OFFENSIVE ALLIANCES IN GRAPHS

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Abstract

A set S is an offensive alliance if for every vertex v in its boundary $N(S) - S$ it holds that the majority of vertices in v's closed neighbourhood are in S. The offensive alliance number is the minimum cardinality of an offensive alliance. In this paper we explore the bounds on the offensive alliance and the strong offensive alliance numbers (where a strict majority is required). In particular, we show that the offensive alliance number is at most 2/3 the order and the strong offensive alliance number is at most 5/6 the order.

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1. Introduction

In real life, an alliance is a collection of entities such that the union is stronger than the individual. The alliance can be either to protect against attack, or to assert collective will against others. This motivated the definition of defensive and offensive alliances in graphs, given in [5].

In this paper we study the offensive alliances. Informally, given a graph $G = (V, E)$, we say a set S is an offensive alliance if every other vertex that is adjacent to S is outgunned by S: more of its neighbours are in S than are not. Formally, we denote by $N(v)$ the (open) neighbourhood of a vertex v and $N[v] = \{v\} \cup N(v)$. Similarly, for $S \subseteq V$ we denote $N(S) = \bigcup_{v \in S} N(v)$

and $N[S] = S \cup N(S)$. We define the boundary ∂S as the set $N(S) - S$. Then S is an *offensive alliance* if:

$$\text{for all } v \in \partial S : |N[v] \cap S| \geq |N[v] - S|.$$

It is a *strong offensive alliance* if the inequality is strict for all vertices in the boundary. Equivalently, we define the *excess* of a vertex relative to S by

$$ex(v; S) = |N[v] \cap S| - |N[v] - S|.$$

So for an offensive alliance the excess of each vertex in the boundary is at least 0, and for a strong offensive alliance the excess is at least 1.

Then in [5] the offensive alliance and strong offensive alliance numbers of a graph G were defined as follows:

> $a_o(G)$ *is the minimum cardinality of a nonempty offensive alliance, and* $\hat{a}_o(G)$ *is the minimum cardinality of a nonempty strong offensive alliance.*

This definition specifically allows for S to be only local. Consider, for example the graph obtained from the disjoint union of $K_4 \cup K_{1,3}$ with one end-vertex of the star and one vertex of the clique identified (shown in Figure 1). The two end-vertices form an offensive alliance, so that $a_o(G) \leq 2$. In fact, $a_o(G) = 2$ and $\hat{a}_o(G) = 3$. (The unique minimum strong offensive alliance is the vertices of degree 3 in the clique.) We say that an offensive alliance is *global* if every vertex is affected. That is, $S \cup \partial S = V$ (S is a dominating set).
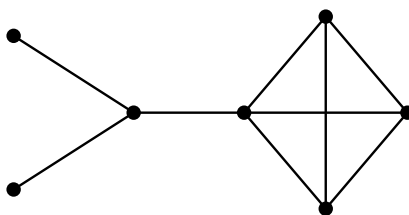

Figure 1. A graph G with $a_o(G) = 2$ and $\hat{a}_o(G) = 3$

The property of being an offensive alliance is not hereditary and hence the parameter $a_o(G)$ is not monotonic. That is, removal of edges or vertices can both increase or decrease the parameter.

Linial et al. [6] considered a more general setting where associated with every vertex is a set $Dom(v)$. They defined a monopoly M as a set of vertices such that for every vertex v, the majority of the vertices in $Dom(v)$ are in M. They defined a *self-ignoring monopoly* as a set M such that for every vertex not in M, the majority of the vertices in $Dom(v)$ are in M. They proved several asymptotics on the minimum size of monopolies and self-ignoring monopolies where $Dom(v)$ is the set of neighbours, or more generally, the vertices at a particular distance, or within a particular distance. When the domain of a vertex is its closed neighbourhood, a self-ignoring monopoly is precisely a global offensive alliance.

They pointed out applications in fault-tolerant computing where the processors adopt majority voting when there is a conflict on distributed data. Other applications include distributed networks, distributed databases, resource allocation, and system-level diagnosis; see [8].

This problem is also related to the "unfriendly graph partition problem" introduced by Aharoni et al. [1] and (under a different name) by Luby [7]. The goal there is to partition the vertex set into two sets such that for each vertex the majority of its neighbors are in the opposite set. If one requires strict majority, then this would be a partition into two offensive alliances, which does not always exist. However, an unfriendly partition is achieved by simply maximising the number of edges between the two sets. (Life is more interesting for infinite graphs, see [10].) The "friendly" version (equivalent to a partition into what [5] calls defensive alliances) is studied in [4, 9].

The offensive alliance number is also related to the parameter signed domination introduced by Dunbar et al. [2]. Signed domination entails a partition of the vertex set into positive and negative vertices. The requirement for the positive set P is that for every vertex, $|N[v] \cap P| > |N[v] - P|$. The signed domination number $\gamma_S(G)$ of a graph G of order n is the minimum of $2|P| - n$ taken over all valid partitions. It follows that

$$(1) \qquad \hat{a}_o(G) \leq (n + \gamma_S(G))/2.$$

In this paper we explore the elementary properties of the offensive alliance numbers, including their values for several families of graphs. We then establish upper bounds on the offensive alliance and strong offensive alliance

numbers. Thereafter we consider graphical operations and how the alliance numbers for these relate to those of the constituents.

2.   Bounds and Calculations

We start with a primitive lower bound in terms of the minimum degree $\delta(G)$. This bound follows from considering any vertex in the boundary.

Observation 1. *For all graphs G, $a_o(G) \geq (\delta(G) + 1)/2$ and $\hat{a}_o(G) > (\delta(G) + 1)/2$.*

Examples of equality in the above bounds are the complete and complete bipartite graphs (though for the strong offensive alliance number the star is an exception):

Corollary 2. *For $n \geq 1$, $a_o(K_n) = \lceil n/2 \rceil$ and $\hat{a}_o(K_n) = \lceil (n + 1)/2 \rceil$.*
For $1 \leq r \leq s$, $a_o(K_{r,s}) = \lceil (r + 1)/2 \rceil$.
For $2 \leq r \leq s$, $\hat{a}_o(K_{r,s}) = \lceil r/2 + 1 \rceil$ but $\hat{a}_o(K_{1,s}) = \lceil s/2 + 1 \rceil$.

At the other extreme, there is the following upper bound since every vertex cover is an offensive alliance. We use $\alpha(G)$ to denote the vertex cover number of G.

Observation 3. *For all graphs G, $a_o(G) \leq \alpha(G)$.*
If $\delta(G) \geq 2$, then $\hat{a}_o(G) \leq \alpha(G)$.

In general, $\hat{a}_o(G)$ is at most the minimum cardinality of a vertex cover that contains all end-vertices. More generally, one can define $\alpha_k(G)$ as the minimum cardinality of a set whose removal brings the maximum degree to at most k. Then:

Observation 4. *For all graphs G, $a_o(G) \leq \alpha_{\lceil \delta/2 \rceil - 1}(G)$ and $a_o(G) \leq \alpha_{\lfloor \delta/2 \rfloor - 1}(G)$.*

If the graph has small maximum degree $\Delta(G)$, there are lower bounds that echo Observation 3.

Observation 5. *Let G be a connected graph. If $\Delta(G) \leq 2$ then $a_o(G) = \alpha(G)$, and if $\Delta(G) \leq 3$ then $\hat{a}_o(G) \geq \alpha(G)$.*

Proof. Let S be an offensive alliance and let $v \in \partial S$. Then by the restriction on the degree of v, $N(v) \subseteq S$. It follows that ∂S is independent, and that there is no vertex at distance 2 from S. That is, S is a vertex cover. ∎

Examples of equality in the above bounds are the paths and cycles:

Observation 6. *For $n \geq 1$, $a_o(P_n) = \lfloor n/2 \rfloor$ and $\hat{a}_o(P_n) = \lfloor n/2 \rfloor + 1$. For $n \geq 3$, $a_o(C_n) = \hat{a}_o(C_n) = \lceil n/2 \rceil$.*

The two alliance parameters are equal for the cycle and indeed equal for any Eulerian graph, as then every offensive alliance is automatically strong:

Observation 7. *If every vertex of graph G has even degree, then $a_o(G) = \hat{a}_o(G)$.*

Equality in Observation 5 also holds for cubic graphs:

Observation 8. *For any connected cubic graph G, $\hat{a}_o(G) = \alpha(G)$.*

Hence the strong offensive alliance number is NP-hard. Another consequence of Observation 5 is:

Corollary 9. *Let G be a connected graph. Then $a_o(G) = 1$ iff G is a star, and $\hat{a}_o(G) = 1$ iff $G = K_1$.*

Proof. For the first part, note that $a_o(G) = 1$ implies that every vertex in ∂S is an end-vertex. ∎

In particular, this confirms that there is no upper bound for the strong offensive alliance number in terms of the offensive alliance number.

Finally in this section we note a simple sufficient condition for every offensive alliance to be global.

Observation 10. *Let G be connected. If for all vertices v it holds that*

$$\delta(\langle N(v) \rangle) \geq (\deg v - 3)/2,$$

then every strong offensive alliance is global. Moreover, if the inequality is always strict, then every offensive alliance is global.

Proof. Suppose alliance S is not global. Then there exists a vertex $v \in \partial S$ with a neighbor $w \notin N[S]$. So the two sets $N[w] \cap N[v]$ and $S \cap N[v]$ are disjoint. Hence, if S is a strong offensive alliance,

$$2 + \delta \langle N(v) \rangle \leq |N[w] \cap N[v]| < |N[v]|/2 = (\deg v + 1)/2.$$

This inequality is the negative of the inequality stated in the theorem. The proof for offensive alliances is similar. ∎

In fact, for an alliance S to be nonglobal in a connected graph, the set of vertices which do not satisfy the above inequality must contain a cut-set (since ∂S is a cut-set).

3.   Maximum Values

In this section we consider the maximum offensive alliance and strong offensive alliance numbers of a graph.

Theorem 1. *For all graphs G of order $n \geq 2$, $a_o(G) \leq 2n/3$.*

Proof. Let $G = (V, E)$. The result is trivial if G has an isolated vertex. So assume that $\delta(G) \geq 1$. Color the vertex set V with three colors such that the number of monochromatic edges (both ends the same color) is as small as possible. Then any vertex is incident with at least double the number of nonmonochromatic edges as monochromatic edges. (If a green vertex has more green neighbours than red neighbours, then we can recolor it red, a contradiction.) So any two colour classes form an offensive alliance. ∎

We know of three examples of equality in the above bound: K_3, $K_{2,2,2}$ and the graph formed as follows: take three disjoint triangles T_1, T_2, T_3 and add 3 edges so that there is a triangle containing one vertex of each of T_1, T_2 and T_3.

The theorem can also be deduced from a general bound:

Observation 11. *If G has n vertices and domination number γ, then $a_o(G) \leq (n + \gamma)/2$.*

Proof. Let S be a minimum dominating set and partition $V - S$ into two sets maximizing the number of edges joining the two sets. Then the smaller of these and S is an offensive alliance. ∎

Asymptotically, the maximum offensive alliance number for connected graphs that we know is 5/8 of the order. Define a graph G_k on $8k$ vertices as follows. Start with a cycle on $3k$ vertices: $v_0, \ldots, v_{3k-1}, v_0$. Then for each pair v_i, v_{i+1} of consecutive vertices on the cycle, introduce a vertex x_i adjacent to both (all arithmetic modulo $3k$). Finally, for each x_{3j} add two new vertices such that they and it induce a triangle T_j. Every vertex of G_k has degree 2 or 4. Graph G_4 is depicted in Figure 2.
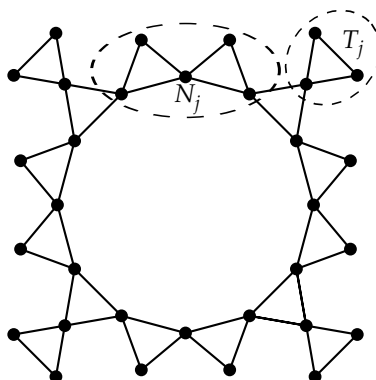

Figure 2. The graph G_4

Consider an offensive alliance S of G_k. By Observation 10, S is global. The vertex set of G_k can be partitioned into the T_j and $N_j = N[v_{3j+2}]$ ($0 \leq j \leq k-1$). From each T_j at least two vertices are in S. (If either degree-2 vertex is not in S, then the other two vertices of the triangle must be in S.) Further, from each N_j at least 3 vertices are in S. (If v_{3j+2} not in S, then at least 3 of its neighbours are. If v_{3j+2} in S, then S contains at least one more from each triangle.) Since $V(G) - \{x_0, \ldots, x_{3k-1}\}$ is a vertex cover of size $5k$, it follows that:

$$a_o(G_k) = 5k.$$

By a similar argument to the above theorem, it follows that:

Theorem 2. *If every vertex of graph G has odd degree, then $a_o(G) \leq n/2$.*

Proof. Color the vertex set V with two colors such that the number of monochromatic edges (both ends the same color) is as small as possible. Then each colour class forms an offensive alliance. ∎

It is to be noted that Theorem 1 holds even if one requires a global offensive alliance. On the other hand, a global strong offensive alliance requires all leaves, and hence may require almost all the vertices. The following theorem provides an upper bound on the strong offensive alliance number.

Theorem 3. *For all graphs G of order $n \geq 3$, $\hat{a}_o(G) \leq 5n/6$. Moreover, if G has minimum degree at least 2, then $\hat{a}_o(G) \leq 3n/4$.*

Proof. For the case where G has minimum degree at least 2, one uses the same proof technique as in Theorem 1 above, except that one colors with four colors instead of three.

For the general result, one proceeds in a similar fashion but with considerably more details, which we omit. ∎

We know three graphs where $\hat{a}_o(G)$ is 5/6 the order. Take K_2 and add two feet to each vertex; or take K_3 and add one foot to each vertex; or take K_3 and adding three feet to each vertex.

The largest we know of is asymptotically 4/5 the order. For integer $n \geq 3$, define a graph J_n as follows. Start with a wheel (the join of a cycle on n vertices and a central vertex). The for each vertex v on the cycle, introduce a path on 4 vertices and join v to the two central vertices. We will denote the subgraph induced by the 4 vertices and v as H_v. Thus J_n has $5n + 1$ vertices. The graph J_7 is illustrated in Figure 3.
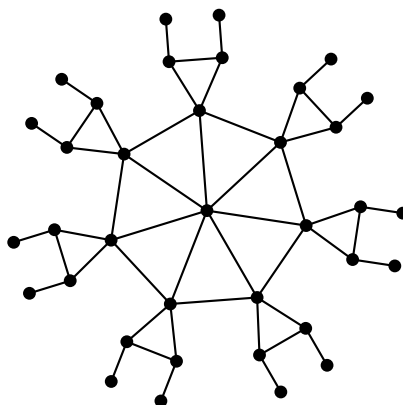

Figure 3. The graph J_7

By the comment after Observation 10, every strong offensive alliance of J_n is global. (Only the central vertex fails to satisfy the inequality in the

hypothesis.) Thus one needs four vertices from each H_v. (One must have the two leaves; if miss either degree-3 vertex then must have all its neighbors.) Thus $\hat{a}_o(J_n) \geq 4n$. This value is achievable by an alliance S with all vertices except the center of the wheel and one degree-3 vertex from each H_v.

In general, as the minimum degree increases, the upper bound tends to half the order. This can be proven directly by probabilistic methods, but it amounts to using the methods of Füredi and Mubayi [3], and so one can simply read it from their results:

Theorem 4. *For graph with order n and minimum degree δ, $a_o(G) \leq \hat{a}_o(G) \leq n(1/2 + c_\delta)$ where $c_\delta \to 0$ as $\delta \to \infty$.*

Proof. In [3] it is proven that $\gamma_s(G) \leq O(n\sqrt{\log(\delta+1)/(\delta+1)})$ so that $\gamma_s(G) \leq no(\delta)$. The result then follows from Inequality 1. ∎

3..1   Trees

For a tree T on n vertices, the maximum offensive alliance number is $\lfloor n/2 \rfloor$, as this is an upper bound on $\alpha(T)$. Equality is obtained for the path. It can be shown that the only other examples of equality are $K_{1,3}$ with one edge subdivided once, and $K_{1,4}$ with two edges each subdivided once.

For the strong offensive alliance number, the bound is higher.

Theorem 5. *For a tree T on n vertices, $\hat{a}_o(T) \leq \lceil 3n/4 \rceil$.*

Proof. Suppose the maximum degree of T is at most 3. Let L denote the number of leaves. Then, since the sum of the degrees is $2(n-1)$, the number of vertices of degree 3 is $L-2$. Hence $L \leq n/2+1$. Now, an offensive alliance is obtained by a vertex cover that includes the leaves, and so

$$\hat{a}_o(T) \leq L + (n - L)/2 \leq 3n/4 + 1/2,$$

as required.

Now, suppose T has two vertices v_1 and v_2 of degree at least 4. Then let S_1 denote the vertices that are separated from v_2 by the removal of v_1, and let S_2 be the vertices that are separated from v_1 by the removal of v_2. Then both sets are strong offensive alliances, and are disjoint, and so $\hat{a}_o(T) < n/2$.

Finally, suppose T has exactly one vertex v of degree $d \geq 4$. Let its neighbors be n_i and the component of $T - v$ containing n_i be T_i. So taking

all vertices in the $\lceil d/2+1 \rceil$ smallest T_i, one obtains a strong offensive alliance. Hence $\hat{a}_o(T) \leq (n-1)\lceil d/2+1 \rceil/d$, which is less than $3n/4$ for $d = 4$ or $d \geq 6$. So we may assume that $d = 5$.

By a similar argument to that in the first paragraph, there is a strong offensive alliance S_i for T_i that contains n_i and has cardinality at most $3|T_i|/4 + 1/2$. (Note that in T_i, vertex n_i has degree one or two; if the latter case use the fact that in any bipartite graph, one can find a vertex cover containing a specified vertex with order at most $\lceil n/2 \rceil$.)

So taking the S_i in the 4 smallest components, one obtains a strong offensive alliance. The union of all the S_i has cardinality at most $3(n-1)/4 + 5$. So $\hat{a}_o(T) \leq (4/5) \times (3(n-1)/4 + 1/2)$, which is less than $\lceil 3n/4 \rceil$ for $n \geq 9$. The case $n \leq 8$ is easily checked.   ∎

Equality achieved by any tree with all vertices of degree 3 or 1 such that the vertices of degree 3 have a perfect matching.

4.   Graph Operations

In this section we consider the parameters for the result of various graph operations.

A trivial observation is that the (strong) offensive alliance number is the minimum of the two numbers when the graph is the union of two components:

$$a_o(G \cup H) = \min(a_o(G), a_o(H)) \text{ and } \hat{a}_o(G \cup H) = \min(\hat{a}_o(G), \hat{a}_o(H)).$$

We consider next the join. In general there is no upper bound for $a_o(G+H)$ in terms of $a_o(G)$ and $a_o(H)$ (for example, consider the join of two stars). Nor in fact is there an upper bound even if one of the pieces is a single vertex. One can at least show that if $\delta(G) \geq 1$ and $\Delta(G) \leq 2$, then $a_o(G + K_1) = \gamma(G) + 1$. This gives the offensive alliance number of the wheel.

We consider next cartesian products. Again, there is no general upper bound for the parameter for $G \times H$ in terms of the parameters for G and H. So we consider the special case of the grid. We need the following two observations.

Observation 12. *Let $G = C_m \times C_n$ for $m, n \geq 4$ and let S be an offensive alliance. Then each component of $G - S$ is a subgraph of the star on 4 edges.*

Proof. Let $v \in \partial S$. Then at least 3 of v's neighbours are in S. If exactly 3, then let w be the neighbour not in S. If $w \in \partial S$ then $\{v, w\}$ induces a component in $G - S$. So we may assume that $w \notin \partial S$.

Vertex w has two neighbours, say x and y, that are adjacent to vertices in $N(v)$ and hence are in ∂S. Then w is their only neighbour not in S. Similarly, the fourth neighbour of w is in ∂S and so the component of $V - S$ containing v is a subset of $K_{1,4}$. ∎

The following is an extension of the vertex cover upper bound. The 3-packing number $\rho_4(G)$ is the maximum number of vertices that are pairwise at least distance 4 apart.

Lemma 13. *If G is r-regular for $r \geq 3$ and triangle-free, then $a_o(G) \leq n - (r+1)\rho_4(G)$.*

Proof. Consider any 3-packing P, and let $S = V - N[P]$. It follows that $\partial S = N(P)$ and thus that S is an offensive alliance. ∎

Theorem 6. *If n and m are both a multiple of 4, then $a_o(C_n \times C_m) = \frac{3}{8}nm$.*

Proof. A 3-packing of cardinality $nm/8$ is obtained by taking from the even-numbered copies of C_m every fourth vertex, staggered by two each time. (See the doubly ringed vertices in Figure 4.) So the above value is an upper bound.

On the other hand, let S be an offensive alliance, and let M be the number of edges between S and ∂S. Trivially, $|M| \leq 4|S|$. On the other hand, by considering each of the five possible components of $G - S$ in turn, it follows that $|M| \geq 12|V - S|/5$. This means that $|S| \geq 3|V|/8$. ∎
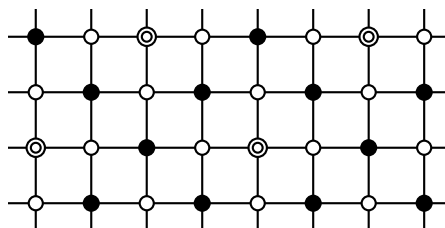

Figure 4. The grid $C_4 \times C_8$: The black vertices form a strong offensive alliance.

Similar asymptotics hold for grids which are the product of two paths.

Finally we consider the *corona* $G \circ K_1$ of a graph G, which is obtained by adding an end-vertex adjacent to all vertices. This operation can dramatically decrease the alliance number. Example: if $\Delta(G) \leq 2$ then $a_o(G \circ K_1) \leq 3$.

5.   Open Questions

Among the open problems raised by the results in this paper, the following are of particular interest.

1. What is the real upper bound for the strong offensive alliance number? What are the extremal graphs? What is the maximum asymptotically for the two parameters?

2. The relationship between these parameters and between them and other parameters is important. Under what conditions is every offensive alliance global? When is $a_o(G) = \hat{a}_o(G)$? When is $a_o(G) = \alpha(G)$?

3. Can one determine the exact values for any other classes of graphs (e.g. grids)? Or at least get good bounds (e.g. outerplanar graphs)?

References

[1] R. Aharoni, E.C. Milner and K. Prikry, *Unfriendly partitions of graphs*, J. Combin. Theory (B) **50** (1990) 1–10.

[2] J.E. Dunbar, S.T. Hedetniemi, M.A. Henning and P.J. Slater, *Signed domination in graphs*, in: "Graph Theory, Combinatorics and Algorithms" Y. Alavi and A. Schwenk, eds. (Wiley, 1995) 311–321.

[3] Z. Füredi and D. Mubayi, *Signed domination in regular graphs and set-systems*, J. Combin. Theory (B) **76** (1999) 223–239.

[4] M.U. Gerber and D. Kobler, *Partitioning a graph to satisfy all vertices*, European J. Oper. Res. **125** (2000) 283–291.

[5] S.M. Hedetniemi, S.T Hedetniemi and P. Kristiansen, *Alliances in graphs*, J. Combin. Math. Combin. Comput., to appear.

[6] N. Linial, D. Peleg, Y. Rabinovitch and M. Saks, *Sphere packings and local majorities in graphs*, In 2nd ISTCS, 141–149. IEEE Computer Soc. Press, June 1993.

[7] M. Luby, *A simple parallel algorithm for the maximal independent set problem*, SIAM J. Comput. **15** (1986) 1036–1053.

[8] D. Peleg, *Local majorities, coalitions and monopolies in graphs: a review*, Theoret. Comput. Sci. **282** (2002) 231–257.

[9] K.H. Shafique and R.D. Dutton, *On satisfactory partitioning of graphs*, Congress. Numer. **154** (2002) 183–194.

[10] S. Shelah and E.C. Milner, *Graphs with no unfriendly partitions*, in: A Tribute to Paul Erdős, A. Baker et al. eds. (Cambridge University Press, 1990) 373–384.