $i = 1, \ldots, m$ identify the vertex $a^i$ with $u$, the vertex $b^i$ with $v$ and join the half-edge $f_i$ with $e_{i+1}$ (index modulo $m$).

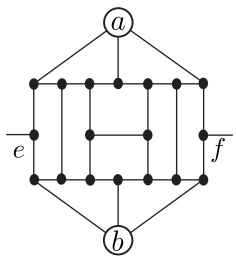


Figure 2: The splitted dodecahedron

In the resulting graph, any chain of at least 7 faces contains an $\geq m$-vertex.

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