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LINEAR FORESTS AND ORDERED CYCLES

Guantao Chen[1], Ralph J. Faudree[2], Ronald J. Gould[3], Michael S. Jacobson[4], Linda Lesniak[5] and Florian Pfender[3,6]

[1] *Georgia State University, Atlanta, GA 30303*
[2] *University of Memphis, Memphis, TN 38152*
[3] *Emory University, Atlanta, GA 30322*
[4] *University of Louisville, Louisville, KY 40292 and*
[5] *Drew University, Madison, NJ 07940*
[6] *Technische Universität Berlin, Berlin, Germany*

Abstract

A collection $L = P^1 \cup P^2 \cup \ldots \cup P^t$ $(1 \leq t \leq k)$ of t disjoint paths, s of them being singletons with $|V(L)| = k$ is called a (k,t,s)-*linear forest*. A graph G is (k,t,s)-*ordered* if for every (k,t,s)-linear forest L in G there exists a cycle C in G that contains the paths of L in the designated order as subpaths. If the cycle is also a hamiltonian cycle, then G is said to be (k,t,s)-*ordered hamiltonian.* We give sharp sum of degree conditions for nonadjacent vertices that imply a graph is (k,t,s)-ordered hamiltonian.

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1. Introduction

Over the years hamiltonian graphs have been widely studied. A variety of related properties have also been considered. Some of the properties are weaker, for example traceability in graphs, while others are stronger, for example hamiltonian connectedness. Recently a new strong hamiltonian property was introduced in [7] and further studied in [5], [2], and [3].

We say a graph G on n vertices, $n \geq 3$ is k-*ordered* for an integer k, $1 \leq k \leq n$, if for every sequence $S = (x_1, x_2, \ldots, x_k)$ of k distinct vertices in G, there exists a cycle that contains all the vertices of S in the designated order. A graph is *k-ordered hamiltonian* if for every sequence S of k vertices there exists a hamiltonian cycle which encounters S in its designated order. Hu, Tian and Wei [4] considered a different question; when is it possible to find a long cycle passing through a collection of paths?

In this paper we combine these two ideas. In order to treat this in generality, we say L is a *(k,t,s)-linear forest* if L is a collection $L = P^1 \cup P^2 \cup \ldots \cup P^t$ $(1 \leq t \leq k)$ of t disjoint paths, s of them being singletons such that $|V(L)| = k$. A graph G is *(k,t,s)-ordered* if for every (k,t,s)-linear forest L in G there exists a cycle C in G that contains the paths of L in the designated order as subpaths. Further, if the paths of L are each oriented and C can be chosen to encounter the paths of L in the designated order and according to the designated orientation on each path, then we say G is *strongly (k,t,s)-ordered.* If C is a hamiltonian cycle then we say G is *(k,t,s)-ordered hamiltonian* and *strongly (k,t,s)-ordered hamiltonian*, respectively. Note that saying G is (s,s,s)-ordered is the same as saying G is s-ordered.

We will think of all cycles being directed. For a cycle C and vertices $x, y \in V(C)$, we denote the $x - y$ path on C following the direction of C by xCy.

As usual, we will denote the minimum degree of a graph G by $\delta(G)$, and the minimum degree sum of two non adjacent vertices in a graph G by $\sigma_2(G)$.

We will say that a graph G on at least $2k$ vertices is *k-linked*, if for every vertex set $T = \{x_1, x_2, \ldots, x_k, y_1, y_2, \ldots, y_k\}$ of $2k$ vertices, there are k disjoint $x_i - y_i$ paths. The property remains the same if we allow repetition in T, and ask for k internally disjoint $x_i - y_i$ paths. Thus, as an easy consequence, every k-linked graph is k-ordered and $(2k-s,k,s)$-ordered.

An important theorem about k-linked graphs is the following theorem of Bollobás and Thomason [1]:

Theorem 1. *Every $22k$-connected graph is k-linked.*

The following lemmas will be used later.

Lemma 1. *If a $2k$-connected graph G has a k-linked subgraph H, then G is k-linked.*

Proof. Let $T = \{x_1, x_2, \ldots, x_k, y_1, y_2, \ldots, y_k\}$ be a set of $2k$ vertices in $V(G)$. Since G is $2k$-connected, there are $2k$ disjoint paths from T to $V(H)$.

Choose the paths from T to $V(H)$ such that each path contains exactly one element of $V(H)$ (if $x_i \in T \cap V(H)$ then the corresponding path consists only of this one vertex). Now we can connect these paths in the desired way inside H, since H is k-linked. ■

Lemma 2. *If G is a graph, $v \in V(G)$ with $d(v) \geq 2k-1$, and if $G-v$ is k-linked, then G is k-linked.*

Proof. Let $T = \{x_1, x_2, \ldots, x_k, y_1, y_2, \ldots, y_k\}$ be a set of $2k$ vertices in $V(G)$. If $v \notin T$, we can find disjoint $x_i - y_i$ paths inside $G-v$. Thus we may assume that $v = x_1$. If $y_1 \in N(v)$, we can find disjoint $x_i - y_i$ paths for all $i \geq 2$ in $G - v - y_1$, since $G - v - y_1$ is $(k-1)$-linked. Adding the path vy_1 completes the desired set of paths in G. If $y_1 \notin N(v)$, then there exists a vertex $x_1' \in N(v) - T$, since $d(v) \geq 2k-1$. We can find disjoint $x_i - y_i$ paths for $i \geq 2$ and a $x_1' - y_1$ path in $G - v$, which we can then extend to an $x_1 - y_1$ path in G. ■

Further, we will use a Theorem of Mader [6] about dense graphs:

Theorem 2. *Every graph G with $|V(G)| = n \geq 2k-1$, and $|E(G)| \geq (2k-3)(n-k+1)+1$ has a k-connected subgraph.*

Corollary 3. *Every graph G with $|V(G)| = n \geq 2k-1$, and $|E(G)| \geq 2kn$ has a k-connected subgraph.*

2. Degree Conditions

In this section we examine minimum degree conditions sufficient to insure a graph is either (k,t,s)-ordered hamiltonian or strongly (k,t,s)-ordered hamiltonian. Sharp results for $s = t = k$ were shown in [5], [2] and [3]:

Theorem 4 [5]**.** *Let $k \geq 2$ be a positive integer and let G be a graph of order n, where $n \geq 11k-3$. Then G is k-ordered hamiltonian if $\delta(G) \geq \left\lceil \frac{k}{2} \right\rceil + \lfloor \frac{n}{2} \rfloor - 1$.*

Theorem 5 [3]**.** *Let $k \geq 3$ be a positive integer and let G be a graph of order $n \geq 2k$. If $\sigma_2(G) \geq n + \frac{3k-9}{2}$, then G is k-ordered hamiltonian.*

As a first step, we prove the following theorem:

Theorem 6. *Let s, t, k be integers with $0 \le s < t < k$ or $s = t = k \ge 3$. If G is a (strongly) (k, t, s)-ordered graph on $n \ge k$ vertices with*

$$\sigma_2(G) \ge \begin{cases} n + k - t & \text{if } s = 0, \\ n + k - t + s - 1 & \text{if } s > 0, \end{cases}$$

then G is (strongly) (k, t, s)-ordered hamiltonian.

As a corollary, we obtain the following theorem.

Theorem 7. *For $k \ge 1$ and $1 \le t \le k$, if G is a (strongly) (k, t, s)-ordered graph on $n \ge k$ vertices with $\delta(G) \ge \frac{n+k-t+s}{2}$, then G is (strongly) (k, t, s)-ordered hamiltonian.*

In the same spirit, we will prove another theorem, which is not needed for our main result, Theorem 10.

Theorem 8. *Let s, t, k be integers with $1 < t/2 < s \le t \le k$. If G is a (strongly) (k, t, s)-ordered graph on $n \ge 11k$ vertices with*

$$\sigma_2(G) \ge n + k - \frac{t+3}{2},$$

then G is (strongly) (k, t, s)-ordered hamiltonian.

***Proof of Theorem* 6 *and Theorem* 8.** Since G is (strongly) (k, t, s)-ordered, we may choose a longest cycle C containing the paths of a given (k, t, s)-linear forest L in the designated order and with the designated orientations (if there are any) on each path. We need to show that C is hamiltonian.

Let $L = P^1 \cup P^2 \cup \ldots \cup P^t$, and $x_1, \ldots, x_t, y_1, \ldots, y_t \in V(C)$, such that $P^i = x_i C y_i$ for all $1 \le i \le t$. Note that $x_i = y_i$ if P_i is a singleton. Let $R^i = y_i C x_{i+1}$ for $1 \le i \le t-1$, and $R^t = y_t C x_1$. Let $R = \bigcup_i R^i$.

Suppose C is not hamiltonian and let H be a component of $G - C$.

Claim 1. *No R^i contains more than one vertex adjacent to H.*

Suppose there exists an interval R^i with at least two vertices adjacent to H. Without loss of generality we may assume that R^1 is such an interval. Pick two of these vertices v_1, v_2 such that there are no other adjacencies of H in

$v_1Cv_2 \subset R^1$. Note that $r = |v_1Cv_2| - 2 \geq 1$, otherwise C can be extended by at least one vertex.

Let $u_1 \in N(v_1) \cap H$, let $u_2 \in N(v_2) \cap H$. Note that we allow $u_1 = u_2$. Consider now $X = (N(u_1) \cup N(u_2)) \cap C$. There cannot be two vertices consecutive on R in X, otherwise C can be extended by at least one vertex. Further, X does not contain any vertices of $v_1^+Cv_2^-$ by our choice of v_1, v_2. Note that $R \setminus v_1^+Cv_2^-$ consists of $t-s+1$ paths, and $|C \setminus R| = k - 2t + s$, thus

$$d(u_1) + d(u_2) \leq 2|X| + d_H(u_1) + d_H(u_2)$$

$$\leq 2\left(|H| - 1 + \frac{|R| - r + t - s + 1}{2} + k - 2t + s\right).$$

Now concentrate on v_1^+ and v_2^-. There cannot be two consecutive vertices in $R \setminus v_1^+Cv_2^-$, such that one is adjacent to v_1^+ and the other adjacent to v_2^-, otherwise the whole segment $v_1^+Cv_2^-$ could be inserted between those two vertices, and a longer cycle through u_1 could be found. Thus,

$$d(v_1^+) + d(v_2^-) \leq 2\Big(r - 1 + \frac{|R| - r + 1 + t - s}{2} + k - 2t + s + n - |C| - |H|\Big).$$

But now,

$$2(n + k - t) \leq d(v_1^+) + d(u_1) + d(v_2^-) + d(u_2)$$

$$\leq 2(n + k - t - 1 + |R| + k - 2t + s - |C|) = 2(n + k - t - 1),$$

a contradiction. Therefore, there can be at most one vertex adjacent to H in each R^i.

To prove Theorem 6, observe that the degree condition forces G to be complete or $(k-t+s+1)$-connected. If G is complete we are done. So we may assume that G is $(k-t+s+1)$-connected. Since $|C - R| = k - 2t + s$, there are at least $t+1$ vertices adjacent to H in R. Thus, there exists an R^i with two such vertices, a contradiction proving Theorem 6.

To prove Theorem 8, we first prove the following claim.

Claim 2. *H is the only component of $G - C$.*

Otherwise, let H' be a different component, let $v_1 \in H, v_2 \in H'$. For $i = 1, 2$, let

$$\begin{aligned} a_i &= |\{v \in N(v_i) \cap (C \setminus L)\}|, \\ b_i &= |\{v \in N(v_i) : v = x_j \text{ or } v = y_j \text{ for some } j \text{ with } x_j \neq y_j\}|, \\ c_i &= |\{v \in N(v_i) : v = x_j = y_j \text{ for some } j\}|. \end{aligned}$$

We know that $a_i + b_i + 2c_i \leq t$, since by Claim 1, v_i can have at most one neighbor in each R_j. Further, $b_i \leq 2(t-s)$. Thus,

$$\begin{aligned} 2d(v_1) &\leq 2(|H| - 1 + k - 2t + s + a_1 + b_1 + c_1) \\ &= 2|H| + k + a_1 + k - t - 2 + (b_1 - 2(t-s)) + (a_1 + b_1 + 2c_1 - t) \\ &\leq 2|H| + k + a_1 + k - t - 2. \end{aligned}$$

Similarly,

$$2d(v_2) \leq 2|H'| + k + a_2 + k - t - 2.$$

Therefore,

$$n+k-\frac{t+3}{2} \leq d(v_1)+d(v_2) \leq |H|+|H'|+k+\frac{a_1+a_2}{2}+k-t-2 \leq n+k-t-2,$$

a contradiction, proving the claim.

The degree condition forces G to be complete or $(k - \frac{t-1}{2})$-connected. If G is complete we are done. So we may assume that G is $(k - \frac{t-1}{2})$-connected. Since $|C - R| = k - 2t + s$, there are at least $\frac{3t+1}{2} - s$ neighbors of H in R.

Claim 3. *For some i, $1 \leq i \leq t$, the following is true: $x_i = y_i$ and H has two neighbors in $y_{i-1}Cx_{i+1}^- \setminus x_i$.*

Let h_i count the number of neighbors of H in $y_{i-1}Cx_i \cup y_iCx_{i+1}^-$. We know that $h_i \in \{0, 1, 2\}$ for all $1 \leq i \leq t$. Further, $\sum_i h_i \geq 3t + 1 - 2s - (t - s)$, since the sum counts every neighbor of H in $\{x_i : x_i \neq y_i\}$ once and all other neighbors of H in R twice. Thus, at least $(t - s) + 1$ of the h_i are equal to 2. Therefore, $h_i = 2$ for some i with $x_i = y_i$. The vertex x_i cannot be one of the two neighbors of H by Claim 1, establishing the claim.

Let i be as in Claim 3, let $y \in y_{i-1}Cx_i^-$ and $z \in y_i^+Cx_{i+1}^-$ be the two neighbors of H. If $y^+z^+ \in E$, then $yHzC^-y^+z^+Cy$ is a longer cycle. Thus, $y^+z^+ \notin E$ and, since y^+ and z^+ are not in $N(H)$,

$$|C| \geq 2 + \frac{d(y^+) + d(z^+)}{2} > \frac{n+k}{2} - \frac{t}{4} + 1.$$

This implies that

$$|R| = |C| - k + 2t - s > \frac{n-k}{2} > 5k.$$

Now let $u \in H$, $v \in C - N(H)$. Then

$$\begin{aligned} d(v) \geq n + k - \frac{t+3}{2} - d(u) &\geq n + k - \frac{t+3}{2} - (k - 2t + s) - t - |H| \\ &\geq |C| - 1 - s + \frac{t-1}{2}. \end{aligned}$$

Therefore, v is adjacent to all but at most $\frac{s}{2}$ vertices on C.

For the final contradiction we differentiate two cases.

Case 1. Suppose $y^+ \neq x_i$ or $z^+ \neq x_{i+1}$.

Let $w \in \{y^+, z^+\} - \{x_i, x_{i+1}\}$. Let $N = N(x_i) \cap N(x_{i+1}) \cap N(w)$. Since none of the vertices x_i, x_{i+1}, w is adjacent to H, each is adjacent to all but at most $\frac{s}{2}$ vertices of the cycle. Thus, $|N| \geq |C| - \frac{3s}{2}$.

Claim 4. *For some j, $|N \cap y_jCx_{j+1}| \geq 4$.*

Otherwise,

$$5k < |R| \leq 3t + |R| - |N| \leq 3t + \frac{3s}{2},$$

a contradiction.

Let j be as in the last claim, and let $v_1, v_2, v_3, v_4 \in N \cap y_jCx_{j+1}$ be the first four of these vertices in that order.

If $v_4 \in y^+Cx_i$, define a new cycle as follows: $C' = zC^-v_4x_{i+1}CyHz$.

If $v_4 \in z^+Cx_{i+1}$, let $C' = zC^-x_iv_4CyHz$.

Otherwise observe that there is at most one neighbor x of H in v_1Cv_4.

For $j \neq i$, define the new cycle C' as follows:
If $x \in v_1 C v_2$, let $C' = zC^- x_i v_3 x_{i+1} C v_2 w v_4 C y H z$.
If $x \in v_3 C v_4$, let $C' = zC^- x_i v_2 x_{i+1} C v_1 w v_3 C y H z$.
Otherwise, let $C' = zC^- x_i v_2 C v_3 x_{i+1} C v_1 w v_4 C y H z$.
For $i = j$, a very similar construction works:
Let $C' = zC^- v_4 w v_1 C^- x_i v_2 C v_3 x_{i+1} C y H z$.
In any case, no vertex in $C - C'$ is adjacent to H, so all of them have high degree to C and thus high degree to $R \cap C'$. Therefore, we can insert them one by one into C' creating a longer cycle, a contradiction, completing Case 1.

Case 2. Suppose $y^+ = x_i, z^+ = x_{i+1}$.

Let $N' = N(x_i) \cap N(x_{i+1})$. Then $|N'| \geq |C| - s$.

Claim 5. *For some l, $|N' \cap y_l C x_{l+1}| \geq 5$.*

Otherwise,

$$5k < |R| \leq 4t + |R| - |N'| \leq 4t + s,$$

a contradiction.

Let l be as in the last claim, and let $z_1, z_2, z_3, z_4, z_5 \in N' \cap y_l C x_{l+1}$ be the first five of these vertices in that order. At most one of them is adjacent to H, say z_2. Now a very similar argument as in the last case gives the desired contradiction, just replace x_i by z_1, x_{i+1} by z_5, and w by z_4. One possible cycle would then be (for $l < j < i$): $C' = zC^- x_i z_2 C z_3 x_{i+1} C z_1 v_2 C v_3 z_5 C v_1 z_4 v_4 C y H z$. ■

Theorem 9. *If $s = t = k \geq 3$ or $0 \leq s < t < k$, and G is a graph of order $n \geq \max\{178t + k, 8t^2 + k\}$ with*

$$\sigma_2(G) \geq \begin{cases} n + k - 3 & \text{if } s = 0, \\ n + k + s - 4 & \text{if } 0 < 2s \leq t, \\ n + k + \frac{t-9}{2} & \text{if } 2s > t, \end{cases}$$

then G is strongly (k, t, s)-ordered.

***Proof of Theorem* 9.** To simplify the proof, we will first use an induction argument on k. The statement is obviously true for the base cases

$(s = 0, t = 1, k = 2)$ and $(s = t = k = 3)$, since G then is 2-connected. Suppose the statement is true for all $k \leq k_0$. We need to show the statement for $k = k_0 + 1$. So, let G be a graph of order $n \geq \max\{178t + k, 8t^2 + k\}$ satisfying the degree condition for some triple (k, t, s). We need to show that for any (k, t, s)-linear forest L in G, we can find a cycle passing through it in the designated order and direction. Let L be such a forest. Delete all inner vertices of the paths from $V(G)$, and replace the paths by edges to create a new graph G' and a new linear forest L'. If there are any paths of three or more vertices in G, this will reduce the order of G and the order of L. Finding a cycle in G' through L' yields a cycle in G through L. Since $k' = 2t - s, n' = n - (k - k') \geq \max\{178t + k', 8t^2 + k'\}$, and

$$\sigma_2(G') \geq \sigma_2(G) - 2(k - k') \geq \begin{cases} n' + k' - 3 & \text{if } s = 0, \\ n' + k' + s - 4 & \text{if } 0 < 2s \leq t, \\ n' + k' + \frac{t-9}{2} & \text{if } 2s > t, \end{cases}$$

there is such a cycle in G' if $k' < k$, by the induction hypothesis. Thus, we may assume that $k' = k$, and so $L = L'$, meaning that L consists only of paths with one or two vertices.

Claim 1. *G has a t-linked subgraph H.*

All vertices of G with $d(v) < \frac{n}{2}$ have to be adjacent. If there are at least $2t$ of them, this clique is H. Otherwise $|E(G)| \geq (n - 2t)\frac{n}{4} \geq 44tn$, which implies by Corollary 3 that G contains a $22t$-connected subgraph H. By Theorem 1, H is t-linked.

Claim 2. *G is t-linked (and thus $(2t - s, t, s)$-ordered) or $V(G) = V(A) \cup V(B)$, where $|A| \leq |B| + 2t - 1$, B is t-linked, and A is either t-linked or complete.*

If G is $2t$-connected, then G is t-linked by Lemma 1. So assume there is a cut set K with $|K| < 2t$. Let A' and B' be two components of $G - K$ with $|A'| \leq |B'|$. Let $v \in A', w \in B'$. Then

$$n + 2t - s - 3 \leq d(v) + d(w) \leq |A'| + |B'| + 2|K| - 2 \leq n + 2t - 3,$$

so u and v can miss a total of at most s possible adjacencies. Since $|B'| > \frac{n}{2} - t$, this ensures B' to be $22t$-connected and thus t-linked. If A' is complete, we are done. Otherwise, the degree sum condition insures $|A'| \geq \frac{n-2t-s+1}{2}$, so A' is $22t$-connected and thus t-linked. To find A and B, we now partition the vertices of K as follows one-by-one: Add any vertex $u \in K$ with degree $d_{B'}(u) \geq 2t-1$ to B', and add the remaining vertices to A'. The result will be as desired, as can be seen step by step: If u has high ($\geq 2t-1$) degree to B', adding it to B' will leave B' t-linked by Lemma 2. If u has low degree to B', it must be either adjacent to all of A' or have high degree to A' by the degree sum condition. In both cases, A' stays complete (if $|A'| < 2t$), or A' stays t-linked (note that a complete graph on $2t$ vertices is t-linked), again by Lemma 2. This proves the claim.

Case 1. Suppose $t < 2s$.

First, we may assume that $t \geq 3$. Otherwise, $t = s \leq 2$, and there is nothing to prove. We will use A' and B' as defined in the proof of Claim 2 above. There is a vertex $v \in B'$ with $d_A(v) = 0$: For every vertex $w \in A'$ we have $d_{B'}(w) = 0$, and for every $w \in A \cap K$ we have $d_{B'}(w) \leq 2t-2$. Since there are at most $2t-1$ vertices in $A \cap K$, at most $(2t-2)(2t-1) < |B'|$ vertices can have $d_A(v) > 0$.

Therefore, by the degree sum condition, we have $d_B(w) \geq 2t - s + \frac{t-5}{2}$ for every $w \in A$. Let $L = \{x_1y_1, x_2y_2, \ldots, x_ty_t\}$, where $x_i = y_i$ if the path is a singleton, and all paths are directed from x_i to y_i (remember: all paths are either edges or singletons by the induction hypothesis). We need to find paths from y_i to x_{i+1}. Let

$$\begin{aligned}
L_A &= L \cap A, \\
L_B &= L \cap B, \\
L'_A &= \{x_i \in L_A | y_{i-1} \in L_B\} \cup \{y_i \in L_A | x_{i+1} \in L_B\}, \\
L'_B &= \{x_i \in L_B | y_{i-1} \in L_A\} \cup \{y_i \in L_B | x_{i+1} \in L_A\}, \\
S_A &= \{x_i \in L_A | y_{i-1} \in L_B\} \cap \{y_i \in L_A | x_{i+1} \in L_B\}, \\
S_B &= \{x_i \in L_B | y_{i-1} \in L_A\} \cap \{y_i \in L_B | x_{i+1} \in L_A\}.
\end{aligned}$$

By these definitions we get

$$|L'_A| + |S_A| = |L'_B| + |S_B|.$$

For $x_i \in L'_A$, let $N'(x_i) = (N(x_i) \cap B) - (L - \{y_{i-1}\})$.
For $y_i \in L'_A$, let $N'(y_i) = (N(y_i) \cap B) - (L - \{x_{i+1}\})$.

For $X \subset L'_A$, let

$$N'(X) = \bigcup_{x_i \in X} N'(x_i) \cup \bigcup_{y_i \in X} N'(y_i).$$

For $t = s = 3$, there is nothing to prove. For $t = 3, s = 2$, we get for every nonempty $X \subset L'_A$,

$$|N'(X)| \geq 3 - |L_B| + |X| + |X \cap S_A| \geq |X| + |X \cap S_A|.$$

For $t \geq 4$ we get for every nonempty $X \subset L'_A$,

$$\begin{aligned} |N'(X)| &\geq 2t - s + \frac{t-5}{2} - |L_B| + |X| + |X \cap S_A| - |S_B| \\ &= |X| + |X \cap S_A| + |L_A| - |S_B| + \frac{t-5}{2} \\ &\geq |X| + |X \cap S_A| + |L'_A| - |S_B| + \frac{t-5}{2} \\ &= |X| + |X \cap S_A| + \frac{|L'_A| - |S_B| + |L'_B| - |S_A|}{2} + \frac{t-5}{2} \\ &\geq |X| + |X \cap S_A| + \frac{t-5}{2}. \end{aligned}$$

Thus, $|N'(X)| \geq |X| + |X \cap S_A|$, and thus by Hall's Theorem, we can find disjoint neighbors for all $x_i, y_i \in L'_A$ in $N'(x_i)$ or $N'(y_i)$, respectively. Using that B is t-linked and that A is t-linked or complete, we can now find the desired cycle.

Case 2. Suppose $s = 0$.

The degree condition forces G to be $(2t-1)$-connected. If G is $2t$-connected, then it is t-linked and we are done. If G has a cut set K of size $2t-1$, the degree condition forces $G - K$ to consist of two complete components A' and B', both of which are adjacent to all ertices in K. It is easy to see that such a graph is t-linked.

Case 3. Suppose $0 < s \leq t/2$.

The degree condition forces G to be $(2t-2)$-connected. If G is $2t$-connected, then it is t-linked and we are done. If G has a cut set K of size $2t-2$, the degree condition forces $G - K$ to consist of two complete components A'

and B', both of which are adjacent to all vertices in K. It is easy to see that such a graph is $(2t-s,t,s)$-ordered. If K has size $2t-1$, G has a very similar structure. Again, it is straightforward to verify the claim. ■

Theorem 10. *If $0 \leq s \leq t \leq k$, and G is a graph of order $n \geq \max\{178t+k, 8t^2+k\}$ with*

$$\sigma_2(G) \geq \begin{cases} n+k-3 & \text{if } s=0, t\geq 3, \\ n+k+s-4 & \text{if } 0<2s\leq t, t\geq 3, \\ n+k+\frac{t-9}{2} & \text{if } 2s>t\geq 3, \\ n+k-2 & \text{if } s\leq 1, t=2, \\ n+k-1 & \text{if } s=0, t=1, \\ n & \text{if } s=t\leq 2, \end{cases}$$

then G is strongly (k,t,s)-ordered hamiltonian.

Proof. Apply Theorem 6 and Theorem 9. ■

3. Sharpness

Theorem 6 is sharp for $s=0$, illustrated by the following graph: Let $A = K_{\frac{n+k-t-1}{2}}$, and B be a set of $\frac{n-k+t+1}{2}$ isolated vertices. Add all edges between A and B. For n sufficiently large, G is strongly (k,t,s)-ordered, and $\sigma_2(G) = n+k-t-1$. But G is not strongly (k,t,s)-ordered hamiltonian, since no hamiltonian cycle can contain a (k,t,s)-linear forest L which completely lies inside A: Every hamiltonian cycle has exactly $k-t-1$ edges in A, one edge less than L.

The following graph shows sharpness of Theorem 9, $s=0$. Let G consist of three complete graphs: $A = K_{\frac{n-k+2}{2}}, K = K_{k-2}, B = K_{\frac{n-k+2}{2}}$. Add all edges between A and K and all edges between K and B. The degree sum condition is just missed, but G is not $(k,t,0)$-ordered: Let $x_1 \in A$, $y_t \in B$, $\langle L - \{x_1, y_t\}\rangle = K$.

The following graph shows sharpness of Theorem 9, $t \geq 2s \geq 2$. Let G consist of four complete graphs: $S = K_s, T = K_{k-s}, A = K_{2s-1}, B = K_{n-k-2s+1}$. Add all edges from A, all edges between T and B. For every vertex $s_i \in S$, pick two vertices $u_i, v_i \in T$. Add all edges between S and T but the edges s_iu_i, s_iv_i. We have $\sigma_2(G) = n+k+s-5$, but if we pick

$V(L) = V(S) \cup V(T)$, such that $x_{2i} = y_{2i} = s_i, x_{2i+1} = u_i, y_{2i-1} = v_i$ for all $i \leq s$, there is no cycle passing through L in the designated order and direction.

The following graph shows sharpness of Theorem 9, $2s > t$. Let G consist of four complete graphs: $S = K_{\lceil \frac{t}{2} \rceil}, T = K_{k-\lceil \frac{t}{2} \rceil}, A = K_{t-1}, B = K_{n-k-2s+1}$. Add all edges from A, all edges between T and B. For every vertex $s_i \in S$, pick two vertices $u_i, v_i \in T$, with the exception that $v_{i+1} = u_i$ for $1 \leq i \leq s - \lceil \frac{t}{2} \rceil$. Add all edges between S and T but the edges $s_i u_i, s_i v_i$. We have $\sigma_2(G) = n + k + \lfloor \frac{t}{2} \rfloor - 5$, but if we pick $V(L) = V(S) \cup V(T)$, such that $x_{2i} = y_{2i} = s_i, x_{2i+1} = u_i, y_{2i-1} = v_i$ for all $i \leq \lceil \frac{t}{2} \rceil$, there is no cycle passing through L in the designated order and direction.

4. Note Added in Proofs

Very recently, Thomas and Wollan [8] have improved the bound in Theorem 1 to the following.

Theorem 11. *If a graph G is $2k$-connected and has at least $5k|V(G)|$ edges, then G is k-linked.*

Corollary 12. *Every $10k$-connected graph is k-linked.*

Using these results in place of Theorem 1 will improve some of the bounds on n.

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