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ON THE DOMINATION NUMBER OF PRISMS OF GRAPHS

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Abstract

For a permutation π of the vertex set of a graph G, the graph πG is obtained from two disjoint copies G_1 and G_2 of G by joining each v in G_1 to $\pi(v)$ in G_2. Hence if $\pi = 1$, then $\pi G = K_2 \times G$, the prism of G. Clearly, $\gamma(G) \le \gamma(\pi G) \le 2\gamma(G)$. We study graphs for which $\gamma(K_2 \times G) = 2\gamma(G)$, those for which $\gamma(\pi G) = 2\gamma(G)$ for at least one permutation π of $V(G)$ and those for which $\gamma(\pi G) = 2\gamma(G)$ for each permutation π of $V(G)$.

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1. Introduction

We generally follow the notation and terminology of [4]. Specifically, in a graph $G = (V(G), E(G))$, if $S, T \subseteq V(G)$, then the set of all edges of G

with one endvertex in S and the other endvertex in T is denoted by $E(S,T)$. Further, $N(v) = \{u \in V(G) : uv \in E(G)\}$ and $N[v] = N(v) \cup \{v\}$ denote the *open* and *closed neighbourhoods*, respectively, of a vertex v of G. The *closed neighbourhood of a set* $S \subseteq V(G)$, denoted by $N[S]$, is the set $\cup_{s \in S} N[s]$ and the *open neighbourhood* $N(S)$ *of* S is $\cup_{s \in S} N(s)$. We will also need the set $N\{S\} = N(S) - S$. If $s \in S$, then the *private neighbourhood of* s relative to S, denoted by $\mathrm{pn}(s,S)$, is the set $N[s] - N[S - \{s\}]$. A vertex in $\mathrm{pn}(s,S)$ is called a *private neighbour*, abbreviated *pn*, of s *relative to* S.

As usual $\gamma(G)$ denotes the domination number of G. The set $S \subseteq V(G)$ is called a γ-*set* if it is a dominating set with $|S| = \gamma(G)$. For $A, B \subseteq V(G)$, we abbreviate "A dominates B" to "$A \succ B$"; if $B = V(G)$ we write $A \succ G$ and if $B = \{b\}$ we write $A \succ b$. The subgraph induced by B is denoted by $\langle B \rangle$. A *universal* vertex is one that is adjacent to all other vertices of the graph. The *double star* $S(k,l)$ is the graph obtained by joining the central vertices of the stars $K_{1,k}$ and $K_{1,l}$.

For a permutation π of the vertex set of a graph G, the graph πG is obtained from two disjoint copies G_1 and G_2 of G by joining each v in G_1 to $\pi(v)$ in G_2, that is, $V(\pi G) = V(G_1) \cup V(G_2)$ and $E(G) = E(G_1) \cup E(G_2) \cup \{\{v, \pi(v)\} : v \in V(G_1), \pi(v) \in V(G_2)\}$. Hence if $\pi = 1$, then $\pi G = K_2 \times G$; this graph is sometimes referred to as the *prism* of (or over) G. Thus we may also think of πG, where π is an arbitrary permutation of $V(G)$, as a type of prism of G. If C_5 has vertex sequence $0,1,2,3,4$ and π is the permutation $i \to 2i (\mathrm{mod}\, 5)$, then πC_5 is the Petersen graph; other Petersen-type graphs are obtained similarly. Clearly, $\gamma(G) \le \gamma(\pi G) \le 2\gamma(G)$.

We are mostly interested in graphs where the domination number of (some of) the prisms is equal to twice the domination number of the graph. We investigate graphs for which $\gamma(K_2 \times G) = 2\gamma(G)$, called *prism doublers*, those for which $\gamma(\pi G) = 2\gamma(G)$ for some but not all permutations π of $V(G)$, called *partial doublers*, and those for which $\gamma(\pi G) = 2\gamma(G)$ for each permutation π of $V(G)$, called *universal doublers*. As we shall see, the double star $S(2,2)$ is an example of a graph which satisfies $\gamma(\pi G) = 2\gamma(G)$ for $\pi = 1$ but not for all permutations π of $V(G)$, i.e., a prism doubler but not a universal doubler, P_5 is an example of a graph that satisfies $\gamma(\pi G) = 2\gamma(G)$ for at least one permutation π of $V(G)$ but not for $\pi = 1$, i.e., a partial doubler, and C_6 is an example of a universal doubler. In addition, the graph G obtained from C_4 by joining one of its vertices to two new vertices is an example of a graph that satisfies $\gamma(\pi_k G) = \gamma(G) + k$ for some permutation π_k of $V(G)$, $0 \le k \le \gamma(G)$.

For any vertex v of G, we denote the corresponding vertex in the subgraph G_i, $i = 1, 2$, of πG by v_i. Similarly, any set $X \subseteq V(G)$ will be denoted by X_i when considered in the subgraph G_i of πG. Conversely any set $X_i \subseteq V(G_i)$ (vertex $v_i \in V(G_i)$ respectively) will be denoted by X (v respectively) in G.

2. Universal Doublers

The following lemma collects some useful facts about graphs with isolated vertices.

Lemma 1. (a) (Ore, see [6]) *If $\gamma(G) > |V(G)|/2$, then G has an isolated vertex.*

(b) *If G has an isolated vertex v and π is a permutation of $V(G)$ such that $\pi(v) = v$, then $\gamma(\pi G) < 2\gamma(G)$.*

Proof. (b) Suppose G has an isolated vertex v. Let G' be the (possibly empty) subgraph induced by the other vertices of G. Suppose π is a permutation of $V(G)$ with $\pi(v) = v$, and let π' be the permutation of $V(G')$ induced by π. Then $\gamma(\pi G) = \gamma(\pi' G') + 1 < 2\gamma(G') + 2 = 2\gamma(G)$. ■

Proposition 2. *A graph G is a universal doubler if and only if for each $X \subseteq V(G)$ with $0 < |X| < \gamma(G)$, $|V(G) - N[X]| \geq 2\gamma(G) - |X|$.*

Proof. Consider any $X \subseteq V(G)$ with $0 < |X| < \gamma(G)$, and let $Y = V(G) - N[X]$. Suppose $|Y| < 2\gamma(G) - |X|$. If $|Y| < \gamma(G)$, let D be any γ-set of G, and if $|Y| \geq \gamma(G)$, let D be any dominating set of G with $|D| = |Y|$. Let π be any permutation of $V(G)$ such that $\pi(Y) \subseteq D$. Then $W = X_1 \cup D_2 \succ \pi G$ (since $X_1 \succ V(G_1) - Y_1$ and $D_2 \succ V(G_2) \cup Y_1$), and $|W| = |X_1| + |D_2| < 2\gamma(G)$.

Conversely, let π be a permutation of $V(G)$ such that $\gamma(\pi G) < 2\gamma(G)$ and let $W = X_1 \cup D_2$ be a γ-set of πG, where $X_1 = W \cap V(G_1)$ and $D_2 = W \cap V(G_2)$. Without loss of generality let $|X_1| < \gamma(G)$.

If $X_1 \neq \phi$, then $|D_2| = \gamma(\pi G) - |X| < 2\gamma(G) - |X|$. Since $D_2 \succ Y_1$ and each vertex of D_2 covers at most one vertex of Y_1, we have $|Y_1| \leq |D_2|$, so $|Y| < 2\gamma(G) - |X|$ as desired.

If $X_1 = \phi$, then $D_2 \succ V(\pi G)$, which means $D_2 = V(G_2)$, since each vertex of G_2 covers just one vertex of G_1. Then $|V(G)| = |D_2| < 2\gamma(G)$, so by Lemma 1(a), G has an isolated vertex, say v. Let $u = \pi^{-1}(v)$. Then $N_{\pi G}[v_2] = \{v_2, u_1\}$, so if we define $X'_1 = \{u_1\}$ and $D'_2 = D_2 - \{v_2\}$, then

$W' = X_1' \cup D_2'$ is a γ-set of πG with $X_1' \neq \phi$, reducing to the previous case. ∎

The following corollary gives some degree properties of universal doublers. First, recall that $S \subseteq V(G)$ is a *packing* if the vertices in S have mutually disjoint closed neighbourhoods, and that a packing which dominates G is called an *efficient* dominating set [4, p. 108].

Corollary 3. *Let G be a universal doubler. Then G has no isolated vertices and every vertex of G that is contained in a minimum dominating set has degree at least $\gamma(G)$.*

If G has an efficient dominating set, then $\gamma(G) \leq \sqrt{|V(G)| + 0.25} - 0.5$. Otherwise, for any nonempty packing X contained in a minimum dominating set of G we have $\gamma(G) \leq |V(G)|/(|X| + 2)$.

Proof. Let G satisfy the hypothesis. By Lemma 1(b), G has no isolated vertex.

Let w be a vertex of G that is contained in some minimum dominating set D of G. If $\gamma(G) = 1$, then since w is not isolated, we have $\deg(w) \geq \gamma(G)$. If $\gamma(G) > 1$, let $X = D - \{w\}$. Then $0 < |X| < \gamma(G)$, so by Proposition 2, $|V(G) - N[X]| \geq 2\gamma(G) - |X| = \gamma(G) + 1$. Since D is a dominating set, we see $\{w\} \succ V(G) - N[X]$, which implies $\deg(w) \geq \gamma(G)$.

Finally, suppose that X satisfies the hypotheses. If X dominates G, then since X is a packing, $|V(G)| \geq \gamma(G)(\gamma(G) + 1)$, from which the desired bound follows. Otherwise Proposition 2 applies to X, giving $|V(G)| - |X|(\gamma(G) + 1) \geq 2\gamma(G) - |X|$ as needed. ∎

Any vertex of P_6 or C_6 dominates at most three vertices and thus leaves at least three vertices undominated; hence by Proposition 2, if $G \in \{P_6, C_6\}$, then $\gamma(\pi G) = 4$ for each permutation π of $V(G)$. The next corollary allows us to produce more universal doublers.

Corollary 4. *Suppose G is an r-regular graph that has an efficient dominating set and $r \geq \gamma(G)$. Then $\gamma(\pi G) = 2\gamma(G)$ for each permutation π of $V(G)$.*

Proof. Let $X \subseteq V(G)$, with $0 < |X| < \gamma(G)$; then $|N[X]| \leq (r + 1)|X|$. Since G has an efficient dominating set, $|V(G)| = (r+1)\gamma(G)$. Thus $|V(G) - N[X]| \geq (r + 1)(\gamma(G) - |X|)$, which is at least $2\gamma(G) - |X|$ since $r \geq \gamma(G)$, and the conclusion follows from Proposition 2. ∎

We use circulant graphs to obtain examples of universal doublers. Let r, k be positive integers, and let $n = k(r+1)$. Let p be the largest odd divisor of $r+1$, and let $m = (r+1)/p$. Define a graph $G_{r,k}$ as follows. Let $V(G_{r,k}) = \{0, 1, \ldots, n-1\}$. Two distinct vertices v, w are adjacent if there is some i with $|i| \leq (p-1)/2$ and $v - w \equiv i \pmod{pk}$. (Less formally, for each vertex v, the closed neighbourhood $N[v]$ consists of m runs of p vertices, with equal spacing between the runs and one run centered on v.) Then $G_{r,k}$ is r-regular; since $G_{r,k}$ has the efficient dominating set $\{ip : 0 \leq i \leq k-1\}$, $\gamma(G_{r,k}) = k$. For $r \geq k$, Corollary 4 implies $\gamma(\pi G_{r,k}) = 2\gamma(G_{r,k})$ for each permutation π of $V(G_{r,k})$.

The graph $G_{r,k}$ is connected except when $r = 2^j - 1$ for some positive integer j, when $p = 1$ and $G_{r,k}$ consists of k disjoint copies of the complete graph K_{r+1}. When $r > 1$, and r, k are odd, we can change the edge set to obtain a connected graph: say distinct v, w are adjacent if either $|v - w| = n/2$ or there is i with $|i| \leq (r-1)/2$ and $v - w \equiv i \pmod{n}$. The resulting graph $G^*_{r,k}$ is connected and r-regular with an efficient dominating set $\{i(r+1) : 0 \leq i \leq k-1\}$; thus $\gamma(G^*_{r,k}) = k$. For $r \geq k$, Corollary 4 implies $\gamma(\pi G^*_{r,k}) = 2\gamma(G^*_{r,k})$ for each permutation π of $V(G_{r,k})$.

Also, the 3-cube $G = (K_2)^3$ satisfies the hypotheses of Corollary 4 with $r = 3$ and $\gamma(G) = 2$, hence $\gamma(\pi G) = 4$ for each permutation π of $V(G)$.

3. Prism Doublers

In this section we consider graphs G where the domination number of the (usual) prism of G is twice that of G. We begin with a lemma which will allow us to study dominating sets of G instead of those of its prism.

Lemma 5. *The set X with $X \cap V(G_1) = A_1$, $X \cap V(G_2) = B_2$ dominates $K_2 \times G$ if and only if in G, every vertex not in $A \cup B$ is adjacent to a vertex in A and to a vertex in B.*

Proof. Suppose $X \succ K_2 \times G$ and consider $v \in V(G) - (A \cup B)$. Now, $v_1 \notin B_1$, thus $B_2 \not\succ v_1$ and therefore $A_1 \succ v_1$, that is, v is adjacent in G to a vertex in A. Similarly, since $v_2 \notin A_2$, v is adjacent to a vertex in B. Conversely, consider any $x_i \in V(G_i)$. If $x_i \in A_i \cup B_i$, then obviously $X \succ x_i$. If $x_i \notin A_i \cup B_i$, then x is adjacent to a vertex in A and to a vertex in B and hence $A_1 \succ x_1$, $B_2 \succ x_2$. ■

If A and B are sets such that $A \cup B$ satisfies the conditions of Lemma 5, we say that (A, B) is a *dominating pair* of G. Clearly $\gamma(K_2 \times G) = \min\{|A| + |B| : (A, B)$ is a dominating pair of $G\}$. Say that a dominating pair (A, B) is *minimum* if $|A| + |B| = \gamma(K_2 \times G)$. Given a set A, we say we *extend* A to a dominating pair of G if we find a set B such that (A, B) is such a pair. Note that (A, ϕ) is a dominating pair if and only if $A = V(G)$, and (A, A) is a dominating pair if and only if $A \succ G$. More generally we have the following result which we formulate as a corollary for easy reference.

Corollary 6. *The pair (A, B) is a dominating pair of G if and only if $V(G) - N[A] \subseteq B$ and $V(G) - N[B] \subseteq A$.*

If $\gamma(G) = 1$ and G has at least two vertices, then $\gamma(K_2 \times G) = 2$. Now consider the case $\gamma(G) = 2$; we assume that G has no isolated vertices, for otherwise it is easy to see that $\gamma(K_2 \times G) < 2\gamma(G)$.

Construct the class $\mathcal{H}$ of graphs as follows. For any $H \in \mathcal{H}$, $V(H) = S \cup T \cup \{u, v, w\}$ and $E(H)$ consists of edges such that

- $N(u) = S$, $N(v) = T$, $N(w) = S \cup T$,
- $E(\langle S \rangle)$ and $E(\langle T \rangle)$ consist of any edges such that at least one of $\langle S \rangle$ and $\langle T \rangle$ has a universal vertex,
- $E(S, T)$ is arbitrary.

Note that if (without loss of generality) $x \in S$ is a universal vertex of $\langle S \rangle$, then $\{x, v\}$ is a γ-set of H for any $H \in \mathcal{H}$, while $(\{u, v\}, \{w\})$ is a dominating pair of H. Hence $\gamma(H) = 2$ and $\gamma(K_2 \times H) \leq 3$.

Proposition 7. *If $\gamma(G) = 2$, then $\gamma(K_2 \times G) = 4$ if and only if $G \notin \mathcal{H}$ and for each γ-set $X = \{x_1, x_2\}$ of G, $|\mathrm{pn}(x_i, X)| \geq 2$ for each i, where $|\mathrm{pn}(x_i, X)| = 2$ implies that $\mathrm{pn}(x_i, X)$ does not dominate $G - x_j$, $i \neq j$.*

Proof. Suppose $\gamma(K_2 \times G) = 4$; clearly $G \notin \mathcal{H}$. Let $X = \{x, x'\}$ be any γ-set of G. If $\mathrm{pn}(x, X) = \{y\}$, then $(\{x', y\}, \{x'\})$ is a dominating pair of G and $\gamma(K_2 \times G) \leq 3$, a contradiction. Hence each vertex in X has at least two X-pns. If $Y = \mathrm{pn}(x, X)$ dominates $G - x'$ and $|Y| = 2$, then $(Y, \{x'\})$ is a dominating pair, again a contradiction.

Conversely, suppose $\gamma(K_2 \times G) < 4$ and suppose that (A, B) is a minimum dominating pair of G such that $|A| \leq |B|$. If A is empty then $B = V(G)$ so $|V(G)| \leq 3$. Then $\gamma(G) = 2$ and Lemma 1(a) imply G has an isolated vertex, and the conclusion follows. So we may assume $|A| = 1$ and $|B| \leq 2$, say $A = \{w\}$, $Y = V(G) - N[w]$. If $Y = \{y\}$, then $\{y, w\}$ is a γ-set of G in which y has itself as its only pn and we are done.

By Corollary 6, $Y \subseteq B$ and so $|Y| \leq |B| \leq 2$. Hence we may assume that $|Y| = 2$; say $Y = \{u, v\}$. Then $B = \{u, v\}$ and since (A, B) is a dominating pair, $B \succ G - w$ and $w \succ G - \{u, v\}$. Therefore, if there is a vertex y in G that dominates $\{u, v\}$, then $\{y, w\}$ is a γ-set of G such that $\mathrm{pn}(y, \{y, w\}) = B \succ G - w$. If there is no such vertex y, then $d(u, v) \geq 3$. If $d(u, v) \geq 5$, then in any shortest (u, v)-path there are at least two vertices that are not dominated by $\{u, v\}$, contradicting the fact that $B \succ G - w$. Hence $3 \leq d(u, v) \leq 4$.

If $d(u, v) = 4$, then w is the unique vertex of G not dominated by $\{u, v\}$, and w lies on each (u, v)-path. Since $\gamma(G) = 2$, it follows that to dominate $\{u, v, w\}$, there is at least one vertex that dominates (without loss of generality) both u and w, as well as each central vertex on any (u, w)-path of length two. Thus $G \in \mathcal{H}$ with $E(S, T) = \phi$. Similarly, if $d(u, v) = 3$, then $G \in \mathcal{H}$ with $E(S, T) \neq \phi$. ■

This result can be generalized as follows. (Recall that $N\{X\} = N[X] - X$.)

Theorem 8. *A graph G is a prism doubler if and only if for each pair of sets $X, Y \subseteq V(G)$ with $0 < |X| < \gamma(G)$ and $Y = V(G) - N[X]$, either*

(a) $|Y| \geq 2\gamma(G) - |X|$, *or*

(b) $|Y| = 2\gamma(G) - |X| - d$ *for some d, $1 \leq d \leq |X|$, and at least d vertices (necessarily in $N[X]$) are required to dominate $N\{X\} - N[Y]$.*

Proof. Suppose $\gamma(K_2 \times G) = 2\gamma(G)$ and consider any $X \subseteq V(G)$ with $0 < |X| < \gamma(G)$. Note that $(X, X \cup Y)$ is a dominating pair of G. If $|Y| \geq 2\gamma(G) - |X|$, we are done. If $|Y| < 2\gamma(G) - 2|X|$, then $|X| + |X \cup Y| < 2|X| + 2\gamma(G) - 2|X| = 2\gamma(G)$. Hence we assume that $2\gamma(G) - 2|X| \leq |Y| \leq 2\gamma(G) - |X| - 1$; say $|Y| = 2\gamma(G) - |X| - d$, where $1 \leq d \leq |X|$. Suppose there is $Z \subseteq N[X]$ such that $|Z| \leq d - 1$ and $Z \succ N\{X\} - N[Y]$. Then $N\{X\} \subseteq N[Y] \cup N[Z]$, hence $V(G) - N[Y \cup Z] = V(G) - (N[Y] \cup N[Z]) \subseteq X$ and so by Corollary 6, $(X, Y \cup Z)$ is a dominating pair of G. But $|X| + |Y \cup Z| \leq |X| + (2\gamma(G) - |X| - d) + (d - 1) = 2\gamma(G) - 1$, a contradiction. Hence (b) holds.

Conversely, suppose $\gamma(K_2 \times G) < 2\gamma(G)$ and consider any minimum dominating pair (X, D) of G. Since $|X| + |D| < 2\gamma(G)$, we may assume

$|X| < \gamma(G)$ and $|D| < 2\gamma(G) - |X|$. If X is empty then $D = V(G)$ so $\gamma(K_2 \times G) = |V(G)|$. Thus $|V(G)| < 2\gamma(G)$, so by Lemma 1(a), G has an isolated vertex, say v. Then $(\{v\}, D - \{v\})$ is a minimum dominating pair, so we may assume X is nonempty. Let $Y = V(G) - N[X]$. By Corollary 6, $Y \subseteq D$ and so $|Y| < 2\gamma(G) - |X|$; hence (a) does not hold. If $|Y| < 2\gamma(G) - 2|X|$, then (b) does not hold either (because d is not in the stated range) and we are done. Hence suppose $|Y| = 2\gamma(G) - |X| - d$ for some d with $1 \leq d \leq |X|$. Let $Z = D - Y$; then $Y \cup Z = D \succ V(G) - X$ and so $Z \succ N\{X\} - N[Y]$. Further, $|Z| = |D| - |Y| < (2\gamma(G) - |X|) - (2\gamma(G) - |X| - d) = d$. Hence (b) does not hold. ■

We now apply Proposition 7 and Theorem 8 to paths and cycles to show that with the exception of P_3, C_3 (which have $\gamma = 1$), P_6 and C_6 (see Section 2), no path or cycle is a prism doubler. Let P_n, C_n have vertex sequence $1, 2, \ldots, n$.

- Note that P_4, C_4 have γ-sets in which a vertex has only one pn; hence by Proposition 7, $\gamma(K_2 \times P_4), \gamma(K_2 \times C_4) < 4$. Also, $P_5 \in \mathcal{H}$, hence $\gamma(K_2 \times P_5) < 4$. For any two non-adjacent vertices x and y of C_5, $X = \{x, y\}$ is a γ-set of C_5, where x has two pns (one of which is x) and $\mathrm{pn}(x, X)$ dominates $V(G) - \{y\}$. Thus $\gamma(K_2 \times C_5) < 4$.
- For $n \geq 7$, write $n = 4i + r$ with $0 \leq r \leq 3$. For $G \in \{P_n, C_n\}$, let $X = \{4j - 1 : j = 1, \ldots, i\}$, except for P_{4i} let $X = \{4j - 1 : j = 1, \ldots, i - 1\} \cup \{4i\}$. Then $|X| = i$ in all cases. Let $Y = V(G) - N[X]$. Except for $G = P_{4i}$, $Y = \{4j - 3 : j = 1, \ldots, i\} \cup \{4i + j : 1 \leq j \leq r\}$. For P_{4i}, $Y = \{4j - 3 : j = 1, \ldots, i\} \cup \{4i - 2\}$. Thus $|Y| = i + r$ except for P_{4i}, when $|Y| = i + 1$. Then using $\gamma(G) = \lceil n/3 \rceil$, it is straightforward to verify $|X| < \gamma(G)$ and $|Y| < 2\gamma(G) - |X|$.
  In all cases, X contains neither vertices at distance less than four nor, when $G = P_n$, any vertex adjacent to an end vertex. Therefore $Y \succ V(G) - X$. In particular, $N\{X\} - N[Y] = \phi$, so X does not satisfy the conditions of Theorem 8, and thus $\gamma(K_2 \times G) < 2\gamma(G)$.

As an example of a prism doubler that is not a universal doubler, consider the double star $S(k, l)$ with $k, l \geq 2$. Proposition 7 shows that $\gamma(K_2 \times S(k, l)) = 4$. However, if $k = 2$, then Figure 1 illustrates a permutation π such that $\gamma(\pi S(k, l)) = 3$. A similar result holds for the graph $K_n(k_1, \ldots, k_n)$, $k_i \geq n$, obtained by joining k_i new vertices to vertex v_i of K_n with $V(K_n) = \{v_1, \ldots, v_n\}$.

We now consider prism doublers with the additional properties that they are regular and have efficient dominating sets; say that such a graph is a *perfect doubler*.

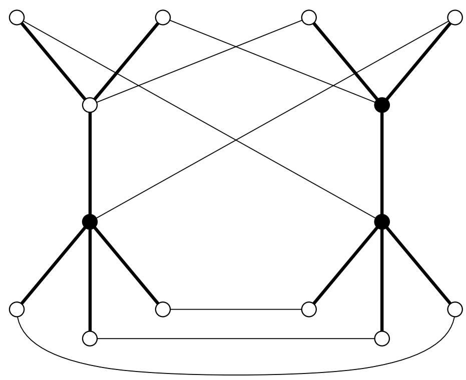

Figure 1. $\gamma(\pi G) = 3$

Above we have shown that C_{3i} is a perfect doubler only when $i \leq 2$. Here is an infinite family of examples: for each positive integer m, set $G_m = K_2 \times C_{4m}$. Then G_m is a 3-regular graph with $8m$ vertices, and has an efficient dominating set $\{(0, 4j), (1, 4j + 2) : 0 \leq j \leq m - 1\}$ of size $2m$. It can be proved by induction that G_m is a perfect doubler for each m. Also, as shown in the previous section, the graphs $G_{r,k}$ and $G^*_{r,k}$ are perfect doublers when $r \geq k$.

A more interesting family of examples is provided by the hypercubes. For a positive integer p, consider the $(2^p - 1)$-dimensional binary hypercube $G_p = (K_2)^{2^p-1}$. This graph is $(2^p - 1)$-regular and has efficient dominating sets (also known as perfect single-error-correcting codes), for example the Hamming code $\mathcal{H}_p$ (see for Example [7, p. 423]). Thus $\gamma(G_p) = 2^{2^p-1-p}$. It has been shown (in [5] and independently in [8]) that $\gamma(K_2 \times G_p) = \gamma((K_2)^{2^p}) = 2^{2^p-p} = 2\gamma(G_p)$.

The ternary hypercubes with efficient dominating sets are also perfect doublers [1]. That is, for $p \geq 1$, let $T_p = (K_3)^{(3^p-1)/2}$. Then $\gamma(T_p) = 3^{((3^p-1)/2)-p}$ and $\gamma(K_2 \times T_p) = 2\gamma(T_p)$.

Here is a version of Theorem 8 useful for regular graphs with efficient dominating sets.

Corollary 9. *Let G be an r-regular graph with an efficient dominating set. Suppose that for each subset X of $V(G)$ with $\frac{r-1}{r}\gamma(G) < |X| < \gamma(G)$ and $(r-1)\gamma(G) < |N\{X\}|$, there does not exist any set W satisfying $V(G) - N[X] \subseteq W \subseteq V(G)$ and $|W| < 2\gamma(G) - |X|$ and $W \succ V(G) - X$. Then G is a perfect doubler.*

Proof. Suppose that G is an r-regular graph with an efficient dominating set, but not a perfect doubler. Then there is some $X \subseteq V(G)$ with $0 < |X| < \gamma(G)$ for which conditions (a) and (b) of Theorem 8 both fail. Set $Y = V(G) - N[X]$.

Since (a) does not hold, $|Y| < 2\gamma(G) - |X|$. As $|V(G)| = (r+1)\gamma(G)$, this implies $(r-1)\gamma(G) < |N[X]| - |X| = |N\{X\}|$. The r-regularity of G gives $|N\{X\}| \le r|X|$, so $\frac{r-1}{r}\gamma(G) < |X|$.

Since (b) does not hold, $|Y| = 2\gamma(G) - |X| - d$ for some d, $1 \le d \le |X|$, and some vertex set S satisfies $|S| < d$ and $S \succ N\{X\} - N[Y]$. Set $W = Y \cup S$. Then $|W| < 2\gamma(G) - |X|$ and $W \succ V(G) - X$. ■

We need the following definition from [3] to give another example of a perfect doubler. Let k be an integer, $k > 1$, and let $S = \{1, 2, \ldots, 2k-1\}$. The vertices of the *odd graph* O_k are the subsets of S of cardinality $k-1$, and two vertices of O_k are adjacent if they are disjoint sets.

Thus O_k is a k-regular graph with $C(2k-1, k-1)$ vertices. For example, $O_2 \cong K_3$ and O_3 is isomorphic to the Petersen graph. Biggs has shown [2, Sections 3j, 21b, 21j] that if O_k has an efficient dominating set, then k is even. Such sets are known to exist for $k = 2, 4, 6$ and it is conjectured [3] that there are no more.

We next show that O_2 and O_4 are perfect doublers. We will need the following result, equivalent to [3, Section 2.3]: the distance between vertices v, w of O_k is

$$(1) \qquad d(v,w) = \begin{cases} 2|v \cap w| + 1 & \text{if } |v \cap w| < (k-1)/2, \\ 2(k-1-|v \cap w|) & \text{otherwise.} \end{cases}$$

This implies that O_k has diameter $k-1$.

Proposition 10. *For $G \in \{O_2, O_4\}$, $\gamma(K_2 \times G) = 2\gamma(G)$.*

Proof. Since $O_2 \cong K_3$, it suffices to consider O_4, which is a 4-regular graph with 35 vertices, diameter 3, and domination number 7; an efficient dominating set of O_4 is {123, 145, 167, 246, 257, 347, 356}. Here and later, we write vertices of O_4 as strings of length 3.

By Corollary 9, we need only examine vertex sets X with $\frac{3}{4}\gamma(O_4) < |X| < \gamma(O_4)$, which means $|X| = 6$. We also may assume $3\gamma(O_4) < |N\{X\}|$, and then $|X| = 6$ implies $|N[X]| \geq 28$. Since O_4 is 4-regular and $|X| = 6$, $|N[X]| \leq 30$.

For a vertex set X, choose a maximum size packing X' inside X. For distinct vertices v, w of X', it follows from (1) that $|v \cap w| = 1$.

With $X \subseteq V(O_4)$ of size 6, consider the possibility $|X'| \geq 5$. At least one of the indices $1, \ldots, 7$ must then occur at least 15/7 times among the vertices of X'; that is, some index occurs at least three times. Without loss of generality, we may assume that 123, 145, 167 are in X'. No other vertex of X' can then contain 1 (otherwise such a vertex will have at least two indices in common with one of 123, 145 or 167). Since vertices of X' are not adjacent, some index occurs at least twice in the remaining members of X', so we may assume 246 and 257 are in X'. If $|X'| = 6$, it is not difficult to see that the last member of X' is either 347 or 356.

If $|N[X]| = 30$ then $X = X'$, and by the previous paragraph, X consists of all but one vertex, say z, of an efficient dominating set of O_4. Then $V(O_4) - N[X] = N[z]$ has 5 elements. Note that $N[z]$ dominates a vertex $w \in V(O_4) - N[z]$ if and only if $d(z, w) = 2$, that is, if and only if (by (1)) $|z \cap w| = 2$. Thus $N[z]$ dominates 12 vertices in $V(O_4) - N[z]$ and 17 vertices in total. Two more vertices will cover at most 10 more vertices, so it is not possible to find a set W of 7 vertices including those of $N[z]$ that covers the 29 vertices of $V(O_4) - X$.

For the remainder of the proof, it is helpful to consider the situation where there is a vertex not in X that is covered by more than one vertex in X. Without loss of generality, we may assume that 135 and 357 are in X; these both cover 246. The vertices that are at distance 3 from both 135 and 357 may be divided into three families: $\mathcal{F}_1 = \{234, 236, 346\}$, $\mathcal{F}_2 = \{245, 256, 456\}$, and $\mathcal{F}_3 = \{127, 147, 167\}$.

If $|N[X]| = 29$ then $V(O_4)$ contains one vertex doubly covered by X and 28 that are singly covered. Thus one internal distance of the set X is 2, and the other distances are 3. We may assume 135, 357 are the members of X at distance 2; then the remaining four vertices of X are in $\mathcal{F}_1 \cup \mathcal{F}_2 \cup \mathcal{F}_3$. Since the internal distances of each $\mathcal{F}_i$ are all 2, this is not possible.

If $|N[X]| = 28$, then from $V(O_4) - N[X] \subseteq W$ and $|W| < 2\gamma(O_4) - |X|$ we see that $W = V(O_4) - N[X]$.

If some vertex not in X is multiply covered by X, we may again assume that $135, 357 \in X$ and $246 \notin X$. It then follows from $|N[X]| \geq 28$ that three vertices of X, say x_1, x_2, x_3, are at distance 3 from each other and from each of 135, 357. Since the internal distances of each $\mathcal{F}_i$ are all 2, each $\mathcal{F}_i$ contains one x_j. However, the four neighbours of 246 are 135 and 357 (which are in X), 157 (which is adjacent to every member of $\mathcal{F}_1$), and 137 (adjacent to every member of $\mathcal{F}_2$). Thus $N(246) \subseteq N[X]$, so $W = V(O_4) - N[X]$ does not cover 246.

The only other conceivable way to have $|N[X]| = 28$ is for X to have one internal distance of 1, with the other distances being 3. Without loss of generality, we may assume 123 and 567 are the adjacent members of X. Then the other vertices v_1, v_2, v_3, v_4 of X have the form $e_i 4 e_j$ with $e_i \in \{1, 2, 3\}$ and $e_j \in \{5, 6, 7\}$. But this implies that for some h, k we have $|v_h \cap v_k| = 2$, and then $d(v_h, v_k) = 2$. So $|N[X]| = 28$ cannot be achieved in this way. ■

Our work leads us to believe that something along the following lines is true.

Conjecture 11. *Let G be a regular graph with an efficient dominating set. If every sufficiently large packing in G extends to an efficient dominating set, then G is a perfect doubler.*

4. Partial Doublers

Recall that if $\gamma(\pi G) = 2\gamma(G)$ for some but not all permutations π of $V(G)$, then G is called a partial doubler. As noted above, $\gamma(K_2 \times P_5) < 4$. However, if π is the transposition which maps the central vertex of P_5 to one of its neighbours and vice versa, then $\gamma(\pi P_5) = 4$, so P_5 is a partial doubler. A natural question which now arises is whether, for each integer $k \geq 2$, there exists a graph G and permutations π_1, π_2 such that $\gamma(\pi_1 G) = \gamma(G) = k$ and $\gamma(\pi_2 G) = 2k$, or more generally, permutations π_i, $0 \leq i \leq \gamma(G)$, such that $\gamma(\pi_i G) = \gamma(G) + i$. We answer this question by constructing a large class $\mathcal{G}$ of such graphs.

Construction. Let G be any isolate-free bipartite graph of order n with bipartition (X, Y), $s = |X|$, $t = |Y|$, $s \leq t$, and note that X is a (not necessarily minimum) dominating set of G such that each edge is incident with exactly one vertex in X. Construct G^* with $V(G^*) = X \cup Y \cup A \cup B$ as follows. (The construction is illustrated in Figure 2 for $G = C_4$, where the black vertices are in X and the grey vertices are in Y.) First replace each edge $xy \in E(G)$ with $k \geq 2$ (not necessarily fixed) multiple edges, then subdivide each of these new edges once (the white vertices of degree two in Figure 2, where $k = 2$ in each case). Denote the set of new degree two vertices incident with x and y by A^{xy}. Join each vertex $x \in X$ to a set B^x of new vertices (the white vertices of degree one in Figure 2), where $|B^x| \geq 2$ and $\Sigma_{x \in X}(|B^x| - 2) \geq t$. Let $A = \cup_{xy \in E(G)} A^{xy}$ and $B = \cup_{x \in X} B^x$. Any graph G^* obtained in this way is in $\mathcal{G}$.

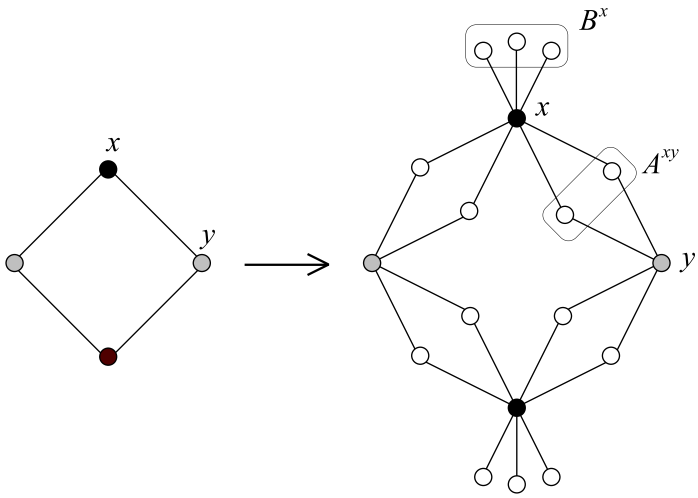

Figure 2. An example of a graph $C_4^* \in \mathcal{G}$

Note that $X \cup Y$ dominates G^*. Moreover, if D is any γ-set of G^*, then $X \subseteq D$ to dominate the set B of leaves. Since $V(G^*) - N[X] = Y$ and $N(y) \cap N(y') = \phi$ for distinct $y, y' \in Y$, at least t vertices are required to dominate Y. Thus $\gamma(G^*) = s + t = n$. For each k with $0 \leq k \leq n$ we now define a permutation π_k of $V(G^*)$; we shall prove that $\gamma(\pi_k G^*) = \gamma(G^*) + k$.

Let $\pi_s = 1$, the identity. For each i with $1 \leq i \leq s$ we define π_{s-i} recursively by means of transpositions. Choose $x \in X$, $y \in Y$ such that $\pi_{s-i+1}(x) = x$, $\pi_{s-i+1}(y) = y$, let ρ_i be the transposition (x, y) and define $\pi_{s-i} = \rho_i \circ \pi_{s-i+1}$. (Since $|X| = s$ and $\pi_s = 1$, this choice of x and y is always possible.) Similarly, for each j with $1 \leq j \leq t$ we define π_{s+j} as follows. Choose any $y \in Y$ such that $\pi_{s+j-1}(y) = y$. Further, choose $x \in X$ such that π_{s+j-1} fixes three distinct vertices $u, v, w \in B^x$. Since $\Sigma_{x \in X}(|B^x| - 2) \geq t$, this choice of x is always possible. Let σ_j be the transposition (y, u) and define $\pi_{s+j} = \sigma_j \circ \pi_{s+j-1}$. The graphs $\pi_s C_4^*$, $\pi_{s-1} C_4^*$ and $\pi_{s+1} C_4^*$ corresponding to C_4^* of Figure 2 are illustrated in Figure 3.

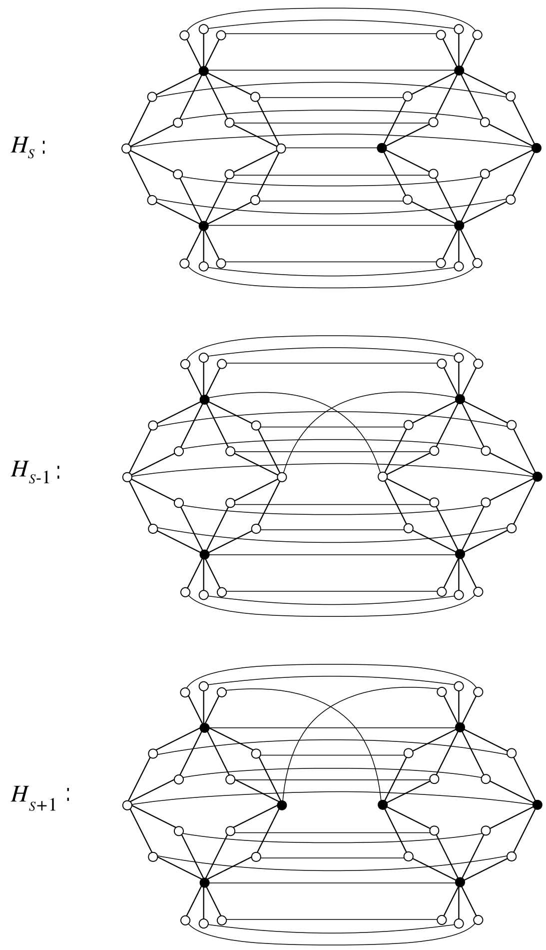

Figure 3. Graphs $H_s = \pi_s C_4^*$, $H_{s-1} = \pi_{s-1} C_4^*$ and $H_{s+1} = \pi_{s+1} C_4^*$

Theorem 12. *For each $k \in \{0, \ldots, n\}$, $\gamma(\pi_k G^*) = \gamma(G^*) + k$.*

Proof. We begin by noting that the result holds for $\pi_s = 1$, for if D is any γ-set of $K_2 \times G^*$, then $X_1 \cup X_2 \subseteq D$ to dominate the vertices in $B_1 \cup B_2$. This leaves the vertices in $Y_1 \cup Y_2$ undominated, and it it easy to see that at least t vertices are required to dominate $Y_1 \cup Y_2$. But Y_2 dominates $Y_1 \cup Y_2$; hence $\gamma(K_2 \times G^*) = 2s + t = \gamma(G^*) + s$.

For arbitrary i, $1 \le i \le s$, consider any γ-set D of $\pi_{s-i}G^*$ and note that as above, $X_1 \cup X_2 \subseteq D$ to dominate $B_1 \cup B_2$. Let $X' = \{x \in X : \pi_{s-i}(x) \in Y\}$ and $Y' = \pi_{s-i}(X')$; observe that $|X'| = |Y'| = i$. In $\pi_{s-i}G^*$, $X_1 \cup X_2$ dominates all vertices except $(Y_1 - Y_1') \cup (Y_2 - Y_2')$; again it is easy to see that at least $|Y - Y'| = t - i$ vertices are required to dominate these. Using (for example) $Y_2 - Y_2'$, we obtain $\gamma(\pi_{s-i}G^*) = 2s + t - i = \gamma(G^*) + s - i$.

Finally, for arbitrary j, $1 \le j \le t$, let D be a γ-set of $\pi_{s+j}G^*$ with $|D \cap (X_1 \cup X_2)|$ maximum, consider any $x \in X$ and suppose $|\{x_1, x_2\} \cap D| < 2$. By definition of π_{s+j} there are at least two vertices $v, w \in B^x$ that are fixed by π_{s+j}. Hence if (say) $x_1 \notin D$, then to dominate v_1 and w_1, $\{v_1, v_2\} \cap D \neq \phi$ and $\{w_1, w_2\} \cap D \neq \phi$. But then $(D - \{v_1, v_2, w_1, w_2\}) \cup \{x_1, x_2\}$ is a γ-set which contradicts the choice of D. Therefore $X_1 \cup X_2 \subseteq D$. As in the case of $\pi_s G^*$, this leaves the vertices in $Y_1 \cup Y_2$ undominated. Let $Y' = \{y \in Y : \pi_{s+j}(y) \in B\}$ and note that $|Y'| = j$. Then for any $y', y'' \in Y'$, $N[y'] \cap N[y''] = \phi$ and in $\pi_{s+j}G^*$, $N[Y_1'] \cap N[Y_2'] = \phi$. Therefore a set Z of at least j vertices are needed to dominate Y_1' in $\pi_{s+j}G^*$, and j vertices distinct from those in Z to dominate Y_2'. Further, at least $|Y - Y'| = t - j$ vertices are required to dominate $(Y_1 - Y_1') \cup (Y_2 - Y_2')$. Hence

$$\begin{aligned}\gamma(\pi_{s+j}G^*) &\ge |X_1| + |X_2| + 2j + t - j \\ &= 2s + j + t \\ &= \gamma(G^*) + s + j.\end{aligned}$$

Obviously $X_1 \cup X_2 \cup Y_1' \cup Y_2$ dominates $\pi_{s+j}G^*$ and so $\gamma(\pi_{s+j}G^*) = \gamma(G^*) + s + j$, as required. ■

Theorem 12 also holds if we require $\Sigma_{x \in X}(|B^x| - 1) \ge t$ in the construction of G^*, but the proof is technically more difficult. The simplest example is obtained by taking $G = K_2$ with $V(G) = \{x, y\}$, replacing xy with $K_{2,2}$ and joining x to two new vertices u and v. Then $\pi_0 = (x, y)$, $\pi_1 = 1$ and $\pi_2 = (y, u)$.

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