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CYCLE-PANCYCLISM IN BIPARTITE TOURNAMENTS I

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Abstract

Let T be a hamiltonian bipartite tournament with n vertices, γ a hamiltonian directed cycle of T, and k an even number. In this paper, the following question is studied: What is the maximum intersection with γ of a directed cycle of length k? It is proved that for an even k in the range $4 \le k \le \frac{n+4}{2}$, there exists a directed cycle $\mathcal{C}_{h(k)}$ of length $h(k)$, $h(k) \in \{k, k-2\}$ with $|A(\mathcal{C}_{h(k)}) \cap A(\gamma)| \ge h(k) - 3$ and the result is best possible.

In a forthcoming paper the case of directed cycles of length k, k even and $k > \frac{n+4}{2}$ will be studied.

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1. Introduction

The subject of pancyclism has been studied by several authors (e.g. [1, 2, 5, 10, 12, 14]). Three types of pancyclism have been considered. A digraph D is pancyclic if it has directed cycles of all the possible lengths; D is vertex-pancyclic if given any vertex v there are directed cycles of every length containing v; and D is arc-pancyclic if given any arc e there are directed cycles of every length containing e.

It is well known that a hamiltonian bipartite tournament is pancyclic, and vertex-pancyclic (with only very few exceptions) but not necessarily arc-pancyclic (see e.g. [3, 11, 13]). Within the concept of cycle-pancyclism the following question is studied: Given a directed cycle γ of a digraph D, find the maximum number of arcs which a directed cycle of length k (if such a directed cycle exists) contained in $D[V(\gamma)]$ (the subdigraph of D induced by $V(\gamma)$) has in common with γ. Cycle-pancyclism in tournaments has been studied in [6, 7, 8] and [9]. In this paper, the cycle-pancyclism in bipartite tournaments is investigated. In order to do so, it is sufficient to consider a hamiltonian bipartite tournament T where γ is a hamiltonian directed cycle (because we are looking for directed cycles of length k contained in $D[V(\gamma)]$ whose arcs intersect the arcs of γ the most possible). We will assume (without saying it explicitly in Lemmas, Theorems or Corollaries) that we are working in a hamiltonian bipartite tournament with a vertex set $V = \{0, 1, \ldots, n-1\}$ and an arc set A. Also we assume without loss of generality that $\gamma = (0, 1, \ldots, n-1, 0)$ is a hamiltonian directed cycle of T; k will be an even number; $\mathcal{C}_{h(k)}$ will denote a directed cycle of length $h(k)$ with $h(k) \in \{k, k-2\}$ and $\mathcal{I}\left(\mathcal{C}_{h(k)}\right) = |A(\mathcal{C}_{h(k)}) \cap A(\gamma)|$. This paper is the first part of the study of the existence of a directed cycle $\mathcal{C}_{h(k)}$ where $\mathcal{I}(\mathcal{C}_{h(k)})$ is the maximum. For general concepts we refer the reader to [4].

2. Preliminaries

A chord of a cycle $\mathcal{C}$ is an arc not in $\mathcal{C}$ but with both terminal vertices in $\mathcal{C}$. The length of a chord $f = (u, v)$ of $\mathcal{C}$ denoted $\ell(f)$ is equal to the length of $\langle u, \mathcal{C}, v\rangle$, where $\langle u, \mathcal{C}, v\rangle$ denotes the uv-directed path contained in $\mathcal{C}$. We say that f is a c-chord if $\ell(f) = c$ and $f = (u, v)$ is an a $-c$-chord if $\ell(\langle v, \mathcal{C}, u\rangle) = c$. Observe that if f is a c-chord, then it is also an a $-(n-c)$-chord. All the chords considered in this paper are chords of γ. We will denote by $\mathcal{C}_k$ a directed cycle of length k. In what follows all notation is taken modulo n.

For any a, $2 \leq a \leq n-2$, denote by t_a the largest integer such that $a + t_a(k-2) < n-1$. The important case of t_{k-1} is denoted by t in the rest of the paper. Let r be defined as follows: $r = n - [k - 1 + t(k-2)]$. Notice that: If $a \leq b$ then $t_a \geq t_b$; $t \geq 0$ and $3 \leq r \leq k-1$, r is odd.

Lemma 2.1. *If the a-chord with an initial vertex* 0 *(*0 *being an arbitrary vertex of* T*) is in* A*, then at least one of the two following properties holds.*

(i) *There exists a directed cycle* $\mathcal{C}_k$ *with* $\mathcal{I}(\mathcal{C}_k) = k-2$.

(ii) *For every* $0 \leq i \leq t_a$*, the* $a + i(k-2)$*-chord with an initial vertex* 0 *is in* A*.*

Proof. Suppose that (ii) in the lemma is false, and let

$$j = \min\{i \in \{1, 2, \ldots, t_a\} \mid (a + i(k-2), 0) \in A\},$$

then

$$\mathcal{C}_k = (0, a+(j-1)(k-2)) \cup \langle a+(j-1)(k-2), \gamma, a+j(k-2)\rangle \cup (a+j(k-2), 0)$$

is a directed cycle with $\mathcal{I}(\mathcal{C}_k) = k-2$. ■

Corollary 2.2. *At least one of the two following properties holds*

(i) *There exists a directed cycle* $\mathcal{C}_k$ *with* $\mathcal{I}(\mathcal{C}_k) \geq k-2$*.*

(ii) *For every* $0 \leq i \leq t$*, every* $(k-1+i(k-2))$*-chord is in* A*.*

Proof. Clearly, for any vertex 0, $(0, k-1) \in A$ since otherwise $(k-1, 0) \in A$ and $\mathcal{C}_k = \langle 0, \gamma, k-1\rangle \cup (k-1, 0)$ is a directed cycle with $\mathcal{I}(\mathcal{C}_k) = k-1$ and thus (i) holds. Now applying Lemma 2.1 with $a = k-1$ we have that (i) or (ii) holds. ■

3. The Cases $k = 4, 6, 8$

Theorem 3.1. *There exists a directed cycle* $\mathcal{C}_4$ *with* $\mathcal{I}(\mathcal{C}_4) \geq 2$*.*

Proof. It follows from Corollary 2.2 for $k = 4$ that we may assume that for every $0 \leq i \leq t_3$, every $(3+2i)$-chord is in A; now recall $3 \leq r \leq k-1$, r is odd and $r = n - [k-1+t(k-2)]$. Hence $r = 3$ and we conclude that $\mathcal{C}_4 = (0, 3+2t_3, 3+2t_3+1, 3+2t_3+2, 0)$ has $\mathcal{I}(\mathcal{C}_4) = 3$. ■

Theorem 3.2. *There exists a directed cycle* $\mathcal{C}_{h(6)}$ *with* $\mathcal{I}(\mathcal{C}_{h(6)}) \geq h(6) - 2$*.*

Proof. It follows from Corollary 2.2 for $k = 6$ that we can assume that for every i, $0 \leq i \leq t_5$, every $(5+4i)$-chord is in A; recall $3 \leq r \leq 5$, and r is odd; so $r \in \{3, 5\}$. When $r = 3$, $\mathcal{C}_4 = (0, 5+4t_5) \cup \langle 5+4t_5, \gamma, 0\rangle$ satisfies $\mathcal{I}(\mathcal{C}_4) = 3$; and when $r = 5$, $\mathcal{C}_6 = (0, 5+4t_5) \cup \langle 5+4t_5, \gamma, 0\rangle$ satisfies $\mathcal{I}(\mathcal{C}_6) = 5$. ■

Theorem 3.3. *There exists a directed cycle $\mathcal{C}_{h(8)}$ with $\mathcal{I}(\mathcal{C}_{h(8)}) \geq h(8) - 3$.*

Proof. It follows from Corollary 2.2 for $k = 8$ that we may assume that for every i, $0 \leq i \leq t_7$ every $(7 + 6i)$-chord is in A; recall $3 \leq r \leq k - 1 = 7$ and r is odd, so $r \in \{3, 5, 7\}$. When $r = 7$ we obtain $\mathcal{C}_8 = (0, 7 + 6t_7) \cup \langle 7 + 6t_7, \gamma, 0\rangle$ a directed cycle with $\mathcal{I}(\mathcal{C}_8) = 7$. When $r = 5$ we have $\mathcal{C}_6 = (0, 7 + 6t_7) \cup \langle 7 + 6t_7, \gamma, 0\rangle$ is a directed cycle with $\mathcal{I}(\mathcal{C}_6) = 5$. When $r = 3$, since $n \geq 2k - 4 = 12$ and $r = 3$ we have $t_7 \geq 1$ and every -9-chord is in A (notice that $(0, n - 9) \in A$ because $n - 9 = 7 + (t_7 - 1)6$ and 0 is an arbitrary vertex) in particular $(n - 2, n - 11) \in A$; now observe that we can assume $(n - 7, 0) \in A$ (otherwise $(0, n - 7) \in A$ and $\mathcal{C}_8 = (0, n - 7) \cup \langle n - 7, \gamma, 0\rangle$ is a directed cycle with $\mathcal{I}(\mathcal{C}_8) = 7$); also observe that $r = 3$ implies $(0, n - 3) \in A$. We conclude that $\mathcal{C}_8 = (n - 2, n - 11) \cup \langle n - 11, \gamma, n - 7\rangle \cup (n - 7, 0, n - 3, n - 2)$ is a directed cycle with $\mathcal{I}(\mathcal{C}_8) = 5$. ■

4. The Case $n = 2k - 4$

Theorem 4.1. *If $n = 2k - 4$, then there exists a directed cycle $\mathcal{C}_{h(k)}$ with $\mathcal{I}(\mathcal{C}_{h(k)}) = h(k) - 1$.*

Proof. Consider the arc between 0 and $k - 3$; when $(0, k - 3) \in A$ we have $\mathcal{C}_k = (0, k - 3) \cup \langle k - 3, \gamma, 0\rangle$ a directed cycle with $\mathcal{I}(\mathcal{C}_k) = k - 1$ and when $(k - 3, 0) \in A$ we obtain $\mathcal{C}_{k-2} = (k - 3, 0) \cup \langle 0, \gamma, k - 3\rangle$ a directed cycle with $\mathcal{I}(\mathcal{C}_{k-2}) = k - 3$. ■

5. The Cases $r = k - 1$ and $r = k - 3$

Theorem 5.1. *If $r = k - 1$ or $r = k - 3$, then there exists a directed cycle $\mathcal{C}_{h(k)}$ with $\mathcal{I}(\mathcal{C}_{h(k)}) = h(k) - 1$.*

Proof. If $r = k - 1$, then $(0, n - (k - 1)) \in A$ and $\mathcal{C}_k = (0, n - (k - 1)) \cup \langle n - (k - 1), \gamma, 0\rangle$ is a directed cycle with $\mathcal{I}(\mathcal{C}_k) = k - 1$. If $r = k - 3$, then $(0, n - (k - 3)) \in A$ and $\mathcal{C}_{k-2} = (0, n - (k - 3)) \cup \langle n - (k - 3), \gamma, 0\rangle$ is a directed cycle with $\mathcal{I}(\mathcal{C}_{k-2}) = k - 3$. ■

Corollary 5.2. *If $t = 0$, then there exists a directed cycle $\mathcal{C}_{h(k)}$ with $\mathcal{I}(\mathcal{C}_{h(k)}) = h(k) - 1$.*

Proof. If $t = 0$, then $n = k - 1 + r$, where $k - 3 \leq r \leq k - 1$, r is odd (because $n \geq 2k - 4$), so $r \in \{k-3, k-1\}$. If $r = k - 3$, then the assertion follows by Theorem 4.1. If $r = k - 1$ and thus $n = 2k - 2$, then we can distinguish the cases $(0, k-1) \in A$ and $(0, k-1) \notin A$. If $(0, k-1) \in A$, then $\mathcal{C}_k = (0, k-1) \cup \langle k-1, \gamma, n \rangle$ is a cycle with $\mathcal{I}(\mathcal{C}_k) = k - 1$. The other case follows analogously. ■

6. The General Case

In this section, we assume $r \leq k - 5$, $t \geq 1$, and $k \geq 10$ (so $n \geq 16$) in view of the results in previous sections.

Lemma 6.1. *If the $(k - 1 + \alpha)$-chord, $\alpha \leq r + 1$, with an initial vertex 0 is in A, then at least one of the two following properties holds.*

(i) *There exists a directed cycle $\mathcal{C}_k$ with $\mathcal{I}(\mathcal{C}_k) = k - 2$.*

(ii) *For every $0 \leq i \leq t - 1$, the $k - 1 + \alpha + i(k-2)$-chord with an initial vertex 0 is in A.*

It follows directly from Lemma 2.1; only observe that since $3 \leq r \leq k - 5$ we have $k - 1 + r + 1 + (t-1)(k-2) \leq k + k - 5 + (t-1)(k-2) = k - 1 + t(k-2) - 2 \leq k - 1 + t(k-2) + r - 5 = n - 5$.

Lemma 6.2. *At least one of the two following properties holds*

(i) *There exists a directed cycle $\mathcal{C}_{h(k)}$ with $\mathcal{I}(\mathcal{C}_{h(k)}) \geq h(k) - 3$.*

(ii) *All the following chords are in A:* (a) *Every $(k-1)$-chord.* (b) *Every $(-r)$-chord.* (c) *Every $(k-3)$-chord and* (d) *Every $-(r+2)$-chord.*

Proof. Assume that (i) is false. Let us prove that (ii) holds. The proof of (a) follows directly from Corollary 2.2. The proof of (b) follows from Corollary 2.2, observing that $n - r = k - 1 + t(k-2)$. To prove (c) assume that there is a $-(k-3)$-chord, say $f = (y, x)$. It follows from (a) that $(x - 2, y) \in A$, and it follows from (b) that $(x - 2 + r, x - 2) \in A$. Hence, there exists a vertex z in $\langle x - 2 + r, \gamma, y - 1 \rangle$ such that $(z, x - 2) \in A$ and $(x - 2, z + 2) \in A$. We conclude

$$\mathcal{C}_{k-2} = (y, x) \cup \langle x, \gamma, z \rangle \cup (z, x-2) \cup (x-2, z+2) \cup \langle z+2, \gamma, y \rangle$$

is a directed cycle with $\mathcal{I}(\mathcal{C}_{k-2}) = k - 5 = (k-2) - 3$.

Finally to prove (d), let (y, x) be a $(r+2)$-chord. If follows from (c) and Lemma 2.1 that every $k-3+i(k-2)$-chord is in A for $0 \leq i \leq t$ in particular $(x+k-2, x+k-2+k-3+(t-1)(k-2)) \in A$, now observe that $x+k-2+k-3+(t-1)(k-2) = y+r+2+k-2+k-3+(t-1)(k-2) = y+r+k-1+t(k-2) = y+n = y$, so $(x+k-2, y) \in A$; we conclude that $\mathcal{C}_k = (y, x) \cup \langle x, \gamma, x+k-2\rangle \cup (x+k-2, y)$ is a directed cycle with $\mathcal{I}(\mathcal{C}_k) = k-2$. ■

Lemma 6.3. *Let $0 \leq i \leq r+1$, i being even. If all the $-r$-chords, $-(r+2)$-chords, $(k-3+i)$-chords and $(k-1+i)$-chords are in T, then at least one of the following properties holds.*

(i) *There exists a directed cycle $\mathcal{C}_k$ with $\mathcal{I}(\mathcal{C}_k) \geq k-3$.*

(ii) *All the $-(2r-i+1)$-chords, $-(2r-i+3)$-chords and $-(2r-i+5)$-chords are in T.*

Proof. Assume that the hypothesis of the Lemma holds and (i) is false. Let us prove that (ii) holds.

Since all the $[(k-3)+i]$-chords and all the $[(k-1)+i]$-chords are in T, it follows from Lemma 6.1 (taking $\alpha = i-2$) that every $[k-3+i+(t-1)(k-2)]$-chord is in T, and that (taking $\alpha = i$) every $[k-1+i+(t-1)(k-2)]$-chord is in T. Thus the following arcs are in T: $(r, 0)$, $(r+2, 0)$, $(0, k-1+(t-1)(k-2)+i)$ and $(0, k-1+(t-1)(k-2)+i-2)$.

Let $x_1 = r-1$, $x_2 = r+1$, $x_3 = k-1+(t-1)(k-2)+i-2$, $x_4 = x_3+2$, $x_5 = x_4+k-2$, $x_6 = x_5+2$, $x_7 = x_5-2$, $x_8 = x_7-2$. Therefore $(0, x_3) \in A$ and $(0, x_4) \in A$.

Observe that: $\ell\langle x_5, \gamma, 0\rangle = n - x_5 = r-i$ (because $n - x_5 = k-1+t(k-2)+r-(k-1)-i-t(k-2) = r-i$), $\ell\langle x_6, \gamma, 0\rangle = r-i-2$, $\ell\langle x_6, \gamma, x_1\rangle = 2r-i-3$ (because $\ell\langle x_6, \gamma, x_1\rangle = \ell\langle x_6, \gamma, 0\rangle + r-1 = r-i-2+r-1$), $\ell\langle x_7, \gamma, x_1\rangle = \ell\langle x_6, \gamma, x_1\rangle + 4 = 2r-i+1$, $\ell\langle x_7, \gamma, x_2\rangle = \ell\langle x_7, \gamma, x_1\rangle + 2 = 2r-i+3$, $\ell\langle x_8, \gamma, x_2\rangle = \ell\langle x_7, \gamma, x_2\rangle + 2 = 2r-i+5$, $\ell\langle x_4, \gamma, x_7\rangle = \ell\langle x_4, \gamma, x_5\rangle - 2 = k-4$ and $\ell\langle x_3, \gamma, x_8\rangle = \ell\langle x_4, \gamma, x_7\rangle = k-4$.

First, we prove that every $-(2r-i+1)$-chord is in A. Suppose that there exists a $(2r-i+1)$-chord. We can assume without loss of generality that (x_7, x_1) is that chord. Hence $\mathcal{C}_k = (x_7, x_1, x_1+1 = r, 0, x_4) \cup \langle x_4, \gamma, x_7\rangle$ is a directed cycle with $\mathcal{I}(\mathcal{C}_k) = k-3$.

Now we prove that every $-(2r-i+3)$-chord is in A. Assume the contrary; we may assume that (x_7, x_2) is a $(2r-i+3)$-chord. Then $\mathcal{C}_k = (x_7, x_2 = r+1, r+2, 0, x_4) \cup \langle x_4, \gamma, x_7\rangle$ is a directed cycle with $\mathcal{I}(\mathcal{C}_k) = k-3$.

Finally, we prove that every $-(2r-i+5)$-chord is in T. Assuming the opposite, we may consider that (x_8, x_2) is a $(2r-i+5)$-chord. Then $\mathcal{C}_k = (x_8, x_2 = r+1, r+2, 0, x_3) \cup \langle x_3, \gamma, x_8 \rangle$ is a directed cycle with $\mathcal{I}(\mathcal{C}_k) = k-3$. ■

Lemma 6.4. *At least one of the following properties holds.*

(i) *There exists a directed cycle $\mathcal{C}_k$ with $\mathcal{I}(\mathcal{C}_k) \geq k-3$.*

(ii) *For any even vertex x (resp. odd) there exist at most $\frac{k-4}{2}$ consecutive odd (resp. even) vertices in γ which are in-neighbors of x.*

Proof. Assume that (i) does not hold. Assume without loss of generality that $x = 0$. It follows from Corollary 2.2 that the vertices $k-1+i(k-2)$, for $0 \leq i \leq t$ are not in-neighbords of 0.

So, there are at most $\frac{k-4}{2}$ odd vertices consecutive in $\langle k-1, \gamma, 0 \rangle$ which are in-neighbors of 0. Since $(0,1) \in A$, also in $\langle 0, \gamma, k-1 \rangle$ there are at most $\frac{k-4}{2}$ odd vertices consecutive in-neighbors of 0. ■

The following corollary is a directed consequence of this Lemma (only observe that the hypothesis $n \geq 2k-4$ is not needed in the Lemma).

Corollary 6.5. *Let T be a bipartite tournament with n vertices and γ a hamiltonian cycle of T. For each even (resp. odd) vertex x of T such that the number of consecutive odd (resp. even) in-neighbors of x in γ is at least $\frac{k-2}{2}$, $3 \leq k \leq n$, k even, there exists a directed cycle $\mathcal{C}_k$ containing x with $\mathcal{I}(\mathcal{C}_k) \geq k-2$.*

Lemma 6.6. *If every $(k+1)$-chord is in A then at least one of the two following properties holds.*

(i) *There exists a directed cycle $\mathcal{C}_{h(k)}$ with $\mathcal{I}(\mathcal{C}_{h(k)}) \geq h(k)-3$.*

(ii) *For every odd α, $0 < \alpha r < k$; every $-[(\alpha+1)r+1]$-chord is in A. And for every even α, $0 \leq \alpha r < k$; every $-(\alpha+1)r$-chord is in A.*

Proof. For $\alpha = 0$, we can assume that every $-r$-chord is in A (otherwise it follows from Lemma 6.2 that (i) holds and we are done). For $\alpha = 1$, suppose that (x_1, x_0) is a $(2r+1)$-chord, let (x_0, x_2) the $-r$-chord with an initial vertex x_0 and (x_2, x_3) the $[(k-1)+(t-1)(k-2)]$-chord with an initial vertex x_2 (It follows from Corollary 2.2 that we can assume such a chord exists); clearly, $\ell\langle x_1, \gamma, x_2 \rangle = r+1$ and $\ell\langle x_3, \gamma, x_1 \rangle = n-(r+1)-[k-1+(t-1)(k-2)] = k-3$. Now notice that $x_3 \in \langle x_0, \gamma, x_1 \rangle - \{x_0, x_1\}$

(because $r < k-3$ and $k \geq 10$), thus $\mathcal{C}_k = \langle x_3, \gamma, x_1\rangle \cup (x_1, x_0, x_2, x_3)$ is a directed cycle with $\mathcal{I}(\mathcal{C}_k) \geq k-3$.

We have proved the assertion of Lemma 6.6 for $\alpha = 0$ and $\alpha = 1$. To complete the proof, assume (ii) does not hold for some $\alpha' \geq 2$ and we show that (i) holds. Let α be the least integer $\alpha \geq 2$ for which (ii) does not hold. We analyze two possible cases.

Case 1. α is odd.
We have $\alpha \geq 3$, $0 < \alpha r < k$ and there exists an $[(\alpha+1)r+1]$-chord in A. Since $\alpha - 1$ is even, the choice of α implies that every $-((\alpha-1)+1)r$-chord is in A.

Let (x_1, x_0) be an $[(\alpha+1)r+1]$-chord, (x_0, x_2) the $-\alpha r$-chord with an initial vertex x_0 and (x_2, x_3) the $(k+1)+(t-1)(k-2)$-chord with an initial vertex x_2 (it follows from the hypothesis and Lemma 2.1 that these chords are in A). Clearly, $\ell\langle x_1, \gamma, x_2\rangle = (\alpha+1)r + 1 - \alpha r = r+1$ and $\ell\langle x_3, \gamma, x_1\rangle = n - (r+1) - [(k+1)+(t-1)(k-2)] = k-5$. Notice $x_3 \in \langle x_0, \gamma, x_1\rangle - \{x_0, x_1\}$ because $\alpha r < k$, and $\ell\langle x_2, \gamma, x_3\rangle \geq k+1$. We conclude that $\mathcal{C}_{k-2} = \langle x_3, \gamma, x_1\rangle \cup (x_1, x_0, x_2, x_3)$ is a directed cycle with $\mathcal{I}(\mathcal{C}_{k-2}) \geq k-5$.

Case 2. α is even.
We have $\alpha \geq 2$, $0 < \alpha r < k$, and there exists an $(\alpha+1)r$-chord in A. Since $\alpha - 1$ is odd, the choice of α implies that every $-[(\alpha-1+1)r+1]$-chord is in A. Let (x_1, x_0) be an $(\alpha+1)r$-chord, (x_0, x_2) the $-(\alpha r+1)$-chord with an initial vertex x_0, and (x_2, x_3) the $(k+1)+(t-1)(k-2)$-chord with an initial vertex x_2.

Clearly, $\ell\langle x_1, \gamma, x_2\rangle = (\alpha+1)r - \alpha r - 1 = r-1$ and $\ell\langle x_3, \gamma, x_1\rangle = n - (r-1) - [k+1+(t-1)(k-2)] = k-3$. Moreover, $x_3 \in \langle x_0, \gamma, x_1\rangle - \{x_0, x_1\}$ because $\ell\langle x_2, \gamma, x_0\rangle = \alpha r + 1$, $0 < \alpha r + 1 \leq k$ and $\ell\langle x_2, \gamma, x_3\rangle \geq k+1$. We obtain $\mathcal{C}_k = \langle x_3, \gamma, x_1\rangle \cup (x_1, x_0, x_2, x_3)$ a directed cycle with $\mathcal{I}(\mathcal{C}_k) = k-3$. ∎

Lemma 6.7. *At least one of the following properties holds.*

(i) *There exists a directed cycle $\mathcal{C}_{h(k)}$ with $\mathcal{I}(\mathcal{C}_{h(k)}) \geq h(k) - 3$.*

(ii) *For i even $-2 \leq i \leq r+1$; every $-(2r+1-i)$-chord and every $(k-1+i)$-chord is in A.*

Proof. Suppose (i) does not hold, we shall prove that property (ii) holds by induction on i. We start with $i = -2$ and $i = 0$; namely, we prove that the following chords are in A: (a) every $(k-3)$-chord, (b) every $(k-1)$-chord, (c) every $-(2r+3)$-chord and (d) every $-(2r+1)$-chord.

The proof of (a) and (b) follows directly from Lemma 6.2. Let 0 be any vertex of T. It follows from Lemma 6.1 (with $\alpha = 0$) and from Lemma 6.2 (part (b) and (d)) that the following chords are in A: $(0, k-1+(t-1)(k-2))$, $(r+2, 0)$ and $(r, 0)$. Part (c): every $-(2r+3)$-chord is in A. If $(n-r-1, r+2) \in A$, then $\mathcal{C}_k = (n-r-1, r+2, 0, k-1+(t-1)(k-2)) \cup \langle k-1+(t-1)(k-2), \gamma, n-r-1\rangle$ is a directed cycle with $\mathcal{I}(\mathcal{C}_k) = k-3$, a contradiction. (Notice that $k-1+(t-1)(k-2) \in \langle r+2, \gamma, n-r-1\rangle - \{r+2, n-r-1\}$ since $r \le k-5$; moreover $\ell\langle k-1+(t-1)(k-2), \gamma, n-r-1\rangle = n-r-1-(k-1)-(t-1)(k-2) = k-3$).

Part (d): every $-(2r+1)$-chord is in A. If $(n-r-2, r-1) \in A$, then $\mathcal{C}_k = (n-r-2, r-1, r, 0, k-1+(t-1)(k-2)) \cup \langle k-1+(t-1)(k-2), \gamma, n-r-2\rangle$ is a directed cycle with $\mathcal{I}(\mathcal{C}_k) = k-3$, a contradiction. (Notice that since $r \le k-5 < k-1$ and $k \ge 10$ we have $r < k-1+(t-1)(k-2) < n-(r+2)$, and $\ell\langle k-1+(t-1)(k-2), \gamma, n-r-2\rangle = n-(r+2)-[k-1+(t-1)(k-2)] = k-4$). Assume that (ii) in Lemma 6.7 holds for each i' even, $0 \le i' \le i$ and let us prove it for $i+2$; namely, we prove

(α) Every $(k+1+i)$-chord is in A, $0 \le i \le r-1$.

(β) Every $-(2r-1-i)$-chord is in A, $0 \le i \le r-1$.

Proof of **(α).** It follows from the inductive hypothesis that for each j even, $0 \le j \le i$, every $[(k-1)+j]$-chord and every $[(k-3)+j]$-chord is in A (because for $j = 0$ we have proved that every $(k-3)$-chord is in A). It follows from Lemma 6.2 that every $(-r)$-chord and every $-(r+2)$-chord is in A. Therefore it follows from Lemma 6.3 that for even j, $0 \le j \le i$ every $-(2r-j+1)$-chord, $-(2r-j+3)$-chord and $-(2r-j+5)$-chord is in A. That means that for each even j, $-4 \le j \le i \le r-1$ every $-(2r-j+1)$-chord is in A. These are $\frac{i}{2}+3$ chords with initial odd (resp. even) vertices consecutive in γ.

Assume by contradiction that $(x_3, 0)$ is a $-(k+1+i)$-chord, i being even $0 \le i \le r-1$. Let $x_0 = n-(2r-i-1)$. Hence letting $x_2 = 2$ we have that (x_2, x_0) is a $-(2r-i+1)$-chord (we have observed that every $-(2r-i+1)$-chord is in A).

First, we prove that $x_0 \in \langle x_3+1, \gamma, n-1\rangle$: $\ell\langle x_0, \gamma, 0\rangle = 2r-i-1 \ge r \ge 3$, $\ell\langle x_3, \gamma, x_0\rangle = n-(k+1+i+2r-i-1) = k-1+t(k-2)+r-k-2r \ge k-1+k-2-r-k = k-3-r \ge 2$ (remember $3 \le r \le k-5$).

Now, there exists an out-neighbor of x_0, say x, such that x is in $\langle x_2, \gamma, x_3 - 1\rangle$ this is a direct consequence of Lemma 6.4 and the fact that the number even vertices in $\langle x_2, \gamma, x_3 - 1\rangle$ is at least $\frac{k-2}{2}$ (Notice that x_0 is odd, $\ell\langle x_2, \gamma, x_3 - 1\rangle = k + 1 + i - 3 = k + i - 2 \geq k - 2$). Let x_4 be the smallest (the nearest to 0 in γ) such vertex.

Let $x_1 = 0$, we will prove that $x_4 - i - 4 \in \langle x_1, \gamma, x_4 - 3\rangle$. Since for each j, j even $-4 \leq j \leq i \leq r - 1$, every $-(2r - j + 1)$-chord is in A, it follows that

$$\{(2, x_0), (4, x_0), (6, x_0), \ldots, (i + 4, x_0), (i + 6, x_0)\} \subseteq A\,.$$

Hence the selection of x_4 implies $x_4 \geq i + 8$, so $x_4 - i - 4 > 3$.

Finally, since $\ell\langle x_4, \gamma, x_3\rangle + \ell\langle x_1, \gamma, x_4 - i - 4\rangle = k + 1 + i - (i + 4) = k - 3$ it follows that $\mathcal{C}_k = (x_4 - i - 4, x_0, x_4) \cup \langle x_4, \gamma, x_3\rangle \cup (x_3, x_1) \cup \langle x_1, \gamma, x_4 - i - 4\rangle$ is a directed cycle with $\mathcal{I}(\mathcal{C}_k) = k - 3$ (Notice $(x_4 - i - 4, x_0) \in A$ by the choice of x_4 and the fact $x_4 - i - 4 \in \langle x_1, \gamma, x_4 - 3\rangle$).

Proof of (β). Part (β) follows from Lemma 6.3 (taking $i + 2$ instead of i) and the following facts:

Every $(k - 1 + i)$-chord is in A for even i, $-2 \leq i \leq r + 1$ (it follows from part (α)).

Every $(k - 3 + i)$-chord is in A for even i, $0 \leq i \leq r + 1$ (it follows from the assertion of above).

Every $(-r)$-chord and every $-(r + 2)$-chord is in A (it follows from Lemma 6.2). ■

Theorem 6.8. *If $n \geq 2k - 4$, then there exists a directed cycle $\mathcal{C}_{h(k)}$ with $\mathcal{I}(\mathcal{C}_{h(k)}) \geq h(k) - 3$.*

Proof. The case $n = 2k - 4$ is considered in Section 4. Assume $n > 2k - 4$ and suppose by contradiction that there is no directed cycle $\mathcal{C}_{h(k)}$ with $\mathcal{I}(\mathcal{C}_{h(k)}) \geq h(k) - 3$.

It follows from Lemma 6.7 that for each even i, $-2 \leq i \leq r + 1$ every $(k - 1 + i)$-chord is in A, in particular

$$\{(0, k-3), (0, k-1), (0, k+1), (0, k+3), \ldots, (0, k+r-2), (0, k+r)\} \subset A\,. \tag{1}$$

(Notice that $k + r < n - 1$ because $t \geq 1$ and $k \geq 10$).

It follows from Lemma 6.2 that every $(-r)$-chord is in A, and by Lemma 6.7 that every $(k+1)$-chord is in A. Therefore, by Lemma 6.6 we have: For every odd α, $0 < \alpha r < k$, every $-[(\alpha+1)r+1]$-chord is in A. And for every even α, $0 \leq \alpha r < k$, every $-(\alpha+1)r$-chord is in A. Let $\alpha_0 = \max\{\alpha \in \mathbb{N} \mid \alpha r < k\}$. Clearly, $\alpha_0 r < k$. We will analize the two possible cases:

Case 1. α_0 is even.
It follows from Lemma 6.6 that every $-(\alpha_0+1)r$-chord is in A, in particular $((\alpha_0+1)r, 0) \in A$. On the other hand, $\alpha_0 r < k$ implies $(\alpha_0+1)r < k+r$ and the selection of α_0 implies $k < (\alpha_0+1)r < k+r$. Thus $y = (\alpha_0+1)r \in \{k+1, k+3, k+5, \ldots, k+r\}$; thus we have $(y, 0) \in A$ and (1) implies $(0, y) \in A$. A contradiction.

Case 2. α_0 id odd.
It follows from Lemma 6.6 that every $-[(\alpha_0+1)r+1]$-chord is in A, in particular $((\alpha_0+1)r+1, 0) \in A$. On the other hand, $\alpha_0 r < k$ and the choice of α_0 implies $k+1 \leq (\alpha_0+1)r+1 \leq k+r$, $y = (\alpha_0+1)r+1$ is odd and $y \in \{k+1, k+3, k+5, \ldots, k+r\}$. So it follows from (1) that $(0, y) \in A$ and we have proved $(y, 0) \in A$. A contradiction. ■

7. Remarks

In this section, it is proved that the hypothesis of Theorem 6.8 is tight.

Definition 7.1. A digraph D with vertex set V is called *cyclically p-partite complete* ($p \geq 3$) provided one can partition $V = V_0 + V_1 + \cdots + V_{p-1}$ so that (u, v) is an arc of D if and only if $u \in V_i$, $v \in V_{i+1}$ (notation modulo p).

Remark 7.2. The cyclically 4-partite complete digraph T_4 is a bipartite tournament and clearly every directed cycle of T_4 has length $\equiv 0 \pmod 4$. So for $k = 4m+2$, T_4 has no directed cycles of length k and for $k = 4m$, T_4 has no directed cycles of length $k-2$.

Now we consider the following simple lemma.

Lemma 7.3. *Let $\mathcal{C}_{h(k)}$ be a directed cycle with $\mathcal{I}(\mathcal{C}_{h(k)}) = h(k) - 2$. If $f_1 = (0, x_1)$, $f_2 = (y_1, y_2)$ are the arcs of $\mathcal{C}_{h(k)}$ not in γ, then $y_2 = y_1 + n - (h(k) - 2 + x_1)$. Namely, f_2 is a $-(x_1 + (h(k) - 2))$-chord of γ.*

Remark 7.4. For $n \geq 5$, $k \geq 5$, such that $n \neq k + s(k-2) + m(k-4)$ and $n \neq s(k-2) + m(k-4)$ with $s, m \in \mathbb{N}$, there exists a bipartite hamiltonian tournament T_n with no directed cycles $\mathcal{C}_{h(k)}$ with $\mathcal{I}(\mathcal{C}_{h(k)}) = h(k) - 2$.

Proof. Define T_n as follows:
Let

$$C = \{(i, i+k-1+s(k-2)+m(k-4)) \mid i \in \{0, 1, \ldots, n-1\},\ s, m \in \mathbb{N} \text{ with}$$
$$(k-1) + s(k-2) + m(k-4) < n-1\}$$

and

$$F = \{(i, i+k-3+s(k-2)+m(k-4)) \mid i \in \{0, 1, \ldots, n-1\},\ s, m \in \mathbb{N} \text{ with}$$
$$(k-3) + s(k-2) + m(k-4) < n-1\},$$
$$A(T_n) = C \cup F \cup \left(\left\{ (i+j, i) \mid j \in \left\{ 2, 3, \ldots, \left\lfloor \frac{n-1}{2} \right\rfloor \right\} \right\} - (C \cup F) \right)$$
$$\cup \{(i, i+1) \mid i \in \{0, 1, \ldots, n-1\}\} \cup \left\{ \left(i + \frac{n}{2}, i \right) \mid i \in \left\{ 0, 1, \ldots, \frac{n}{2} - 1 \right\} \right\}.$$

Clearly, there is no directed cycle $\mathcal{C}_{h(k)}$ with $\mathcal{I}(\mathcal{C}_{h(k)}) = h(k) - 1$ (Notice that T_n has every $(k-1)$-chord and every $(k-3)$-chord). Now assume for contradiction that $\mathcal{C}_{h(k)}$ is a directed cycle of T_n with $\mathcal{I}(\mathcal{C}_{h(k)}) = k - 2$, and let $f_1 = (0, x_1)$, $f_2 = (y_1, y_2)$ the only arcs of $\mathcal{C}_k$ not in γ. Without loss of generality we can assume $\ell(f_1) < \frac{n}{2}$. The definition of T_n implies that $x_1 = k - 1 + s(k-2) + m(k-4)$ or $x_1 = k - 3 + s(k-2) + m(k-4)$. It follows from Lemma 7.3 that y_2 has one of the following forms:

(a) $y_2 = y_1 + n - [k - 1 + (s+1)(k-2) + m(k-4)]$.

When $k - 1 + (s+1)(k-2) + m(k-4) < n - 1$ we obtain that f_2 is a $-(k-1+(s+1)(k-2)+m(k-4))$-chord, contradicting the definition of T_n.

When $k-1+(s+1)(k-2)+m(k-4) \geq n-1$ we have that $\ell\langle x_1, \gamma, 0\rangle \leq k-1$ and the fact $\mathcal{C}_k - \{(0, x_1), (y_1, y_2)\} \subseteq \langle x_1, \gamma, 0\rangle$ implies $\ell\langle x_1, \gamma, 0\rangle \geq k-3$; and since $\ell\langle x_1, \gamma, 0\rangle$ is odd we have $\ell\langle x_1, \gamma, 0\rangle \in \{k-1, k-3\}$. Now if $\ell\langle x_1, \gamma, 0\rangle = k-1$, then $n = x_1 + k - 1 = k - 1 + s(k-2) + m(k-4) + k - 1 =$

$k + (s+1)(k-2) + m(k-4)$, a contradiction. If $\ell\langle x_1, \gamma, 0\rangle = k-3$, then $n = x_1+k-3 = k-1+s(k-2)+m(k-1)+k-3 = k+s(k-2)+(m+1)(k-4)$, a contradiction.

(b) $y_2 = y_1 + n - [(k-1) + s(k-2) + (m+1)(k-4)]$.

(c) $y_2 = y_1 + n - [(k-3) + (s+1)(k-2) + m(k-4)]$.

(d) $y_2 = y_1 + n - [(k-3) + s(k-2) + (m+1)(k-4)]$.

Cases (b), (c) and (d) can be analized in a completly analogous form as the case (a) to get a contradiction.

It is easy to verify that if $n = k + s(k-2) + m(k-4)$ or $n = s(k-2)+m(k-4)$ with $s, m \in \mathbb{N}$, then T_n (any bipartite hamiltonian tournament with n vertices) has a directed cycle $\mathcal{C}_{h(k)}$ with $\mathcal{I}(\mathcal{C}_{h(k)}) \geq h(k) - 2$. ■

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