

## LINEAR FORESTS AND ORDERED CYCLES

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### Abstract

A collection  $L = P^1 \cup P^2 \cup \dots \cup P^t$  ( $1 \leq t \leq k$ ) of  $t$  disjoint paths,  $s$  of them being singletons with  $|V(L)| = k$  is called a  $(k, t, s)$ -linear forest. A graph  $G$  is  $(k, t, s)$ -ordered if for every  $(k, t, s)$ -linear forest  $L$  in  $G$  there exists a cycle  $C$  in  $G$  that contains the paths of  $L$  in the designated order as subpaths. If the cycle is also a hamiltonian cycle, then  $G$  is said to be  $(k, t, s)$ -ordered hamiltonian. We give sharp sum of degree conditions for nonadjacent vertices that imply a graph is  $(k, t, s)$ -ordered hamiltonian.

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## 1. Introduction

Over the years hamiltonian graphs have been widely studied. A variety of related properties have also been considered. Some of the properties are weaker, for example traceability in graphs, while others are stronger, for example hamiltonian connectedness. Recently a new strong hamiltonian property was introduced in [7] and further studied in [5], [2], and [3].

We say a graph  $G$  on  $n$  vertices,  $n \geq 3$  is  $k$ -ordered for an integer  $k$ ,  $1 \leq k \leq n$ , if for every sequence  $S = (x_1, x_2, \dots, x_k)$  of  $k$  distinct vertices in  $G$ , there exists a cycle that contains all the vertices of  $S$  in the designated

order. A graph is *k-ordered hamiltonian* if for every sequence  $S$  of  $k$  vertices there exists a hamiltonian cycle which encounters  $S$  in its designated order. Hu, Tian and Wei [4] considered a different question; when is it possible to find a long cycle passing through a collection of paths?

In this paper we combine these two ideas. In order to treat this in generality, we say  $L$  is a  $(k, t, s)$ -linear forest if  $L$  is a collection  $L = P^1 \cup P^2 \cup \dots \cup P^t$  ( $1 \leq t \leq k$ ) of  $t$  disjoint paths,  $s$  of them being singletons such that  $|V(L)| = k$ . A graph  $G$  is  $(k, t, s)$ -ordered if for every  $(k, t, s)$ -linear forest  $L$  in  $G$  there exists a cycle  $C$  in  $G$  that contains the paths of  $L$  in the designated order as subpaths. Further, if the paths of  $L$  are each oriented and  $C$  can be chosen to encounter the paths of  $L$  in the designated order and according to the designated orientation on each path, then we say  $G$  is *strongly*  $(k, t, s)$ -ordered. If  $C$  is a hamiltonian cycle then we say  $G$  is  $(k, t, s)$ -ordered hamiltonian and *strongly*  $(k, t, s)$ -ordered hamiltonian, respectively. Note that saying  $G$  is  $(s, s, s)$ -ordered is the same as saying  $G$  is  $s$ -ordered.

We will think of all cycles being directed. For a cycle  $C$  and vertices  $x, y \in V(C)$ , we denote the  $x - y$  path on  $C$  following the direction of  $C$  by  $xCy$ .

As usual, we will denote the minimum degree of a graph  $G$  by  $\delta(G)$ , and the minimum degree sum of two non adjacent vertices in a graph  $G$  by  $\sigma_2(G)$ .

We will say that a graph  $G$  on at least  $2k$  vertices is *k-linked*, if for every vertex set  $T = \{x_1, x_2, \dots, x_k, y_1, y_2, \dots, y_k\}$  of  $2k$  vertices, there are  $k$  disjoint  $x_i - y_i$  paths. The property remains the same if we allow repetition in  $T$ , and ask for  $k$  internally disjoint  $x_i - y_i$  paths. Thus, as an easy consequence, every  $k$ -linked graph is  $k$ -ordered and  $(2k - s, k, s)$ -ordered.

An important theorem about  $k$ -linked graphs is the following theorem of Bollobás and Thomason [1]:

**Theorem 1.** *Every  $22k$ -connected graph is  $k$ -linked.*

The following lemmas will be used later.

**Lemma 1.** *If a  $2k$ -connected graph  $G$  has a  $k$ -linked subgraph  $H$ , then  $G$  is  $k$ -linked.*

**Proof.** Let  $T = \{x_1, x_2, \dots, x_k, y_1, y_2, \dots, y_k\}$  be a set of  $2k$  vertices in  $V(G)$ . Since  $G$  is  $2k$ -connected, there are  $2k$  disjoint paths from  $T$  to  $V(H)$ .

Choose the paths from  $T$  to  $V(H)$  such that each path contains exactly one element of  $V(H)$  (if  $x_i \in T \cap V(H)$  then the corresponding path consists only of this one vertex). Now we can connect these paths in the desired way inside  $H$ , since  $H$  is  $k$ -linked. ■

**Lemma 2.** *If  $G$  is a graph,  $v \in V(G)$  with  $d(v) \geq 2k - 1$ , and if  $G - v$  is  $k$ -linked, then  $G$  is  $k$ -linked.*

**Proof.** Let  $T = \{x_1, x_2, \dots, x_k, y_1, y_2, \dots, y_k\}$  be a set of  $2k$  vertices in  $V(G)$ . If  $v \notin T$ , we can find disjoint  $x_i - y_i$  paths inside  $G - v$ . Thus we may assume that  $v = x_1$ . If  $y_1 \in N(v)$ , we can find disjoint  $x_i - y_i$  paths for all  $i \geq 2$  in  $G - v - y_1$ , since  $G - v - y_1$  is  $(k - 1)$ -linked. Adding the path  $vy_1$  completes the desired set of paths in  $G$ . If  $y_1 \notin N(v)$ , then there exists a vertex  $x'_1 \in N(v) - T$ , since  $d(v) \geq 2k - 1$ . We can find disjoint  $x_i - y_i$  paths for  $i \geq 2$  and a  $x'_1 - y_1$  path in  $G - v$ , which we can then extend to an  $x_1 - y_1$  path in  $G$ . ■

Further, we will use a Theorem of Mader [6] about dense graphs:

**Theorem 2.** *Every graph  $G$  with  $|V(G)| = n \geq 2k - 1$ , and  $|E(G)| \geq (2k - 3)(n - k + 1) + 1$  has a  $k$ -connected subgraph.*

**Corollary 3.** *Every graph  $G$  with  $|V(G)| = n \geq 2k - 1$ , and  $|E(G)| \geq 2kn$  has a  $k$ -connected subgraph.*

## 2. Degree Conditions

In this section we examine minimum degree conditions sufficient to insure a graph is either  $(k, t, s)$ -ordered hamiltonian or strongly  $(k, t, s)$ -ordered hamiltonian. Sharp results for  $s = t = k$  were shown in [5], [2] and [3]:

**Theorem 4** [5]. *Let  $k \geq 2$  be a positive integer and let  $G$  be a graph of order  $n$ , where  $n \geq 11k - 3$ . Then  $G$  is  $k$ -ordered hamiltonian if  $\delta(G) \geq \left\lceil \frac{k}{2} \right\rceil + \left\lfloor \frac{n}{2} \right\rfloor - 1$ .*

**Theorem 5** [3]. *Let  $k \geq 3$  be a positive integer and let  $G$  be a graph of order  $n \geq 2k$ . If  $\sigma_2(G) \geq n + \frac{3k-9}{2}$ , then  $G$  is  $k$ -ordered hamiltonian.*

As a first step, we prove the following theorem:

**Theorem 6.** *Let  $s, t, k$  be integers with  $0 \leq s < t < k$  or  $s = t = k \geq 3$ . If  $G$  is a (strongly)  $(k, t, s)$ -ordered graph on  $n \geq k$  vertices with*

$$\sigma_2(G) \geq \begin{cases} n + k - t & \text{if } s = 0, \\ n + k - t + s - 1 & \text{if } s > 0, \end{cases}$$

*then  $G$  is (strongly)  $(k, t, s)$ -ordered hamiltonian.*

As a corollary, we obtain the following theorem.

**Theorem 7.** *For  $k \geq 1$  and  $1 \leq t \leq k$ , if  $G$  is a (strongly)  $(k, t, s)$ -ordered graph on  $n \geq k$  vertices with  $\delta(G) \geq \frac{n+k-t+s}{2}$ , then  $G$  is (strongly)  $(k, t, s)$ -ordered hamiltonian.*

In the same spirit, we will prove another theorem, which is not needed for our main result, Theorem 10.

**Theorem 8.** *Let  $s, t, k$  be integers with  $1 < t/2 < s \leq t \leq k$ . If  $G$  is a (strongly)  $(k, t, s)$ -ordered graph on  $n \geq 11k$  vertices with*

$$\sigma_2(G) \geq n + k - \frac{t+3}{2},$$

*then  $G$  is (strongly)  $(k, t, s)$ -ordered hamiltonian.*

**Proof of Theorem 6 and Theorem 8.** Since  $G$  is (strongly)  $(k, t, s)$ -ordered, we may choose a longest cycle  $C$  containing the paths of a given  $(k, t, s)$ -linear forest  $L$  in the designated order and with the designated orientations (if there are any) on each path. We need to show that  $C$  is hamiltonian.

Let  $L = P^1 \cup P^2 \cup \dots \cup P^t$ , and  $x_1, \dots, x_t, y_1, \dots, y_t \in V(C)$ , such that  $P^i = x_i C y_i$  for all  $1 \leq i \leq t$ . Note that  $x_i = y_i$  if  $P_i$  is a singleton. Let  $R^i = y_i C x_{i+1}$  for  $1 \leq i \leq t-1$ , and  $R^t = y_t C x_1$ . Let  $R = \bigcup_i R^i$ .

Suppose  $C$  is not hamiltonian and let  $H$  be a component of  $G - C$ .

**Claim 1.** *No  $R^i$  contains more than one vertex adjacent to  $H$ .*

Suppose there exists an interval  $R^i$  with at least two vertices adjacent to  $H$ . Without loss of generality we may assume that  $R^1$  is such an interval. Pick two of these vertices  $v_1, v_2$  such that there are no other adjacencies of  $H$  in

$v_1 C v_2 \subset R^1$ . Note that  $r = |v_1 C v_2| - 2 \geq 1$ , otherwise  $C$  can be extended by at least one vertex.

Let  $u_1 \in N(v_1) \cap H$ , let  $u_2 \in N(v_2) \cap H$ . Note that we allow  $u_1 = u_2$ . Consider now  $X = (N(u_1) \cup N(u_2)) \cap C$ . There cannot be two vertices consecutive on  $R$  in  $X$ , otherwise  $C$  can be extended by at least one vertex. Further,  $X$  does not contain any vertices of  $v_1^+ C v_2^-$  by our choice of  $v_1, v_2$ . Note that  $R \setminus v_1^+ C v_2^-$  consists of  $t - s + 1$  paths, and  $|C \setminus R| = k - 2t + s$ , thus

$$\begin{aligned} d(u_1) + d(u_2) &\leq 2|X| + d_H(u_1) + d_H(u_2) \\ &\leq 2 \left( |H| - 1 + \frac{|R| - r + t - s + 1}{2} + k - 2t + s \right). \end{aligned}$$

Now concentrate on  $v_1^+$  and  $v_2^-$ . There cannot be two consecutive vertices in  $R \setminus v_1^+ C v_2^-$ , such that one is adjacent to  $v_1^+$  and the other adjacent to  $v_2^-$ , otherwise the whole segment  $v_1^+ C v_2^-$  could be inserted between those two vertices, and a longer cycle through  $u_1$  could be found. Thus,

$$d(v_1^+) + d(v_2^-) \leq 2 \left( r - 1 + \frac{|R| - r + 1 + t - s}{2} + k - 2t + s + n - |C| - |H| \right).$$

But now,

$$\begin{aligned} 2(n + k - t) &\leq d(v_1^+) + d(u_1) + d(v_2^-) + d(u_2) \\ &\leq 2(n + k - t - 1 + |R| + k - 2t + s - |C|) = 2(n + k - t - 1), \end{aligned}$$

a contradiction. Therefore, there can be at most one vertex adjacent to  $H$  in each  $R^i$ .

To prove Theorem 6, observe that the degree condition forces  $G$  to be complete or  $(k - t + s + 1)$ -connected. If  $G$  is complete we are done. So we may assume that  $G$  is  $(k - t + s + 1)$ -connected. Since  $|C - R| = k - 2t + s$ , there are at least  $t + 1$  vertices adjacent to  $H$  in  $R$ . Thus, there exists an  $R^i$  with two such vertices, a contradiction proving Theorem 6.

To prove Theorem 8, we first prove the following claim.

**Claim 2.**  $H$  is the only component of  $G - C$ .

Otherwise, let  $H'$  be a different component, let  $v_1 \in H, v_2 \in H'$ . For  $i = 1, 2$ , let

$$\begin{aligned} a_i &= |\{v \in N(v_i) \cap (C \setminus L)\}|, \\ b_i &= |\{v \in N(v_i) : v = x_j \text{ or } v = y_j \text{ for some } j \text{ with } x_j \neq y_j\}|, \\ c_i &= |\{v \in N(v_i) : v = x_j = y_j \text{ for some } j\}|. \end{aligned}$$

We know that  $a_i + b_i + 2c_i \leq t$ , since by Claim 1,  $v_i$  can have at most one neighbor in each  $R_j$ . Further,  $b_i \leq 2(t - s)$ . Thus,

$$\begin{aligned} 2d(v_1) &\leq 2(|H| - 1 + k - 2t + s + a_1 + b_1 + c_1) \\ &= 2|H| + k + a_1 + k - t - 2 + (b_1 - 2(t - s)) + (a_1 + b_1 + 2c_1 - t) \\ &\leq 2|H| + k + a_1 + k - t - 2. \end{aligned}$$

Similarly,

$$2d(v_2) \leq 2|H'| + k + a_2 + k - t - 2.$$

Therefore,

$$n + k - \frac{t + 3}{2} \leq d(v_1) + d(v_2) \leq |H| + |H'| + k + \frac{a_1 + a_2}{2} + k - t - 2 \leq n + k - t - 2,$$

a contradiction, proving the claim.

The degree condition forces  $G$  to be complete or  $(k - \frac{t-1}{2})$ -connected. If  $G$  is complete we are done. So we may assume that  $G$  is  $(k - \frac{t-1}{2})$ -connected. Since  $|C - R| = k - 2t + s$ , there are at least  $\frac{3t+1}{2} - s$  neighbors of  $H$  in  $R$ .

**Claim 3.** *For some  $i$ ,  $1 \leq i \leq t$ , the following is true:  $x_i = y_i$  and  $H$  has two neighbors in  $y_{i-1}Cx_{i+1}^- \setminus x_i$ .*

Let  $h_i$  count the number of neighbors of  $H$  in  $y_{i-1}Cx_i \cup y_iCx_{i+1}^-$ . We know that  $h_i \in \{0, 1, 2\}$  for all  $1 \leq i \leq t$ . Further,  $\sum_i h_i \geq 3t + 1 - 2s - (t - s)$ , since the sum counts every neighbor of  $H$  in  $\{x_i : x_i \neq y_i\}$  once and all other neighbors of  $H$  in  $R$  twice. Thus, at least  $(t - s) + 1$  of the  $h_i$  are equal to 2. Therefore,  $h_i = 2$  for some  $i$  with  $x_i = y_i$ . The vertex  $x_i$  cannot be one of the two neighbors of  $H$  by Claim 1, establishing the claim.

Let  $i$  be as in Claim 3, let  $y \in y_{i-1}Cx_i^-$  and  $z \in y_i^+Cx_{i+1}^-$  be the two neighbors of  $H$ . If  $y^+z^+ \in E$ , then  $yHzC^-y^+z^+Cy$  is a longer cycle. Thus,  $y^+z^+ \notin E$  and, since  $y^+$  and  $z^+$  are not in  $N(H)$ ,

$$|C| \geq 2 + \frac{d(y^+) + d(z^+)}{2} > \frac{n+k}{2} - \frac{t}{4} + 1.$$

This implies that

$$|R| = |C| - k + 2t - s > \frac{n-k}{2} > 5k.$$

Now let  $u \in H$ ,  $v \in C - N(H)$ . Then

$$\begin{aligned} d(v) &\geq n+k - \frac{t+3}{2} - d(u) \geq n+k - \frac{t+3}{2} - (k-2t+s) - t - |H| \\ &\geq |C| - 1 - s + \frac{t-1}{2}. \end{aligned}$$

Therefore,  $v$  is adjacent to all but at most  $\frac{s}{2}$  vertices on  $C$ .

For the final contradiction we differentiate two cases.

*Case 1.* Suppose  $y^+ \neq x_i$  or  $z^+ \neq x_{i+1}$ .

Let  $w \in \{y^+, z^+\} - \{x_i, x_{i+1}\}$ . Let  $N = N(x_i) \cap N(x_{i+1}) \cap N(w)$ . Since none of the vertices  $x_i, x_{i+1}, w$  is adjacent to  $H$ , each is adjacent to all but at most  $\frac{s}{2}$  vertices of the cycle. Thus,  $|N| \geq |C| - \frac{3s}{2}$ .

**Claim 4.** For some  $j$ ,  $|N \cap y_jCx_{j+1}| \geq 4$ .

Otherwise,

$$5k < |R| \leq 3t + |R| - |N| \leq 3t + \frac{3s}{2},$$

a contradiction.

Let  $j$  be as in the last claim, and let  $v_1, v_2, v_3, v_4 \in N \cap y_jCx_{j+1}$  be the first four of these vertices in that order.

If  $v_4 \in y^+Cx_i$ , define a new cycle as follows:  $C' = zC^-v_4x_{i+1}CyHz$ .

If  $v_4 \in z^+Cx_{i+1}$ , let  $C' = zC^-x_iv_4CyHz$ .

Otherwise observe that there is at most one neighbor  $x$  of  $H$  in  $v_1Cv_4$ .

For  $j \neq i$ , define the new cycle  $C'$  as follows:

If  $x \in v_1 C v_2$ , let  $C' = z C^- x_i v_3 x_{i+1} C v_2 w v_4 C y H z$ .

If  $x \in v_3 C v_4$ , let  $C' = z C^- x_i v_2 x_{i+1} C v_1 w v_3 C y H z$ .

Otherwise, let  $C' = z C^- x_i v_2 C v_3 x_{i+1} C v_1 w v_4 C y H z$ .

For  $i = j$ , a very similar construction works:

Let  $C' = z C^- v_4 w v_1 C^- x_i v_2 C v_3 x_{i+1} C y H z$ .

In any case, no vertex in  $C - C'$  is adjacent to  $H$ , so all of them have high degree to  $C$  and thus high degree to  $R \cap C'$ . Therefore, we can insert them one by one into  $C'$  creating a longer cycle, a contradiction, completing Case 1.

*Case 2.* Suppose  $y^+ = x_i, z^+ = x_{i+1}$ .

Let  $N' = N(x_i) \cap N(x_{i+1})$ . Then  $|N'| \geq |C| - s$ .

**Claim 5.** For some  $l$ ,  $|N' \cap y_l C x_{l+1}| \geq 5$ .

Otherwise,

$$5k < |R| \leq 4t + |R| - |N'| \leq 4t + s,$$

a contradiction.

Let  $l$  be as in the last claim, and let  $z_1, z_2, z_3, z_4, z_5 \in N' \cap y_l C x_{l+1}$  be the first five of these vertices in that order. At most one of them is adjacent to  $H$ , say  $z_2$ . Now a very similar argument as in the last case gives the desired contradiction, just replace  $x_i$  by  $z_1$ ,  $x_{i+1}$  by  $z_5$ , and  $w$  by  $z_4$ . One possible cycle would then be (for  $l < j < i$ ):  $C' = z C^- x_i z_2 C z_3 x_{i+1} C z_1 v_2 C v_3 z_5 C v_1 z_4 v_4 C y H z$ . ■

**Theorem 9.** If  $s = t = k \geq 3$  or  $0 \leq s < t < k$ , and  $G$  is a graph of order  $n \geq \max \{178t + k, 8t^2 + k\}$  with

$$\sigma_2(G) \geq \begin{cases} n + k - 3 & \text{if } s = 0, \\ n + k + s - 4 & \text{if } 0 < 2s \leq t, \\ n + k + \frac{t-9}{2} & \text{if } 2s > t, \end{cases}$$

then  $G$  is strongly  $(k, t, s)$ -ordered.

**Proof of Theorem 9.** To simplify the proof, we will first use an induction argument on  $k$ . The statement is obviously true for the base cases



$(s = 0, t = 1, k = 2)$  and  $(s = t = k = 3)$ , since  $G$  then is 2-connected. Suppose the statement is true for all  $k \leq k_0$ . We need to show the statement for  $k = k_0 + 1$ . So, let  $G$  be a graph of order  $n \geq \max\{178t + k, 8t^2 + k\}$  satisfying the degree condition for some triple  $(k, t, s)$ . We need to show that for any  $(k, t, s)$ -linear forest  $L$  in  $G$ , we can find a cycle passing through it in the designated order and direction. Let  $L$  be such a forest. Delete all inner vertices of the paths from  $V(G)$ , and replace the paths by edges to create a new graph  $G'$  and a new linear forest  $L'$ . If there are any paths of three or more vertices in  $G$ , this will reduce the order of  $G$  and the order of  $L$ . Finding a cycle in  $G'$  through  $L'$  yields a cycle in  $G$  through  $L$ . Since  $k' = 2t - s, n' = n - (k - k') \geq \max\{178t + k', 8t^2 + k'\}$ , and

$$\sigma_2(G') \geq \sigma_2(G) - 2(k - k') \geq \begin{cases} n' + k' - 3 & \text{if } s = 0, \\ n' + k' + s - 4 & \text{if } 0 < 2s \leq t, \\ n' + k' + \frac{t-9}{2} & \text{if } 2s > t, \end{cases}$$

there is such a cycle in  $G'$  if  $k' < k$ , by the induction hypothesis. Thus, we may assume that  $k' = k$ , and so  $L = L'$ , meaning that  $L$  consists only of paths with one or two vertices.

**Claim 1.**  $G$  has a  $t$ -linked subgraph  $H$ .

All vertices of  $G$  with  $d(v) < \frac{n}{2}$  have to be adjacent. If there are at least  $2t$  of them, this clique is  $H$ . Otherwise  $|E(G)| \geq (n - 2t)\frac{n}{4} \geq 44tn$ , which implies by Corollary 3 that  $G$  contains a  $22t$ -connected subgraph  $H$ . By Theorem 1,  $H$  is  $t$ -linked.

**Claim 2.**  $G$  is  $t$ -linked (and thus  $(2t - s, t, s)$ -ordered) or  $V(G) = V(A) \cup V(B)$ , where  $|A| \leq |B| + 2t - 1$ ,  $B$  is  $t$ -linked, and  $A$  is either  $t$ -linked or complete.

If  $G$  is  $2t$ -connected, then  $G$  is  $t$ -linked by Lemma 1. So assume there is a cut set  $K$  with  $|K| < 2t$ . Let  $A'$  and  $B'$  be two components of  $G - K$  with  $|A'| \leq |B'|$ . Let  $v \in A', w \in B'$ . Then

$$n + 2t - s - 3 \leq d(v) + d(w) \leq |A'| + |B'| + 2|K| - 2 \leq n + 2t - 3,$$

so  $u$  and  $v$  can miss a total of at most  $s$  possible adjacencies. Since  $|B'| > \frac{n}{2} - t$ , this ensures  $B'$  to be  $22t$ -connected and thus  $t$ -linked. If  $A'$  is complete,

we are done. Otherwise, the degree sum condition insures  $|A'| \geq \frac{n-2t-s+1}{2}$ , so  $A'$  is  $2t$ -connected and thus  $t$ -linked. To find  $A$  and  $B$ , we now partition the vertices of  $K$  as follows one-by-one: Add any vertex  $u \in K$  with degree  $d_{B'}(u) \geq 2t-1$  to  $B'$ , and add the remaining vertices to  $A'$ . The result will be as desired, as can be seen step by step: If  $u$  has high ( $\geq 2t-1$ ) degree to  $B'$ , adding it to  $B'$  will leave  $B'$   $t$ -linked by Lemma 2. If  $u$  has low degree to  $B'$ , it must be either adjacent to all of  $A'$  or have high degree to  $A'$  by the degree sum condition. In both cases,  $A'$  stays complete (if  $|A'| < 2t$ ), or  $A'$  stays  $t$ -linked (note that a complete graph on  $2t$  vertices is  $t$ -linked), again by Lemma 2. This proves the claim.

*Case 1.* Suppose  $t < 2s$ .

First, we may assume that  $t \geq 3$ . Otherwise,  $t = s \leq 2$ , and there is nothing to prove. We will use  $A'$  and  $B'$  as defined in the proof of Claim 2 above. There is a vertex  $v \in B'$  with  $d_A(v) = 0$ : For every vertex  $w \in A'$  we have  $d_{B'}(w) = 0$ , and for every  $w \in A \cap K$  we have  $d_{B'}(w) \leq 2t-2$ . Since there are at most  $2t-1$  vertices in  $A \cap K$ , at most  $(2t-2)(2t-1) < |B'|$  vertices can have  $d_A(v) > 0$ .

Therefore, by the degree sum condition, we have  $d_B(w) \geq 2t-s + \frac{t-5}{2}$  for every  $w \in A$ . Let  $L = \{x_1y_1, x_2y_2, \dots, x_t y_t\}$ , where  $x_i = y_i$  if the path is a singleton, and all paths are directed from  $x_i$  to  $y_i$  (remember: all paths are either edges or singletons by the induction hypothesis). We need to find paths from  $y_i$  to  $x_{i+1}$ . Let

$$\begin{aligned} L_A &= L \cap A, \\ L_B &= L \cap B, \\ L'_A &= \{x_i \in L_A | y_{i-1} \in L_B\} \cup \{y_i \in L_A | x_{i+1} \in L_B\}, \\ L'_B &= \{x_i \in L_B | y_{i-1} \in L_A\} \cup \{y_i \in L_B | x_{i+1} \in L_A\}, \\ S_A &= \{x_i \in L_A | y_{i-1} \in L_B\} \cap \{y_i \in L_A | x_{i+1} \in L_B\}, \\ S_B &= \{x_i \in L_B | y_{i-1} \in L_A\} \cap \{y_i \in L_B | x_{i+1} \in L_A\}. \end{aligned}$$

By these definitions we get

$$|L'_A| + |S_A| = |L'_B| + |S_B|.$$

For  $x_i \in L'_A$ , let  $N'(x_i) = (N(x_i) \cap B) - (L - \{y_{i-1}\})$ .

For  $y_i \in L'_A$ , let  $N'(y_i) = (N(y_i) \cap B) - (L - \{x_{i+1}\})$ .

For  $X \subset L'_A$ , let

$$N'(X) = \bigcup_{x_i \in X} N'(x_i) \cup \bigcup_{y_i \in X} N'(y_i).$$

For  $t = s = 3$ , there is nothing to prove. For  $t = 3, s = 2$ , we get for every nonempty  $X \subset L'_A$ ,

$$|N'(X)| \geq 3 - |L_B| + |X| + |X \cap S_A| \geq |X| + |X \cap S_A|.$$

For  $t \geq 4$  we get for every nonempty  $X \subset L'_A$ ,

$$\begin{aligned} |N'(X)| &\geq 2t - s + \frac{t-5}{2} - |L_B| + |X| + |X \cap S_A| - |S_B| \\ &= |X| + |X \cap S_A| + |L_A| - |S_B| + \frac{t-5}{2} \\ &\geq |X| + |X \cap S_A| + |L'_A| - |S_B| + \frac{t-5}{2} \\ &= |X| + |X \cap S_A| + \frac{|L'_A| - |S_B| + |L'_B| - |S_A|}{2} + \frac{t-5}{2} \\ &\geq |X| + |X \cap S_A| + \frac{t-5}{2}. \end{aligned}$$

Thus,  $|N'(X)| \geq |X| + |X \cap S_A|$ , and thus by Hall's Theorem, we can find disjoint neighbors for all  $x_i, y_i \in L'_A$  in  $N'(x_i)$  or  $N'(y_i)$ , respectively. Using that  $B$  is  $t$ -linked and that  $A$  is  $t$ -linked or complete, we can now find the desired cycle.

*Case 2.* Suppose  $s = 0$ .

The degree condition forces  $G$  to be  $(2t-1)$ -connected. If  $G$  is  $2t$ -connected, then it is  $t$ -linked and we are done. If  $G$  has a cut set  $K$  of size  $2t-1$ , the degree condition forces  $G-K$  to consist of two complete components  $A'$  and  $B'$ , both of which are adjacent to all vertices in  $K$ . It is easy to see that such a graph is  $t$ -linked.

*Case 3.* Suppose  $0 < s \leq t/2$ .

The degree condition forces  $G$  to be  $(2t-2)$ -connected. If  $G$  is  $2t$ -connected, then it is  $t$ -linked and we are done. If  $G$  has a cut set  $K$  of size  $2t-2$ , the degree condition forces  $G-K$  to consist of two complete components  $A'$

and  $B'$ , both of which are adjacent to all vertices in  $K$ . It is easy to see that such a graph is  $(2t - s, t, s)$ -ordered. If  $K$  has size  $2t - 1$ ,  $G$  has a very similar structure. Again, it is straightforward to verify the claim. ■

**Theorem 10.** *If  $0 \leq s \leq t \leq k$ , and  $G$  is a graph of order  $n \geq \max \{178t + k, 8t^2 + k\}$  with*

$$\sigma_2(G) \geq \begin{cases} n + k - 3 & \text{if } s = 0, t \geq 3, \\ n + k + s - 4 & \text{if } 0 < 2s \leq t, t \geq 3, \\ n + k + \frac{t-9}{2} & \text{if } 2s > t \geq 3, \\ n + k - 2 & \text{if } s \leq 1, t = 2, \\ n + k - 1 & \text{if } s = 0, t = 1, \\ n & \text{if } s = t \leq 2, \end{cases}$$

*then  $G$  is strongly  $(k, t, s)$ -ordered hamiltonian.*

**Proof.** Apply Theorem 6 and Theorem 9. ■

### 3. Sharpness

Theorem 6 is sharp for  $s = 0$ , illustrated by the following graph: Let  $A = K_{\frac{n+k-t-1}{2}}$ , and  $B$  be a set of  $\frac{n-k+t+1}{2}$  isolated vertices. Add all edges between  $A$  and  $B$ . For  $n$  sufficiently large,  $G$  is strongly  $(k, t, s)$ -ordered, and  $\sigma_2(G) = n + k - t - 1$ . But  $G$  is not strongly  $(k, t, s)$ -ordered hamiltonian, since no hamiltonian cycle can contain a  $(k, t, s)$ -linear forest  $L$  which completely lies inside  $A$ : Every hamiltonian cycle has exactly  $k - t - 1$  edges in  $A$ , one edge less than  $L$ .

The following graph shows sharpness of Theorem 9,  $s = 0$ . Let  $G$  consist of three complete graphs:  $A = K_{\frac{n-k+2}{2}}, K = K_{k-2}, B = K_{\frac{n-k+2}{2}}$ . Add all edges between  $A$  and  $K$  and all edges between  $K$  and  $B$ . The degree sum condition is just missed, but  $G$  is not  $(k, t, 0)$ -ordered: Let  $x_1 \in A, y_t \in B, \langle L - \{x_1, y_t\} \rangle = K$ .

The following graph shows sharpness of Theorem 9,  $t \geq 2s \geq 2$ . Let  $G$  consist of four complete graphs:  $S = K_s, T = K_{k-s}, A = K_{2s-1}, B = K_{n-k-2s+1}$ . Add all edges from  $A$ , all edges between  $T$  and  $B$ . For every vertex  $s_i \in S$ , pick two vertices  $u_i, v_i \in T$ . Add all edges between  $S$  and  $T$  but the edges  $s_i u_i, s_i v_i$ . We have  $\sigma_2(G) = n + k + s - 5$ , but if we pick

$V(L) = V(S) \cup V(T)$ , such that  $x_{2i} = y_{2i} = s_i, x_{2i+1} = u_i, y_{2i-1} = v_i$  for all  $i \leq s$ , there is no cycle passing through  $L$  in the designated order and direction.

The following graph shows sharpness of Theorem 9,  $2s > t$ . Let  $G$  consist of four complete graphs:  $S = K_{\lceil \frac{t}{2} \rceil}, T = K_{k - \lceil \frac{t}{2} \rceil}, A = K_{t-1}, B = K_{n-k-2s+1}$ . Add all edges from  $A$ , all edges between  $T$  and  $B$ . For every vertex  $s_i \in S$ , pick two vertices  $u_i, v_i \in T$ , with the exception that  $v_{i+1} = u_i$  for  $1 \leq i \leq s - \lceil \frac{t}{2} \rceil$ . Add all edges between  $S$  and  $T$  but the edges  $s_i u_i, s_i v_i$ . We have  $\sigma_2(G) = n + k + \lceil \frac{t}{2} \rceil - 5$ , but if we pick  $V(L) = V(S) \cup V(T)$ , such that  $x_{2i} = y_{2i} = s_i, x_{2i+1} = u_i, y_{2i-1} = v_i$  for all  $i \leq \lceil \frac{t}{2} \rceil$ , there is no cycle passing through  $L$  in the designated order and direction.

## 4. Note Added in Proofs

Very recently, Thomas and Wollan [8] have improved the bound in Theorem 1 to the following.

**Theorem 11.** *If a graph  $G$  is  $2k$ -connected and has at least  $5k|V(G)|$  edges, then  $G$  is  $k$ -linked.*

**Corollary 12.** *Every  $10k$ -connected graph is  $k$ -linked.*

Using these results in place of Theorem 1 will improve some of the bounds on  $n$ .

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