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CYCLE-PANCYCLISM IN BIPARTITE TOURNAMENTS I

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Abstract

Let T be a hamiltonian bipartite tournament with n vertices, γ a hamiltonian directed cycle of T, and k an even number. In this paper, the following question is studied: What is the maximum intersection with γ of a directed cycle of length k? It is proved that for an even k in the range $4 \leq k \leq \frac{n+4}{2}$, there exists a directed cycle $\mathcal{C}_{h(k)}$ of length $h(k), h(k) \in \{k, k-2\}$ with $|A(\mathcal{C}_{h(k)}) \cap A(\gamma)| \geq h(k) - 3$ and the result is best possible.

In a forthcoming paper the case of directed cycles of length k, k even and $k > \frac{n+4}{2}$ will be studied.

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1. Introduction

The subject of pancyclism has been studied by several authors (e.g. [1, 2, 5, 10, 12, 14]). Three types of pancyclism have been considered. A digraph D is pancyclic if it has directed cycles of all the possible lengths; D is vertexpancyclic if given any vertex v there are directed cycles of every length containing v; and D is arc-pancyclic if given any arc e there are directed cycles of every length containing e.

It is well known that a hamiltonian bipartite tournament is pancyclic, and vertex-pancyclic (with only very few exceptions) but not necessarily

arc-pancyclic (see e.g. [3, 11, 13]). Within the concept of cycle-pancyclism the following question is studied: Given a directed cycle γ of a digraph D, find the maximum number of arcs which a directed cycle of length k (if such a directed cycle exists) contained in $D[V(\gamma)]$ (the subdigraph of D induced by $V(\gamma)$ has in common with γ . Cycle-pancyclism in tournaments has been studied in [6, 7, 8] and [9]. In this paper, the cycle-pancyclism in bipartite tournaments is investigated. In order to do so, it is sufficient to consider a hamiltonian bipartite tournament T where γ is a hamiltonian directed cycle (because we are looking for directed cycles of length k contained in $D[V(\gamma)]$ whose arcs intersect the arcs of γ the most possible). We will assume (without saying it explicitly in Lemmas, Theorems or Corollaries) that we are working in a hamiltonian bipartite tournament with a vertex set $V = \{0, 1, ..., n - 1\}$ and an arc set A. Also we assume without loss of generality that $\gamma = (0, 1, \dots, n-1, 0)$ is a hamiltonian directed cycle of T; k will be an even number; $\mathcal{C}_{h(k)}$ will denote a directed cycle of length h(k) with $h(k) \in \{k, k-2\}$ and $\Im(\mathcal{C}_{h(k)}) = |A(\mathcal{C}_{h(k)}) \cap A(\gamma)|$. This paper is the first part of the study of the existence of a directed cycle $\mathcal{C}_{h(k)}$ where $\mathfrak{I}(\mathfrak{C}_{h(k)})$ is the maximum. For general concepts we refer the reader to [4].

2. Preliminaries

A chord of a cycle \mathcal{C} is an arc not in \mathcal{C} but with both terminal vertices in \mathcal{C} . The length of a chord f = (u, v) of \mathcal{C} denoted $\ell(f)$ is equal to the length of $\langle u, \mathcal{C}, v \rangle$, where $\langle u, \mathcal{C}, v \rangle$ denotes the *uv*-directed path contained in \mathcal{C} . We say that f is a *c*-chord if $\ell(f) = c$ and f = (u, v) is an a - c-chord if $\ell(\langle v, \mathcal{C}, u \rangle) = c$. Observe that if f is a *c*-chord, then it is also an a - (n - c)chord. All the chords considered in this paper are chords of γ . We will denote by \mathcal{C}_k a directed cycle of length k. In what follows all notation is taken modulo n.

For any $a, 2 \leq a \leq n-2$, denote by t_a the largest integer such that $a + t_a(k-2) < n-1$. The important case of t_{k-1} is denoted by t in the rest of the paper. Let r be defined as follows: r = n - [k - 1 + t(k-2)]. Notice that: If $a \leq b$ then $t_a \geq t_b$; $t \geq 0$ and $3 \leq r \leq k-1$, r is odd.

Lemma 2.1. If the a-chord with an initial vertex 0 (0 being an arbitrary vertex of T) is in A, then at least one of the two following properties holds.

(i) There exists a directed cycle C_k with $\mathfrak{I}(C_k) = k - 2$.

(ii) For every $0 \le i \le t_a$, the a + i(k-2)-chord with an initial vertex 0 is in A.

Proof. Suppose that (ii) in the lemma is false, and let

$$j = \min\{i \in \{1, 2, \dots, t_a\} \mid (a + i(k - 2), 0) \in A\},\$$

then

$$\mathbb{C}_k = (0, a + (j-1)(k-2)) \cup \langle a + (j-1)(k-2), \gamma, a + j(k-2) \rangle \cup (a + j(k-2), 0)$$

is a directed cycle with $\mathfrak{I}(\mathfrak{C}_k) = k - 2$.

Corollary 2.2. At least one of the two following properties holds

- (i) There exists a directed cycle \mathfrak{C}_k with $\mathfrak{I}(\mathfrak{C}_k) \geq k-2$.
- (ii) For every $0 \le i \le t$, every (k 1 + i(k 2))-chord is in A.

Proof. Clearly, for any vertex $0, (0, k-1) \in A$ since otherwise $(k-1, 0) \in A$ and $\mathbb{C}_k = \langle 0, \gamma, k-1 \rangle \cup (k-1, 0)$ is a directed cycle with $\mathfrak{I}(\mathbb{C}_k) = k-1$ and thus (i) holds. Now applying Lemma 2.1 with a = k - 1 we have that (i) or (ii) holds.

3. The Cases k = 4, 6, 8

Theorem 3.1. There exists a directed cycle C_4 with $\mathfrak{I}(C_4) \geq 2$.

Proof. It follows from Corollary 2.2 for k = 4 that we may assume that for every $0 \le i \le t_3$, every (3+2i)-chord is in A; now recall $3 \le r \le k-1$, r is odd and r = n - [k - 1 + t(k - 2)]. Hence r = 3 and we conclude that $C_4 = (0, 3 + 2t_3, 3 + 2t_3 + 1, 3 + 2t_3 + 2, 0)$ has $\mathcal{J}(C_4) = 3$.

Theorem 3.2. There exists a directed cycle $\mathcal{C}_{h(6)}$ with $\mathfrak{I}(\mathcal{C}_{h(6)}) \geq h(6) - 2$.

Proof. It follows from Corollary 2.2 for k = 6 that we can assume that for every $i, 0 \le i \le t_5$, every (5 + 4i)-chord is in A; recall $3 \le r \le 5$, and r is odd; so $r \in \{3, 5\}$. When r = 3, $\mathcal{C}_4 = (0, 5 + 4t_5) \cup \langle 5 + 4t_5, \gamma, 0 \rangle$ satisfies $\mathcal{J}(\mathcal{C}_4) = 3$; and when r = 5, $\mathcal{C}_6 = (0, 5 + 4t_5) \cup \langle 5 + 4t_5, \gamma, 0 \rangle$ satisfies $\mathcal{J}(\mathcal{C}_6) = 5$.

Theorem 3.3. There exists a directed cycle $\mathcal{C}_{h(8)}$ with $\mathfrak{I}(\mathcal{C}_{h(8)}) \geq h(8) - 3$.

Proof. It follows from Corollary 2.2 for k = 8 that we may assume that for every $i, 0 \le i \le t_7$ every (7 + 6i)-chord is in A; recall $3 \le r \le k - 1 = 7$ and r is odd, so $r \in \{3, 5, 7\}$. When r = 7 we obtain $\mathbb{C}_8 = (0, 7 + 6t_7) \cup \langle 7 + 6t_7, \gamma, 0 \rangle$ a directed cycle with $\mathfrak{I}(\mathbb{C}_8) = 7$. When r = 5 we have $\mathbb{C}_6 = (0, 7 + 6t_7) \cup \langle 7 + 6t_7, \gamma, 0 \rangle$ is a directed cycle with $\mathfrak{I}(\mathbb{C}_6) = 5$. When r = 3, since $n \ge 2k - 4 = 12$ and r = 3 we have $t_7 \ge 1$ and every -9-chord is in A(notice that $(0, n - 9) \in A$ because $n - 9 = 7 + (t_7 - 1)6$ and 0 is an arbitrary vertex) in particular $(n - 2, n - 11) \in A$; now observe that we can assume $(n - 7, 0) \in A$ (otherwise $(0, n - 7) \in A$ and $\mathbb{C}_8 = (0, n - 7) \cup \langle n - 7, \gamma, 0 \rangle$ is a directed cycle with $\mathfrak{I}(\mathbb{C}_8) = 7$); also observe that r = 3 implies $(0, n - 3) \in A$. We conclude that $\mathbb{C}_8 = (n - 2, n - 11) \cup \langle n - 11, \gamma, n - 7 \rangle \cup (n - 7, 0, n - 3, n - 2)$ is a directed cycle with $\mathfrak{I}(\mathbb{C}_8) = 5$.

4. The Case n = 2k - 4

Theorem 4.1. If n = 2k - 4, then there exists a directed cycle $C_{h(k)}$ with $\mathcal{I}(C_{h(k)}) = h(k) - 1$.

Proof. Consider the arc between 0 and k-3; when $(0, k-3) \in A$ we have $C_k = (0, k-3) \cup \langle k-3, \gamma, 0 \rangle$ a directed cycle with $\mathcal{I}(C_k) = k-1$ and when $(k-3,0) \in A$ we obtain $C_{k-2} = (k-3,0) \cup \langle 0,\gamma,k-3 \rangle$ a directed cycle with $\mathcal{I}(C_{k-2}) = k-3$.

5. The Cases r = k - 1 and r = k - 3

Theorem 5.1. If r = k - 1 or r = k - 3, then there exists a directed cycle $\mathcal{C}_{h(k)}$ with $\mathfrak{I}(\mathcal{C}_{h(k)}) = h(k) - 1$.

Proof. If r = k - 1, then $(0, n - (k - 1)) \in A$ and $\mathcal{C}_k = (0, n - (k - 1)) \cup \langle n - (k - 1), \gamma, 0 \rangle$ is a directed cycle with $\mathcal{I}(\mathcal{C}_k) = k - 1$. If r = k - 3, then $(0, n - (k - 3)) \in A$ and $\mathcal{C}_{k-2} = (0, n - (k - 3)) \cup \langle n - (k - 3), \gamma, 0 \rangle$ is a directed cycle with $\mathcal{I}(\mathcal{C}_{k-2}) = k - 3$.

Corollary 5.2. If t = 0, then there exists a directed cycle $C_{h(k)}$ with $\mathcal{I}(C_{h(k)}) = h(k) - 1$.

Proof. If t = 0, then n = k - 1 + r, where $k - 3 \le r \le k - 1$, r is odd (because $n \ge 2k - 4$), so $r \in \{k - 3, k - 1\}$. If r = k - 3, then the assertion follows by Theorem 4.1. If r = k - 1 and thus n = 2k - 2, then we can distinguish the cases $(0, k - 1) \in A$ and $(0, k - 1) \notin A$. If $(0, k - 1) \in A$, then $\mathcal{C}_k = (0, k - 1) \cup \langle k - 1, \gamma, n \rangle$ is a cycle with $\mathfrak{I}(\mathcal{C}_k) = k - 1$. The other case follows analogously.

6. The General Case

In this section, we assume $r \le k - 5$, $t \ge 1$, and $k \ge 10$ (so $n \ge 16$) in view of the results in previous sections.

Lemma 6.1. If the $(k - 1 + \alpha)$ -chord, $\alpha \le r + 1$, with an initial vertex 0 is in A, then at least one of the two following properties holds.

- (i) There exists a directed cycle \mathcal{C}_k with $\mathfrak{I}(\mathcal{C}_k) = k 2$.
- (ii) For every $0 \le i \le t 1$, the $k 1 + \alpha + i(k 2)$ -chord with an initial vertex 0 is in A.

It follows directly from Lemma 2.1; only observe that since $3 \le r \le k-5$ we have $k - 1 + r + 1 + (t - 1)(k - 2) \le k + k - 5 + (t - 1)(k - 2) = k - 1 + t(k - 2) - 2 \le k - 1 + t(k - 2) + r - 5 = n - 5$.

Lemma 6.2. At least one of the two following properties holds

- (i) There exists a directed cycle $\mathcal{C}_{h(k)}$ with $\mathfrak{I}(\mathcal{C}_{h(k)}) \geq h(k) 3$.
- (ii) All the following chords are in A: (a) Every (k-1)-chord. (b) Every (-r)-chord. (c) Every (k-3)-chord and (d) Every -(r+2)-chord.

Proof. Assume that (i) is false. Let us prove that (ii) holds. The proof of (a) follows directly from Corollary 2.2. The proof of (b) follows from Corollary 2.2, observing that n - r = k - 1 + t(k - 2). To prove (c) assume that there is a -(k - 3)-chord, say f = (y, x). It follows from (a) that $(x - 2, y) \in A$, and it follows from (b) that $(x - 2 + r, x - 2) \in A$. Hence, there exists a vertex z in $\langle x - 2 + r, \gamma, y - 1 \rangle$ such that $(z, x - 2) \in A$ and $(x - 2, z + 2) \in A$. We conclude

$$\mathfrak{C}_{k-2} = (y, x) \cup \langle x, \gamma, z \rangle \cup (z, x-2) \cup (x-2, z+2) \cup \langle z+2, \gamma, y \rangle$$

is a directed cycle with $\mathcal{I}(\mathcal{C}_{k-2}) = k - 5 = (k-2) - 3$.

Finally to prove (d), let (y, x) be a (r + 2)-chord. If follows from (c) and Lemma 2.1 that every k - 3 + i(k - 2)-chord is in A for $0 \le i \le t$ in particular $(x + k - 2, x + k - 2 + k - 3 + (t - 1)(k - 2)) \in A$, now observe that x + k - 2 + k - 3 + (t - 1)(k - 2) = y + r + 2 + k - 2 + k - 3 + (t - 1)(k - 2) =y + r + k - 1 + t(k - 2) = y + n = y, so $(x + k - 2, y) \in A$; we conclude that $\mathcal{C}_k = (y, x) \cup \langle x, \gamma, x + k - 2 \rangle \cup (x + k - 2, y)$ is a directed cycle with $\mathcal{J}(\mathcal{C}_k) = k - 2$.

Lemma 6.3. Let $0 \le i \le r+1$, *i* being even. If all the -r-chords, -(r+2)-chords, (k-3+i)-chords and (k-1+i)-chords are in T, then at least one of the following properties holds.

- (i) There exists a directed cycle \mathcal{C}_k with $\mathfrak{I}(\mathcal{C}_k) \geq k-3$.
- (ii) All the -(2r-i+1)-chords, -(2r-i+3)-chords and -(2r-i+5)-chords are in T.

Proof. Assume that the hypothesis of the Lemma holds and (i) is false. Let us prove that (ii) holds.

Since all the [(k-3)+i]-chords and all the [(k-1)+i]-chords are in T, it follows from Lemma 6.1 (taking $\alpha = i-2$) that every [k-3+i+(t-1)(k-2)]-chord is in T, and that (taking $\alpha = i$) every [k-1+i+(t-1)(k-2)]-chord is in T. Thus the following arcs are in T: (r,0), (r+2,0), (0, k-1+(t-1)(k-2)]-(k-2)+i) and (0, k-1+(t-1)(k-2)+i-2).

Let $x_1 = r - 1$, $x_2 = r + 1$, $x_3 = k - 1 + (t - 1)(k - 2) + i - 2$, $x_4 = x_3 + 2$, $x_5 = x_4 + k - 2$, $x_6 = x_5 + 2$, $x_7 = x_5 - 2$, $x_8 = x_7 - 2$. Therefore $(0, x_3) \in A$ and $(0, x_4) \in A$.

Observe that: $\ell\langle x_5, \gamma, 0 \rangle = n - x_5 = r - i$ (because $n - x_5 = k - 1 + t(k - 2)$ +r - (k - 1) - i - t(k - 2) = r - i), $\ell\langle x_6, \gamma, 0 \rangle = r - i - 2$, $\ell\langle x_6, \gamma, x_1 \rangle = 2r - i - 3$ (because $\ell\langle x_6, \gamma, x_1 \rangle = \ell\langle x_6, \gamma, 0 \rangle + r - 1 = r - i - 2 + r - 1$), $\ell\langle x_7, \gamma, x_1 \rangle = \ell\langle x_6, \gamma, x_1 \rangle + 4 = 2r - i + 1$, $\ell\langle x_7, \gamma, x_2 \rangle = \ell\langle x_7, \gamma, x_1 \rangle + 2 = 2r - i + 3$, $\ell\langle x_8, \gamma, x_2 \rangle = \ell\langle x_7, \gamma, x_2 \rangle + 2 = 2r - i + 5$, $\ell\langle x_4, \gamma, x_7 \rangle = \ell\langle x_4, \gamma, x_5 \rangle - 2 = k - 4$ and $\ell\langle x_3, \gamma, x_8 \rangle = \ell\langle x_4, \gamma, x_7 \rangle = k - 4$.

First, we prove that every -(2r - i + 1)-chord is in A. Suppose that there exists a (2r - i + 1)-chord. We can assume without loss of generality that (x_7, x_1) is that chord. Hence $\mathcal{C}_k = (x_7, x_1, x_1 + 1 = r, 0, x_4) \cup \langle x_4, \gamma, x_7 \rangle$ is a directed cycle with $\mathcal{I}(\mathcal{C}_k) = k - 3$.

Now we prove that every -(2r - i + 3)-chord is in A. Assume the contrary; we may assume that (x_7, x_2) is a (2r - i + 3)-chord. Then $C_k = (x_7, x_2 = r + 1, r + 2, 0, x_4) \cup \langle x_4, \gamma, x_7 \rangle$ is a directed cycle with $\mathfrak{I}(\mathcal{C}_k) = k - 3$.

Finally, we prove that every -(2r-i+5)-chord is in T. Assuming the opposite, we may consider that (x_8, x_2) is a (2r-i+5)-chord. Then $\mathcal{C}_k = (x_8, x_2 = r+1, r+2, 0, x_3) \cup \langle x_3, \gamma, x_8 \rangle$ is a directed cycle with $\mathcal{J}(\mathcal{C}_k) = k-3$.

Lemma 6.4. At least one of the following properties holds.

- (i) There exists a directed cycle \mathcal{C}_k with $\mathfrak{I}(\mathcal{C}_k) \geq k-3$.
- (ii) For any even vertex x (resp. odd) there exist at most $\frac{k-4}{2}$ consecutive odd (resp. even) vertices in γ which are in-neighbors of x.

Proof. Assume that (i) does not hold. Assume without loss of generality that x = 0. It follows from Corollary 2.2 that the vertices k - 1 + i(k - 2), for $0 \le i \le t$ are not in-neighbords of 0.

So, there are at most $\frac{k-4}{2}$ odd vertices consecutive in $\langle k-1, \gamma, 0 \rangle$ which are in-neighbors of 0. Since $(0,1) \in A$, also in $\langle 0, \gamma, k-1 \rangle$ there are at most $\frac{k-4}{2}$ odd vertices consecutive in-neighbors of 0.

The following corollary is a directed consequence of this Lemma (only observe that the hypothesis $n \ge 2k - 4$ is not needed in the Lemma).

Corollary 6.5. Let T be a bipartite tournament with n vertices and γ a hamiltonian cycle of T. For each even (resp. odd) vertex x of T such that the number of consecutive odd (resp. even) in-neighbors of x in γ is at least $\frac{k-2}{2}$, $3 \leq k \leq n$, k even, there exists a directed cycle \mathcal{C}_k containing x with $\mathcal{J}(\mathcal{C}_k) \geq k-2$.

Lemma 6.6. If every (k + 1)-chord is in A then at least one of the two following properties holds.

- (i) There exists a directed cycle $\mathcal{C}_{h(k)}$ with $\mathfrak{I}(\mathcal{C}_{h(k)}) \geq h(k) 3$.
- (ii) For every odd α , $0 < \alpha r < k$; every $-[(\alpha + 1)r + 1]$ -chord is in A. And for every even α , $0 \le \alpha r < k$; every $-(\alpha + 1)r$ -chord is in A.

Proof. For $\alpha = 0$, we can assume that every -r-chord is in A (otherwise it follows from Lemma 6.2 that (i) holds and we are done). For $\alpha = 1$, suppose that (x_1, x_0) is a (2r + 1)-chord, let (x_0, x_2) the -r-chord with an initial vertex x_0 and (x_2, x_3) the [(k - 1) + (t - 1)(k - 2)]-chord with an initial vertex x_2 (It follows from Corollary 2.2 that we can assume such a chord exists); clearly, $\ell\langle x_1, \gamma, x_2 \rangle = r + 1$ and $\ell\langle x_3, \gamma, x_1 \rangle = n - (r + 1) - [k - 1 + (t - 1)(k - 2)] = k - 3$. Now notice that $x_3 \in \langle x_0, \gamma, x_1 \rangle - \{x_0, x_1\}$

(because r < k - 3 and $k \ge 10$), thus $\mathbb{C}_k = \langle x_3, \gamma, x_1 \rangle \cup (x_1, x_0, x_2, x_3)$ is a directed cycle with $\mathfrak{I}(\mathbb{C}_k) \ge k - 3$.

We have proved the assertion of Lemma 6.6 for $\alpha = 0$ and $\alpha = 1$. To complete the proof, assume (ii) does not hold for some $\alpha' \ge 2$ and we show that (i) holds. Let α be the least integer $\alpha \ge 2$ for which (ii) does not hold. We analyze two possible cases.

Case 1. α is odd.

We have $\alpha \geq 3$, $0 < \alpha r < k$ and there exists an $[(\alpha + 1)r + 1]$ -chord in A. Since $\alpha - 1$ is even, the choice of α implies that every $-((\alpha - 1) + 1)r$ -chord is in A.

Let (x_1, x_0) be an $[(\alpha + 1)r + 1]$ -chord, (x_0, x_2) the $-\alpha r$ -chord with an initial vertex x_0 and (x_2, x_3) the (k + 1) + (t - 1)(k - 2)-chord with an initial vertex x_2 (it follows from the hypothesis and Lemma 2.1 that these chords are in A). Clearly, $\ell \langle x_1, \gamma, x_2 \rangle = (\alpha + 1)r + 1 - \alpha r = r + 1$ and $\ell \langle x_3, \gamma, x_1 \rangle = n - (r + 1) - [(k + 1) + (t - 1)(k - 2)] = k - 5$. Notice $x_3 \in \langle x_0, \gamma, x_1 \rangle - \{x_0, x_1\}$ because $\alpha r < k$, and $\ell \langle x_2, \gamma, x_3 \rangle \ge k + 1$. We conclude that $\mathcal{C}_{k-2} = \langle x_3, \gamma, x_1 \rangle \cup (x_1, x_0, x_2, x_3)$ is a directed cycle with $\mathcal{J}(\mathcal{C}_{k-2}) \ge k - 5$.

Case 2. α is even.

We have $\alpha \geq 2$, $0 < \alpha r < k$, and there exists an $(\alpha + 1)r$ -chord in A. Since $\alpha - 1$ is odd, the choice of α implies that every $-[(\alpha - 1 + 1)r + 1]$ -chord is in A. Let (x_1, x_0) be an $(\alpha + 1)r$ -chord, (x_0, x_2) the $-(\alpha r + 1)$ -chord with an initial vertex x_0 , and (x_2, x_3) the (k+1) + (t-1)(k-2)-chord with an initial vertex x_2 .

Clearly, $\ell \langle x_1, \gamma, x_2 \rangle = (\alpha + 1)r - \alpha r - 1 = r - 1$ and $\ell \langle x_3, \gamma, x_1 \rangle = n - (r - 1) - [k + 1 + (t - 1)(k - 2)] = k - 3$. Moreover, $x_3 \in \langle x_0, \gamma, x_1 \rangle - \{x_0, x_1\}$ because $\ell \langle x_2, \gamma, x_0 \rangle = \alpha r + 1$, $0 < \alpha r + 1 \le k$ and $\ell \langle x_2, \gamma, x_3 \rangle \ge k + 1$. We obtain $\mathcal{C}_k = \langle x_3, \gamma, x_1 \rangle \cup (x_1, x_0, x_2, x_3)$ a directed cycle with $\mathfrak{I}(\mathcal{C}_k) = k - 3$.

Lemma 6.7. At least one of the following properties holds.

- (i) There exists a directed cycle $\mathcal{C}_{h(k)}$ with $\mathfrak{I}(\mathcal{C}_{h(k)}) \geq h(k) 3$.
- (ii) For $i even -2 \le i \le r+1$; every -(2r+1-i)-chord and every (k-1+i)-chord is in A.

Proof. Suppose (i) does not hold, we shall prove that property (ii) holds by induction on *i*. We start with i = -2 and i = 0; namely, we prove that the

following chords are in A: (a) every (k-3)-chord, (b) every (k-1)-chord, (c) every -(2r+3)-chord and (d) every -(2r+1)-chord.

The proof of (a) and (b) follows directly from Lemma 6.2. Let 0 be any vertex of T. It follows from Lemma 6.1 (with $\alpha = 0$) and from Lemma 6.2 (part (b) and (d)) that the following chords are in A: (0, k-1+(t-1)(k-2)), (r+2, 0) and (r, 0). Part (c): every -(2r+3)-chord is in A. If $(n-r-1, r+2) \in A$, then $\mathcal{C}_k = (n-r-1, r+2, 0, k-1+(t-1)(k-2)) \cup \langle k-1+(t-1)(k-2), \gamma, n-r-1 \rangle$ is a directed cycle with $\mathcal{J}(\mathcal{C}_k) = k-3$, a contradiction. (Notice that $k-1+(t-1)(k-2) \in \langle r+2, \gamma, n-r-1 \rangle - \{r+2, n-r-1\}$ since $r \leq k-5$; moreover $\ell \langle k-1+(t-1)(k-2), \gamma, n-r-1 \rangle = n-r-1-(k-1)-(t-1)(k-2) = k-3$).

Part (d): every -(2r+1)-chord is in A. If $(n-r-2, r-1) \in A$, then $C_k = (n-r-2, r-1, r, 0, k-1+(t-1)(k-2)) \cup \langle k-1+(t-1)(k-2), \gamma, n-r-2 \rangle$ is a directed cycle with $\mathcal{I}(\mathcal{C}_k) = k-3$, a contradiction. (Notice that since $r \leq k-5 < k-1$ and $k \geq 10$ we have r < k-1+(t-1)(k-2) < n-(r+2), and $\ell \langle k-1+(t-1)(k-2), \gamma, n-r-2 \rangle = n-(r+2)-[k-1+(t-1)(k-2)] = k-4$). Assume that (ii) in Lemma 6.7 holds for each i' even, $0 \leq i' \leq i$ and let us prove it for i+2; namely, we prove

- (α) Every (k+1+i)-chord is in $A, 0 \le i \le r-1$.
- (β) Every -(2r-1-i)-chord is in $A, 0 \le i \le r-1$.

Proof of (α). It follows from the inductive hypothesis that for each j even, $0 \leq j \leq i$, every [(k-1)+j]-chord and every [(k-3)+j]-chord is in A (because for j = 0 we have proved that every (k-3)-chord is in A). It follows from Lemma 6.2 that every (-r)-chord and every -(r+2)-chord is in A. Therefore it follows from Lemma 6.3 that for even j, $0 \leq j \leq i$ every -(2r-j+1)-chord, -(2r-j+3)-chord and -(2r-j+5)-chord is in A. That means that for each even j, $-4 \leq j \leq i \leq r-1$ every -(2r-j+1)-chord is in A. These are $\frac{i}{2} + 3$ chords with initial odd (resp. even) vertices consecutive in γ .

Assume by contradiction that $(x_3, 0)$ is a -(k + 1 + i)-chord, i being even $0 \le i \le r - 1$. Let $x_0 = n - (2r - i - 1)$. Hence letting $x_2 = 2$ we have that (x_2, x_0) is a -(2r - i + 1)-chord (we have observed that every -(2r - i + 1)-chord is in A).

First, we prove that $x_0 \in \langle x_3+1, \gamma, n-1 \rangle$: $\ell \langle x_0, \gamma, 0 \rangle = 2r - i - 1 \ge r \ge 3$, $\ell \langle x_3, \gamma, x_0 \rangle = n - (k + 1 + i + 2r - i - 1) = k - 1 + t(k - 2) + r - k - 2r \ge k - 1 + k - 2 - r - k = k - 3 - r \ge 2$ (remember $3 \le r \le k - 5$). Now, there exists an out-neighbor of x_0 , say x, such that x is in $\langle x_2, \gamma, x_3-1 \rangle$ this is a direct consequence of Lemma 6.4 and the fact that the number even vertices in $\langle x_2, \gamma, x_3 - 1 \rangle$ is at least $\frac{k-2}{2}$ (Notice that x_0 is odd, $\ell \langle x_2, \gamma, x_3 - 1 \rangle = k + 1 + i - 3 = k + i - 2 \ge k - 2$). Let x_4 be the smallest (the nearest to 0 in γ) such vertex.

Let $x_1 = 0$, we will prove that $x_4 - i - 4 \in \langle x_1, \gamma, x_4 - 3 \rangle$. Since for each j, j even $-4 \leq j \leq i \leq r - 1$, every -(2r - j + 1)-chord is in A, it follows that

$$\{(2, x_0), (4, x_0), (6, x_0), \dots, (i+4, x_0), (i+6, x_0)\} \subseteq A.$$

Hence the selection of x_4 implies $x_4 \ge i + 8$, so $x_4 - i - 4 > 3$.

Finally, since $\ell \langle x_4, \gamma, x_3 \rangle + \ell \langle x_1, \gamma, x_4 - i - 4 \rangle = k + 1 + i - (i + 4) = k - 3$ it follows that $\mathbb{C}_k = (x_4 - i - 4, x_0, x_4) \cup \langle x_4, \gamma, x_3 \rangle \cup (x_3, x_1) \cup \langle x_1, \gamma, x_4 - i - 4 \rangle$ is a directed cycle with $\mathfrak{I}(\mathbb{C}_k) = k - 3$ (Notice $(x_4 - i - 4, x_0) \in A$ by the choice of x_4 and the fact $x_4 - i - 4 \in \langle x_1, \gamma, x_4 - 3 \rangle$).

Proof of (β). Part (β) follows from Lemma 6.3 (taking i + 2 instead of i) and the following facts:

Every (k - 1 + i)-chord is in A for even $i, -2 \le i \le r + 1$ (it follows from part (α)).

Every (k-3+i)-chord is in A for even $i, 0 \le i \le r+1$ (it follows from the assertion of above).

Every (-r)-chord and every -(r+2)-chord is in A (it follows from Lemma 6.2).

Theorem 6.8. If $n \ge 2k - 4$, then there exists a directed cycle $\mathcal{C}_{h(k)}$ with $\mathfrak{I}(\mathcal{C}_{h(k)}) \ge h(k) - 3$.

Proof. The case n = 2k - 4 is considered in Section 4. Assume n > 2k - 4 and suppose by contradiction that there is no directed cycle $\mathcal{C}_{h(k)}$ with $\mathcal{I}(\mathcal{C}_{h(k)}) \ge h(k) - 3$.

It follows from Lemma 6.7 that for each even $i, -2 \le i \le r+1$ every (k-1+i)-chord is in A, in particular

$$(1) \qquad \{(0,k-3),(0,k-1),(0,k+1),(0,k+3),\ldots,(0,k+r-2),(0,k+r)\} \subset A$$

(Notice that k + r < n - 1 because $t \ge 1$ and $k \ge 10$).

It follows from Lemma 6.2 that every (-r)-chord is in A, and by Lemma 6.7 that every (k+1)-chord is in A. Therefore, by Lemma 6.6 we have: For every odd α , $0 < \alpha r < k$, every $-[(\alpha+1)r+1]$ -chord is in A. And for every even α , $0 \leq \alpha r < k$, every $-(\alpha+1)r$ -chord is in A. Let $\alpha_0 = \max\{\alpha \in \mathbb{N} \mid \alpha r < k\}$. Clearly, $\alpha_0 r < k$. We will analize the two possible cases:

Case 1. α_0 is even.

It follows from Lemma 6.6 that every $-(\alpha_0+1)r$ -chord is in A, in particular $((\alpha_0+1)r, 0) \in A$. On the other hand, $\alpha_0 r < k$ implies $(\alpha_0+1)r < k+r$ and the selection of α_0 implies $k < (\alpha_0+1)r < k+r$. Thus $y = (\alpha_0+1)r \in \{k+1, k+3, k+5, \ldots, k+r\}$; thus we have $(y, 0) \in A$ and (1) implies $(0, y) \in A$. A contradiction.

Case 2. α_0 id odd.

It follows from Lemma 6.6 that every $-[(\alpha_0 + 1)r + 1]$ -chord is in A, in particular $((\alpha_0 + 1)r + 1, 0) \in A$. On the other hand, $\alpha_0 r < k$ and the choice of α_0 implies $k + 1 \le (\alpha_0 + 1)r + 1 \le k + r$, $y = (\alpha_0 + 1)r + 1$ is odd and $y \in \{k + 1, k + 3, k + 5, \dots, k + r\}$. So it follows from (1) that $(0, y) \in A$ and we have proved $(y, 0) \in A$. A contradiction.

7. Remarks

In this section, it is proved that the hypothesis of Theorem 6.8 is tight.

Definition 7.1. A digraph D with vertex set V is called *cyclically p-partite* complete $(p \ge 3)$ provided one can partition $V = V_0 + V_1 + \cdots + V_{p-1}$ so that (u, v) is an arc of D if and only if $u \in V_i, v \in V_{i+1}$ (notation modulo p).

Remark 7.2. The cyclically 4-partite complete digraph T_4 is a bipartite tournament and clearly every directed cycle of T_4 has length $\equiv 0 \pmod{4}$. So for k = 4m + 2, T_4 has no directed cycles of length k and for k = 4m, T_4 has no directed cycles of length k = 4m, T_4 has no directed cycles of length k = 2.

Now we consider the following simple lemma.

Lemma 7.3. Let $\mathcal{C}_{h(k)}$ be a directed cycle with $\mathfrak{I}(\mathcal{C}_{h(k)}) = h(k) - 2$. If $f_1 = (0, x_1), f_2 = (y_1, y_2)$ are the arcs of $\mathcal{C}_{h(k)}$ not in γ , then $y_2 = y_1 + n - (h(k) - 2 + x_1)$. Namely, f_2 is $a - (x_1 + (h(k) - 2))$ -chord of γ .

Remark 7.4. For $n \ge 5$, $k \ge 5$, such that $n \ne k + s(k-2) + m(k-4)$ and $n \ne s(k-2) + m(k-4)$ with $s, m \in \mathbb{N}$, there exists a bipartite hamiltonian tournament T_n with no directed cycles $\mathcal{C}_{h(k)}$ with $\mathcal{I}(\mathcal{C}_{h(k)}) = h(k) - 2$.

Proof. Define T_n as follows: Let

$$C = \{(i, i+k-1+s(k-2)+m(k-4)) \mid i \in \{0, 1, \dots, n-1\}, s, m \in \mathbb{N} \text{ with} \\ (k-1)+s(k-2)+m(k-4) < n-1\}$$

and

 $F = \{(i, i+k-3+s(k-2)+m(k-4)) \mid i \in \{0, 1, \dots, n-1\}, s, m \in \mathbb{N} \text{ with }$

$$(k-3) + s(k-2) + m(k-4) < n-1\},$$

$$A(T_n) = C \cup F \cup \left(\left\{ (i+j,i) \mid j \in \left\{2,3,\dots, \left\lfloor \frac{n-1}{2} \right\rfloor \right\} \right\} - (C \cup F) \right)$$

$$\cup \{(i,i+1) \mid i \in \{0,1,\dots,n-1\}\} \cup \left\{ \left(i+\frac{n}{2},i\right) \mid i \in \left\{0,1,\dots,\frac{n}{2}-1\right\} \right\}.$$

Clearly, there is no directed cycle $C_{h(k)}$ with $\mathcal{I}(C_{h(k)}) = h(k) - 1$ (Notice that T_n has every (k-1)-chord and every (k-3)-chord). Now assume for contradiction that $C_{h(k)}$ is a directed cycle of T_n with $\mathcal{I}(C_{h(k)}) = k - 2$, and let $f_1 = (0, x_1), f_2 = (y_1, y_2)$ the only arcs of C_k not in γ . Without loss of generality we can assume $\ell(f_1) < \frac{n}{2}$. The definition of T_n implies that $x_1 = k - 1 + s(k - 2) + m(k - 4)$ or $x_1 = k - 3 + s(k - 2) + m(k - 4)$. It follows from Lemma 7.3 that y_2 has one of the following forms:

(a)
$$y_2 = y_1 + n - [k - 1 + (s + 1)(k - 2) + m(k - 4)].$$

When k - 1 + (s + 1)(k - 2) + m(k - 4) < n - 1 we obtain that f_2 is a -(k - 1 + (s + 1)(k - 2) + m(k - 4))-chord, contradicting the definition of T_n .

When $k-1+(s+1)(k-2)+m(k-4) \ge n-1$ we have that $\ell\langle x_1, \gamma, 0 \rangle \le k-1$ and the fact $\mathcal{C}_k - \{(0, x_1), (y_1, y_2)\} \subseteq \langle x_1, \gamma, 0 \rangle$ implies $\ell\langle x_1, \gamma, 0 \rangle \ge k-3$; and since $\ell\langle x_1, \gamma, 0 \rangle$ is odd we have $\ell\langle x_1, \gamma, 0 \rangle \in \{k-1, k-3\}$. Now if $\ell\langle x_1, \gamma, 0 \rangle = k-1$, then $n = x_1+k-1 = k-1+s(k-2)+m(k-4)+k-1 = k-1+s(k-2)+k-1 = k-1$

k + (s+1)(k-2) + m(k-4), a contradiction. If $\ell \langle x_1, \gamma, 0 \rangle = k-3$, then $n = x_1+k-3 = k-1+s(k-2)+m(k-1)+k-3 = k+s(k-2)+(m+1)(k-4)$, a contradiction.

(b)
$$y_2 = y_1 + n - [(k-1) + s(k-2) + (m+1)(k-4)].$$

- (c) $y_2 = y_1 + n [(k-3) + (s+1)(k-2) + m(k-4)].$
- (d) $y_2 = y_1 + n [(k-3) + s(k-2) + (m+1)(k-4)].$

Cases (b), (c) and (d) can be analized in a completly analogous form as the case (a) to get a contradiction.

It is easy to verify that if n = k + s(k-2) + m(k-4) or n = s(k-2) + m(k-4) with $s, m \in \mathbb{N}$, then T_n (any bipartite hamiltonian tournament with n vertices) has a directed cycle $\mathcal{C}_{h(k)}$ with $\mathcal{I}(\mathcal{C}_{h(k)}) \ge h(k) - 2$.

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