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ANALOGUES OF CLIQUES FOR ORIENTED COLORING

William F. Klostermeyer

Department of Computer and Information Sciences
University of North Florida
Jacksonville, FL 32224–2669, U.S.A.

and

Gary MacGillivray

Department of Mathematics and Statistics
University of Victoria
Victoria, Canada

Abstract

We examine subgraphs of oriented graphs in the context of oriented coloring that are analogous to cliques in traditional vertex coloring. Bounds on the sizes of these subgraphs are given for planar, outerplanar, and series-parallel graphs. In particular, the main result of the paper is that a planar graph cannot contain an induced subgraph D with more than 36 vertices such that each pair of vertices in D are joined by a directed path of length at most two.

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1. Introduction

A *homomorphism* of a directed graph (digraph) D_1 to a digraph D_2 is a function f that maps the vertices of D_1 to the vertices of D_2 such that if xy is an arc of D_1, then $f(x)f(y)$ is an arc of D_2. The existence of a homomorphism of D_1 to D_2 is denoted by $D_1 \to D_2$. If there exists a homomorphism $D_1 \to D_2$, then we say that D_1 is *homomorphic* to D_2.

Let D be an oriented graph, i.e., a digraph in which at most one of the arcs uv, vu exists for each pair of vertices u, v. An *oriented k-coloring* of D is a homomorphism of D to some oriented graph T on k vertices. Since arcs joining non-adjacent vertices of T can be added without destroying the existence of a homomorphism $D \to T$, the oriented graph T can be taken to be a tournament. The *oriented chromatic number* of D, denoted $\chi_o(D)$, or simply χ_o, is the smallest integer k such that there is an oriented k-coloring of D. From the definition of an oriented coloring of digraph D, if $u \to v$ and $w \to u$, then each of u, v, w must receive distinct colors. In other words, adjacent vertices receive different colors and all arcs between vertices of color i and color j, $i \neq j$, must be oriented in the same direction.

Recent results concerning oriented chromatic number of digraphs can be found in [2, 3, 4, 6]. Raspaud and Sopena proved that all oriented planar graphs have oriented chromatic number at most eighty [4]. It was shown in [8] that there exists an orientation of a planar graph with oriented chromatic number at least sixteen. Tightening this bound is a significant problem in the domain of oriented coloring.

Complete subgraphs (cliques) on k vertices obviously prevent an undirected graph from being $(k-1)$-colored. Similarly, a subgraph that is a tournament on k vertices is a similar obstruction for oriented coloring. The objective of this paper is to study another fundamental obstruction for oriented coloring, which we shall term *ocliques.* Ocliques will be defined in Section 2. In Sections 3 through 5, we bound the maximum cardinalities of ocliques in outerplanar, planar, and series-parallel graphs.

2. Preliminaries

Recall that the famous Kuratowski/Wagner theorem states that a graph G is planar if and only if it contains neither a K_5 minor nor a $K_{3,3}$ minor [9]. Similarly, a graph is *outerplanar*, i.e., it can be embedded in the plane with all vertices bordering the outside face, if and only if it contains neither a K_4 minor nor a $K_{2,3}$ minor. A graph is *series-parallel* if and only if it contains no K_4 minor.

Let P_n denote a path with n vertices. In a digraph, vertex x *dominates* vertex y if $x \to y$ is an arc. We say that vertices x and y are *adjacent* if x dominates y or y dominates x and in either event we also say that x and y are neighbors. We say that vertex set A dominates vertex set B if, for each pair of vertices $x \in A$, $y \in B$, x dominates y.

Define an *oclique* to be an oriented graph D such that for any two distinct vertices $x, y \in V(D)$ there exists a directed xy path of length at most two or a directed yx path of length at most two. Let $\omega_o(D)$ denote the cardinality of the largest induced subgraph of D that is an oclique (i.e., the number of vertices it contains) in oriented graph D.

The next fact is implicit in Sopena [7].

Fact 1. *Let D be an oriented graph. Then $\chi_o(D) \geq \omega_o(D)$.*

The following results states that, in terms of the number of colors used, ocliques play the same role in oriented colorings as cliques do in colorings.

Corollary 2. *Let D be an oriented graph with n vertices. Then $\chi_o(D) = n$ if and only if D is an oclique.*

Proof. Suppose D is an oclique. By Fact 1, $\chi_o(D) \geq n$. Since the identity mapping is a homomorphism $D \to D$, this implies $\chi_o(D) = n$.

Now suppose D is not an oclique. Then there exist nonadjacent vertices x and y that are not joined by a directed path of length two. Let D' be the digraph that results from identifying x and y, and creating the new vertex z. Since x and y are not joined by a directed path of length two, D' is an oriented graph. The mapping that sends x and y to z and every other vertex of D to its clone in D' is a homomorphism $D \to D'$. Thus, $\chi_o(D) \leq n - 1$. ■

The next proposition is straightforward.

Proposition 3. *If D is an oclique, then $G[D]$ has diameter at most two.*

The following lemma is used several times. The proof is trivial and is omitted.

Lemma 4. *Let D be an oclique whose underlying graph has a cut vertex, v, whose deletion partitions $G[D]$ into components $C_1, C_2 \ldots, C_k, k \geq 2$. Then either $V(C_i)$ dominates v, or v dominates $V(C_i)$, for all $i, 1 \leq i \leq k$.*

We use the term 2-*path* to mean a directed path of length two. If D is a directed graph, then $G = G[D]$ denotes the underlying undirected graph of D. Also, let $dist(u, v)$ denote the length of the shortest directed uv path in digraph D.

3. Outerplanar Ocliques

Sopena [7] proved that every oriented outerplanar graph admits a homomorphism to the quadratic residue tournament on seven vertices ($V = \{0, 1, \ldots, 6\}$ and $uv \in E$ if and only if $u - v =$ 1,2, or 4 mod 7). Thus we have the following.

Fact 5. *Every outerplanar oclique has at most seven vertices.*

Proposition 6. *Let D be an outerplanar oclique with $|V| \geq 6$. Then $G[D]$ has minimum degree two.*

Proof. Since $G[D]$ is outerplanar, it is well-known that $\delta(G[D]) \leq 2$. Suppose x has degree one in $G = G[D]$. Without loss of generality, $x \to y$ in D. Since D is an oclique, the vertex y dominates all vertices in $V - \{x, y\}$. Fix an outerplanar embedding of D. Let z_1, z_2, z_3, z_4 be vertices in $V - \{x, y\}$ such that the arcs $xy, yz_1, yz_2, yz_3, yz_4$ occur consecutively in clockwise order about y in the embedding. But, any directed path of length at most two joining z_1 and z_4 creates a cycle that separates x from z_2 or z_3. ■

An outerplanar oclique on seven vertices, called O_7, is shown in Figure 1. Two additional outerplanar ocliques on seven vertices can be constructed from O_7 by adding an arc (oriented in either direction) between the leftmost vertex and the rightmost vertex. Another pair of ocliques on seven vertices can be formed from O_7 by adding an arc (oriented in either direction) between the third and fifth vertices from the left.

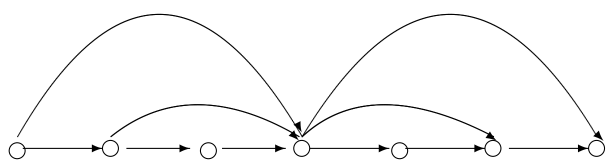

Figure 1. Outerplanar oclique O_7

Proposition 7. *Every outerplanar oclique on seven vertices contains O_7 as a spanning subgraph.*

Proof. Let D be an outerplanar oclique with seven vertices and fix an outerplanar embedding of $G[D]$. We consider two cases.

Case 1. Suppose $G[D]$ has a cut-vertex v. Clearly, each component of $D - v$ must be an outerplanar oclique and each vertex in each component of $G[D] - v$ must be adjacent to v. It is not hard to see that there can be at most two components in $G[D - v]$. Observe that each of these components can have at most three vertices, else $G[D]$ cannot be outerplanar. It follows that D is the digraph O_7.

Case 2. Suppose $G[D]$ is 2-connected. Since $G[D]$ is outerplanar, it has a Hamiltonian cycle C. Let $C = u_0u_1u_2u_3u_4u_5u_6$. Since $G[D]$ has diameter two, there is at least one edge u_i, u_{i+2} (addition modulo 7). Assume without loss of generality that u_0u_2 is an edge in $G[D]$. For u_1 being distance at most two from any other vertex, u_4 and u_5 have to be adjacent to either u_0 or u_2. If we have edges u_4u_2 and u_5u_0, then the distance between u_3 and u_6 is greater than two (in $G[D]$). Therefore u_4 and u_5 are either both adjacent to u_0 or both adjacent to u_2. Assume without loss of generality that u_4 and u_5 are both adjacent to u_0. In order for the distance between u_3 and u_6 to be less than three, we need edge u_0u_3. Thus the degree of u_0 is six.

Consider now D. Without loss of generality, assume u_0u_1 is an edge in D. For u_4, u_5 and u_6 being (oriented) distance at most two from u_1, we necessarily have edges u_4u_0, u_5u_0 and u_6u_0. By symmetry, because of edge u_6u_0, we necessarily have edges u_0u_2 and u_0u_3. Finally, for u_1 and u_3 (respectively, u_4 and u_6) being oriented distance at most two from each other, $u_1u_2u_3$ (respectively $u_4u_5u_6$) is necessarily a directed 2-path (in any direction) so that O_7 is a spanning subgraph of D. ■

4. Planar Graphs

As a consequence of Raspaud and Sopena's theorem, we know that if D is an orientation of a planar graph, then $\omega_o(D) \leq 80$.

The following is quite simple.

Fact 8. *If D is a bipartite planar oclique, then D has at most six vertices.*

Proof. Let $V = A \cup B$ denote the bipartition of D. It is clear that each vertex in A must be adjacent to each vertex in B. Hence one part must contain fewer than three vertices. It is easy to verify that neither $K_{2,p}, p > 4$ nor $K_{1,q}, q > 4$ can be oriented to be an oclique. ■

The following implication of Fact 8 is used several times, so we state it as a corollary.

Corollary 9. *The complete bipartite graph $K_{2,5}$ cannot be oriented to be an oclique.*

Lemma 10. *Let D be a planar oclique with $|V| \geq 10$. Then $G[D]$ has minimum degree at least two.*

Proof. Suppose vertex x has degree one in $G[D]$. Without loss of generality $x \to y$ in D. Since D is an oclique, the vertex y dominates all vertices in $V - \{x, y\}$. Then $G[V - \{x, y\}]$ is an outerplanar oclique and by Fact 5 has at most seven vertices, thus $G[D]$ can have at most nine vertices. ■

The vertex v on 2-path $u \to v \to w$ is designated as the *intermediate vertex.*

Theorem 11. *Let D be an orientation of a triangle-free planar graph such that D is an oclique. Then D has at most fourteen vertices.*

Proof. Suppose the theorem is false and let $G[D]$ be planar and triangle-free, where D is an oclique with at least fifteen vertices. It is known that since the diameter of $G[D]$ is two, $G[D]$ has a vertex, v, of degree at least ten [1, 5]; see also

`http://www.unc.edu/~rpratt/degdiam.html`

Then in D, at least five of v's neighbors are, without loss of generality, dominated by v. These neighbors induce an independent set, since $G[D]$ is triangle-free. We claim that an independent set with five vertices, all dominated by a sixth vertex is a *forbidden configuration* for a planar oclique, meaning that such a subgraph cannot exist in D if D is a planar oclique.

We prove this is a forbidden configuration. Fix a planar embedding of D, and let v dominate $v_1, v_2, \ldots, v_5$ where the v_i vertices are arranged around v in clockwise order. Each pair v_i, v_j must be connected by a 2-path using an intermediate vertex $w_k \neq v$. One can easily verify that at least three such w_k vertices are needed to connect five independent vertices via 2-paths. Suppose that v_1, w_1, v_5 is a 2-path, assuming without loss of generality that $v_1 \to w_1$. This forms a cycle in the underlying graph, v, v_1, w_1, v_5, v.

We consider two cases.

Case 1. Suppose v_2, w_2, v_4 is a 2-path where $w_2 \neq w_1$. Then v_3, w_2, v_5 and v_3, w_2, v_1 are 2-paths (note that if $v_3 \to w_2$, then $w_2 \to v_5, w_2 \to v_1$ and vice versa). But now there can be no 2-path between v_1 and v_4.

Case 2. Suppose v_2, w_1, v_4 is a 2-path. Then v_3, w_1, v_5 is a 2-path, where $v_3 \to w_1$. But now there can be no 2-path between v_1 and v_3. ■

Theorem 12. *If D is a planar oclique, then D has at most* 36 *vertices.*

Proof. Suppose the theorem were false and let D be an oclique with at least 37 vertices. Fix a planar embedding of $G[D]$. Let v be a vertex of degree at most five, which exists because $G[D]$ is planar [9]. By Lemma 10, v has degree at least two. Let $v_1, v_2, \ldots, v_r$, where $2 \leq r \leq 5$, be the neighbors of v. There are at least 31 additional vertices that v is connected to via a 2-path. Since $G[D]$ has diameter two, the subgraph induced by $V(G[D]) - \{v, v_1, \ldots, v_r\}$ is outerplanar and, as such, 3-colorable. Hence there exists an independent set, I, of at least eleven vertices among these 31 or more vertices. Partition $I = \{u_1, u_2, \ldots, u_k\}$ into sets, $S_1, S_2 \ldots, S_r$, according to which v_i vertex is the intermediate vertex on a $v \to v_i \to u_j$ 2-path or a $u_j \to v_i \to v$ 2-path. That is, v_i dominates each vertex in S_i (or vice versa). Of course, a vertex u_j may have several such 2-paths to/from vertex v. We deal with this by re-numbering the v_i vertices as necessary so that we can fix a partition $S_1, S_2 \ldots, S_r$ of I so that the set S_i of vertices with intermediate vertex v_i is at least as large as the set of vertices with intermediate vertex v_{i+q}, for all $q > 0$. Furthermore, we require that if $u_j \in S_i$, then there is no 2-path connecting u_j with v via any v_p where $p < i$.

Since there are at least eleven vertices in the union of the S_i sets, the pigeonhole principle assures us that either (i) $|S_1| \geq 5$ or (ii) $|S_1| = 4$ and $|S_2| = 4$ or (iii) $|S_1| \geq 3$, $|S_2| = 3, |S_3| = 3$ or (iv) $|S_1| = 4, |S_2| \geq 2, |S_3| \geq 2, |S_4| \geq 2, |S_5| \geq 1$ or (v) $|S_1| = 3, |S_2| \geq 2, |S_3| \geq 2, |S_4| \geq 2, |S_5| \geq 1$ or (vi) $|S_1| = 4, |S_2| = 3, |S_3| = 2, |S_4| = 1, |S_5| = 1$ or (vii) $|S_1| = 4, |S_2| = 3, |S_3| = |S_4| = 2$.

(Note that (iv), (v), and (vi) are similar, we separate them to clarify the details as we proceed below). A basic diagram to aid the reader's understanding is given in Figure 2.

Denote the vertices in S_1 as $a_1, a_2, \ldots$ in clockwise order around v_1 and the vertices S_2 as $b_1, b_2, \ldots$ in clockwise order around v_2. Assume, without loss of generality, that v dominates v_1 and thus, v_1 dominates $a_1, a_2, \ldots$. A basic illustration is given in Figure 2.

In case (i), we have the forbidden configuration described in the proof of Theorem 11.

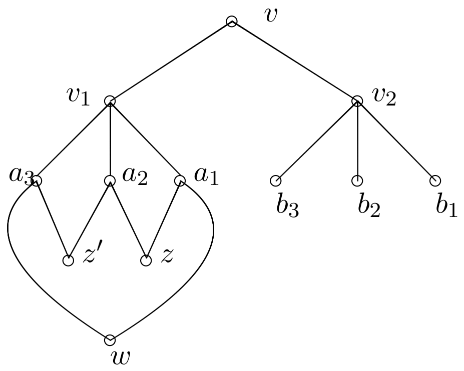

Figure 2. Possible structure of S_1, S_2

In case (ii), $|S_1| = 4, |S_2| = 4$. Vertex v_1 cannot be an intermediate vertex between any two of the b_i vertices in S_2, because in that case, one of the b_i vertices would belong in S_1. Then the 2-path between b_1 and b_3 uses intermediate vertex $z \neq v_1$. Consider the 2-path between a_1 and a_4. First suppose this 2-path uses intermediate vertex $w \neq v_2, w \neq z$. Then there can be no 2-path between a_2 and b_2, as the cycles v_1, a_1, w, a_4, v_1 and v_2, b_1, z, b_3, v_2 are disjoint separating cycles, separating a_2 from b_2. If however, $z = w$, we have the following. A 2-path between a_2 and b_2 must use intermediate vertex z. And though b_1 and b_3 may have 2-paths to the S_1 vertices via intermediate vertex v_1, b_2 cannot, nor can b_2 use z as an intermediate vertex to both a_1 and a_4.

On the other hand, v_2 may be an intermediate vertex between a_1 and a_4. But then case (i) applies as one of a_1 or a_4 could be moved to set S_2.

In case (iii), $|S_1| \geq 3, |S_2| = 3, |S_3| = 3$. Denote the vertices in S_3 as c_1, c_2, c_3. Vertex v_1 cannot be an intermediate vertex between any two of the b_i vertices in S_2 because in that case, one of the b_i vertices would belong in S_1. Likewise v_3 cannot be an intermediate vertex between two vertices in S_2, else a c_i vertex would belong to S_2. Similarly, neither v_1 nor v_2 can serve as intermediate vertices for any pair of vertices in S_3.

Consider the 2-paths between b_1 and b_3, as well as between c_1 and c_3 via intermediate vertices, w, z, respectively. First suppose $w \neq z$. Then it is not possible to form a 2-path between b_2 and c_2. On the other hand, suppose $w = z$. Then b_2 cannot have a 2-path to both c_1 and c_3.

Now examine case (iv), with $|S_1| = 4, |S_2| \geq 2, |S_3| \geq 2, |S_4| \geq 2$, $|S_5| \geq 1$. Denote the vertices in S_3 as c_1, c_2 in clockwise order around v_3. To connect the four vertices in S_1 via 2-paths, at least three intermediate vertices are needed. If only two are used, then a $K_{3,3}$ subgraph results, because v_1 dominates all the vertices in S_1 and one can easily verify that an orientation of $K_{2,4}$ is the only way to connect four independent vertices by 2-paths using two intermediate vertices. For the purposes of the argument in case (iv) (and later in case (vi)), only two such intermediate vertices need to be specified.

Consider the 2-path between a_1 and a_4, using intermediate vertex w. Any two vertices in S_1 may have at most one common neighbor in the set $\{v_2, v_3, v_4, v_5\}$, else $G[D]$ has a $K_{3,3}$ subgraph. We may assume, without loss of generality, that $w \neq v_2, w \neq v_3$ (else we can replace one of S_2, S_3 with S_4). The 2-path between a_1 and a_4 creates a cycle C containing the vertices v_1, a_1, w, a_4. There also exists an intermediate vertex $z \neq w$, $z \neq v_i$, $1 \leq i \leq 5$, such that z is used to form another 2-path between two vertices in S_1 besides a_1 and a_4.

First suppose that z is not adjacent to w. Regardless of the routing of C within the given embedding of $G[D]$, C separates each of b_1, b_2, c_1, c_2 from z. This forces that v_1 is the intermediate vertex on 2-paths between b_1, b_2, c_1, c_2 and z. Then 2-paths cannot be formed between all the vertices in S_2 and S_3.

On the other hand, suppose that each intermediate vertex, other than w itself, between vertices in S_1 is adjacent to w. Denote these intermediate vertices as $z_1, z_2, \ldots, z_q, q \geq 2$. Then in D, w cannot dominate (or be dominated by) the set $\{a_2, a_3, z_1, z_2, \ldots, z_q\}$, else there can be either no 2-path between a_2 and a_4 or no 2-path between a_1 and a_3. Now we have that each vertex in S_2 and S_3 must be adjacent to v_1, which precludes the formation of 2-paths between the vertices in S_2 and S_3 (again, recalling that v_1 cannot be an intermediate vertex between vertices in S_2 and S_3).

Now consider case (v). Extending the argument of case (iv), we show that it is forbidden to have $|S_1| = 3, |S_2| \geq 2, |S_3| \geq 2, |S_4| \geq 2, |S_5| \geq 1$. We use the same terminology as in case (iv) when referring to vertices. Let w be the intermediate vertex on a 2-path between a_1 and a_3. As in case (iv), we may assume, without loss of generality, that $w \neq v_2, w \neq v_3$ (else we can replace one of S_2, S_3 with S_4).

The case when z is not adjacent to w follows identically (with a_3 in the place of a_4) as in case (iv). However, some differences arise when z is adjacent to w. Assume without loss of generality that a_3 dominates w and w dominates a_1 (the argument is symmetric in the other event). In this case, we suppose that w dominates both z and a_2 in D, otherwise the argument proceeds as in case (iv), as each vertex in S_2 and S_3 would have to be adjacent to v_1. Each vertex in S_2 and S_3 must be adjacent to at least one of v_1 or w, in order to have a 2-path to a_2 and z.

If the vertices in S_2 and S_3 use but two intermediate vertices to form 2-paths between them, then each vertex in S_2 and S_3 must be adjacent to both (as an orientation of $K_{2,4}$ is the only way to connect four independent vertices by 2-paths using two intermediate vertices). The resulting graph has a $K_{3,3}$ minor.

So we assume the vertices in $S_2 \cup S_3$ use at least three intermediate vertices to form the 2-paths between them. Of course, none of these intermediate vertices can be v_1. And at most one can be w, so let us use x, x' to denote two intermediate vertices not equal to w. Note that x and x' must each be adjacent to at least one of v_1 or w.

Suppose w is the intermediate vertex on the b_1b_2 2-path and the c_1c_2 2-path. But then the planarity of $G[D]$ makes it impossible to form at least one of the remaining two 2-paths between pairs of the form b_ic_j.

On the other hand, suppose x is the intermediate vertex on the b_1b_2 2-path. Then the cycle (in $G[D]$) $v_2, b_1, w, a_3, v_1, v, v_2$ (or v_2, b_1, v_1, v, v_2) along with the corresponding cycle containing either edge b_2w or b_2v_1 (rather than b_1w or b_1v_1) force c_1 and c_2 to be adjacent to w, in order to have a 2-path to/from x. Which then makes it impossible for both c_1 and c_2 to each have a 2-path to each of a_1 and a_3 (since w dominates only one of a_1, a_3).

The argument for case (vi) is again nearly identical to case (iv). The only difference being the following. Suppose that the intermediate vertex w between a_1 and a_4 is v_2. Then we must replace S_2 as we did in case (iv), but we replace it with both S_4 and S_5 (since we know that w cannot be v_4 or v_5, else $G[D]$ has a $K_{3,3}$ subgraph. In the language of case (iv), we then let c_1 be the lone vertex in S_4 and c_2 be the lone vertex in S_5 and the argument proceeds as in case (iv) (even though, for example, there may be a 2-path between c_1 and c_2 via v_5).

Case (vii) follows exactly as case (iv). ■

5. Series-Parallel Graphs

It is proved in [7] that orientations of graphs having treewidth at most two have oriented chromatic number at most seven. Since series-parallel graphs have treewidth at most two, $\omega_o(D) \leq 7$ for any digraph D whose underlying graph is series-parallel. The proof in [7] is based on graph homomorphisms. We present an alternate, direct proof of this result based on graph minors/subgraphs. Our method of proof yields that Proposition 7 also holds for series-parallel ocliques.

We first state a lemma given in [9], page 251.

Lemma 13 [9]. *Let H be a graph with $\Delta(H) \leq 3$. Then H is a minor of G if and only if G contains a subdivision of H.*

The following is easily verified by inspection.

Fact 14. *No subdivision of $K_{2,3}$, other than $K_{2,3}$ itself, can be oriented to be an oclique.*

One can easily confirm that there exist ocliques whose underlying graph is $K_{2,3}$.

Theorem 15. *Any oclique whose underlying graph is series-parallel has at most seven vertices. Further, there are seven such ocliques on seven vertices, and each contains O_7 as a spanning subgraph.*

Proof. Suppose by way of contradiction that D is an oclique with at least eight vertices; so $G[D]$ is a series-parallel graph. If $G[D]$ contains no $K_{2,3}$ minor, then $G[D]$ is outerplanar, and the first part of the theorem follows from Fact 5. So suppose $G[D]$ contains a $K_{2,3}$ minor. By Lemma 13, we may assume that $G[D]$ contains a $K_{2,3}$ subdivision. Because of Fact 14, we shall focus attention on $K_{2,3}$ itself and not worry about its subdivisions. Let $H = (A, B, E)$ be a $K_{2,3}$ subgraph of $G[D]$. Let $A = \{a_1, a_2\}$ and $B = \{b_1, b_2, b_3\}$. Let $V(G[D]) - V(H) = \{g_1, g_2, \ldots\}$

Some key facts are now stated, which form the heart of the argument.

(1) B induces an independent set in $G[D]$.

***Proof of* (1).** Otherwise, $G[D]$ contains a K_4 minor.

Likewise we have:

(1a) Every path in $G[D]$ joining two distinct vertices of B contains a vertex in A.

Proof. Otherwise there is a K_4 minor.

Note that (1a) implies that if g_i is adjacent to g_j, then $N(g_i) \cap B = N(g_j) \cap B$. It also implies that each g_i is adjacent to at most one vertex of B.
(2) Each g_i vertex must be adjacent to at least one of a_1, a_2.

***Proof of* (2).** This is in order that the diameter of G be at most two and follows from (1a), since no g_i can be adjacent to two vertices from B.

(3) Each g_i vertex must have degree at least two.

***Proof of* (3).** Otherwise, say g_i is adjacent to only a_1, since we know from (2) that g_i must be adjacent to one of a_1, a_2. The logic from Lemma 4 then tells us that a_1 must dominate (or be dominated by) B in D. As B is an independent set and vertices in B have no common neighbors other than vertices in A, it is then not possible to orient the edges in $G[D]$ to form 2-paths between the three vertices in B.

(4) $I = \{g_1, g_2, \ldots\}$ is an independent set.

***Proof of* (4).** As stated above, each vertex in I must be adjacent to at least one of a_1, a_2 in order for D to be an oclique. Let us assume that g_1 is adjacent to g_2. By the note following (1a), if both g_1 and g_2 have a neighbor in B (and each can have at most one), then they are adjacent to the same vertex of B. Thus, there are two possibilities to consider, as (2) assures us that one of the following must occur.

(a) g_1 and g_2 have a common neighbor in A. Then using the logic from (3), each of g_1, g_2 must be adjacent to another vertex from H (and it may be that each is adjacent to a different vertex). Then $G[D]$ contains a K_4 minor.

(b) g_1 is adjacent to a_1 and g_2 is adjacent to a_2. Then either g_1 and g_2 have a neighbor in B, in which case $G[D]$ has a K_4 minor, which violates (1a), or (a) applies.

(5) g_i cannot be adjacent to all of a_1, a_2 and b_1, b_2, b_3.

***Proof of* (5).** Otherwise, $G[D]$ has a K_4 minor.

Combining the above, we deduce that each g_i has degree two. If two g_i's are both adjacent to a_1, and a_2, then $G[D]$ contains a $K_{2,5}$ subgraph. But, since no vertex in $I = \{g_1, g_2, \ldots\}$ can be an intermediate vertex on a 2-path joining two vertices of I, in this case it follows from the structure of $G[D]$ that D cannot be oriented to be an oclique. Hence at most one g_i is adjacent to both a_1 and a_2.

If three g_i's are each adjacent to a_1 and b_1, say, then without loss of generality each must dominate a_1, in order that the g_i's can have 2-paths to $b_q, q \neq 1$. But then, by the structure of $G[D]$, we cannot form 2-paths amongst the three g_i's.

Suppose for the moment that a_1 is not adjacent to a_2 in $G[D]$, and consider the possible neighbors of g_1 and g_2. It follows from our work above that there are three possibilities (up to relabelling the elements of A, B and I):

(i) Both g_1 and g_2 are adjacent to a_1 and b_1;

(ii) g_1 is adjacent to a_1 and b_1, and g_2 is adjacent to a_1 and b_2;

(iii) g_1 is adjacent to a_1 and a_2, and g_2 is adjacent to a_1 and b_1. We show that, in each case, $G[D]$ cannot be oriented as an oclique.

In case (i), without loss of generality, we must have that g_1 and g_2 both dominate a_1, in order for there to be 2-paths to b_2 and b_3. In order for there to be 2-paths from the g_i's to a_2, both g_1 and g_2 must dominate b_1. But now there cannot be a 2-path joining g_1 and g_2. In case (ii), without loss of generality, we must have that g_1 and g_2 both dominate a_1, in order for there to be 2-paths b_3. Again, we have that there cannot be a 2-path joining g_1 and g_2. In case (iii), without loss of generality, g_2 must dominate a_1. In turn, a_1 must dominate b_2, b_3 and g_1 so that 2-paths between g_1 and these vertices are formed. Similarly, g_1 must dominate a_2, which must dominate b_1 and b_3. But now there cannot be a 2-path joining b_2 and b_3.

It follows from the above argument that a_1 and a_2 must be adjacent. We assume from now on that this is the case. As above, one of cases (i), (ii) and (iii) must arise. Cases (ii) and (iii) can be eliminated similarly to the arguments above.

Suppose g_1 and g_2 are both adjacent to a_1 and b_1. In order for g_1 and g_2 to have 2-paths to b_2 and b_3, without loss of generality, both of g_1 and g_2 must dominate a_1, which in turn must dominate b_2 and b_3. There must be a 2-path joining g_1 and g_2. Without loss of generality, again, assume it is $g_1 \to b_1 \to g_2$. Then, a_1 must dominate a_2 so that g_2 can have a 2-path to a_2. In order for there to be a 2-path joining b_2 and b_3, without loss of generality (still), b_2 dominates a_2 and a_2 dominates b_3. This implies that b_1 dominates a_1, so that it can be joined to b_2 by a 2-path. The subgraph induced by the edges whose orientation has been determined so far - all edges in the subgraph induced by $A \cup B \cup \{g_1, g_2\}$ except the edge joining b_1 and a_2 is O_7. Two seven vertex ocliques whose underlying graph is series-parallel and contains $K_{2,3}$ arise from the two choices for the orientation of this edge.

We claim that we must have $|I| = 2$. Suppose I contains a third vertex, g_3. From our work above, g_3 cannot be adjacent to a_1 and b_1. The other possibilities for neighbors of g_3 give rise to cases (ii) or (iii), using g_3 in place of g_2.

It now follows that there are exactly seven ocliques whose underlying graph is series-parallel: the five mentioned in Section 3, and the two arising in the proof above. All of these contain O_7 as a spanning subgraph. This completes the proof. ∎

6. Conclusions

We state our main conjecture.

Conjecture 1. *Let D be an orientation of a planar graph such that for any two vertices $x, y \in V(G)$ there exists an xy directed path or a yx directed path of length at most two. Then D has at most fifteen vertices.*

It is interesting to note that Sopena's planar graph with $\chi_o = 16$ has $\omega_o = 15$ [7].

It is obvious that k-cliques are not the only "obstruction" to k-coloring an undirected graph. Imperfect graphs illustrate this concept. Hadwiger's conjecture perhaps lies at the heart of this issue: it claims that each k-chromatic graph contains a subgraph that "becomes K_k via edge contractions" [9]. It would be interesting to formulate an analogous conjecture for oriented coloring, as there are infinitely many digraphs having $\chi_o > \omega_o$, such as the directed cycle on n vertices, where $n > 5$ and n is not divisible by 3. Such graphs can have $\omega_o = 3$ and $\chi_o = 4$.

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