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VERTEX-DISJOINT COPIES OF K_4^-

Ken-ichi Kawarabayashi*

Department of Mathematics
Faculty of Science and Technology, Keio University
3–14–1 Hiyoshi, Kohoku-ku, Yokohama 223–8522, Japan

e-mail: k_keniti@comb.math.keio.ac.jp

Abstract

Let G be a graph of order n. Let K_l^- be the graph obtained from K_l by removing one edge.

In this paper, we propose the following conjecture:

Let G be a graph of order $n \geq lk$ with $\delta(G) \geq (n-k+1)\frac{l-3}{l-2}+k-1$. Then G has k vertex-disjoint K_l^-.

This conjecture is motivated by Hajnal and Szemerédi's [6] famous theorem.

In this paper, we verify this conjecture for $l = 4$.

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1. Introduction

In this paper, all graphs considered are finite, undirected and without loops or multiple edges. For a graph G, $V(G)$, $E(G)$, $\delta(G)$ and $\chi(G)$ denote the set of vertices and the set of edges, the minimum degree of G and the chromatic number of G, respectively. For a given graph G and $v \in V(G)$, we write $N_G(x)$ the neighborhood of $V(G)$ and $d_G(x) = |N_G(x)|$. For a subset S of $V(G)$, the subgraph induced by S is denoted by $< S >$. For a subgraph H of G, $G - H =< V(G) - V(H) >$ and for a vertex x of G,

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$G - x = < V(G) - \{x\} >$ and also for an edge e of $E(G)$, $G - e$ means the graph obtained from G by removing e. For a graph G, n is always the order of G. With a slight abuse of notation, for a subgraph H of G and a vertex $v \in V(G)$, $N_H(v) = N_G(v) \cap V(H)$ and $d_H(v) = |N_H(v)|$. In addition, for a subgraph H of G and a subset S of $V(G)$, $N_G(S) = \bigcup_{v \in S} N_G(v)$ and when $S \cap V(H) = \emptyset$, $N_H(S) = \bigcup_{v \in S} N_H(v)$. Let F be a given connected graph. Suppose that $|V(G)|$ is a multiple of $|V(F)|$. A spanning subgraph of G is called an F-factor if its components are all isomorphic to F.

There are many results concerning minimum degree conditions for a graph to have an F-factor. Hajnal and Szemerédi [6] proved that for $F = K_l$, $\delta(G) \geq \frac{l-1}{l}n$ suffices. More generally, Alon and Yuster [1] proved an asymptotic result that $\delta(G) \geq (\frac{\chi(F)-1}{\chi(F)} + o(1))n$ assures the existence of an F-factor.

In this paper, we will look at two classes of connected graphs of order 4, namely, K_4^- which is the graph obtained from K_4 by removing one edge (In this paper, we call it D), and the graph obtained from K_4 by removing two edges which have a common vertex (In this paper, we call it S).

For D, the author [7] proved the following.

Theorem 1 [7]. *Let G be a graph of order $4k$ with $\delta(G) \geq \frac{5}{2}k$. Then G has a D-factor.*

What happens if we consider the minimum degree condition for a given graph G of order $n \geq 4k$ to have k vertex-disjoint F ?

In case that F is $K_{1,3}$, Egawa and Ota [3] proved that, if G is a graph of order $n \geq 4k + 6$ with $\delta(G) \geq k + 2$, then G has k vertex-disjoint $K_{1,3}$.

In this paper, we prove the following theorem.

Theorem 2. *Suppose G is a graph of order $n \geq 4k$ with $\delta(G) \geq \frac{n+k}{2}$. Then G has k vertex-disjoint D.*

The condition of $\delta(G)$ is best possible. Consider the graph $G = \overline{K_{k-1}} + (\overline{K_{\frac{n-k+1}{2}}} + \overline{K_{\frac{n-k+1}{2}}})$. It is obvious that G contains at most $k-1$ vertex-disjoint triangles. So G does not have k vertex-disjoint D and the minimum degree is $\frac{n+k}{2} - 1$.

For the case S, as S is a subgraph of D and S has a triangle, we can get the following, and the condition of $\delta(G)$ is also best possible because of the same example as in Theorem 3.

Note that, in [8], it was proved that even the degree sum condition $n+k$ is good enough to have k vertex-disjoint S.

Let e be an edge in K_l. What happens if we consider k vertex-disjoint $(K_l - e)$? Since D is the graph obtained from complete graph K_4 by removing just one edge, for the case $l = 4$, we can get the result that a graph G of order $n \geq 4k$ with $\delta(G) \geq \frac{n+k}{2}$ has k vertex-disjoint $(K_4 - e)$. We propose the following conjecture.

Conjecture 1. Suppose G is a graph with $|V(G)| = n \geq lk$ and $\delta(G) \geq (n-k+1)\frac{l-3}{l-2} + k - 1$, where $l \geq 3$. Then G has k vertex-disjoint $(K_l - e)$.

The condition of $\delta(G)$ is best possible. Consider the graph $G = K_{k-1} + G'$, where G' is K_{l-1}-free graph. It is obvious that G contains at most $k-1$ vertex-disjoint K_{l-1}. So G does not have k vertex-disjoint $(K_l - e)$ and if G' is K_{l-1}-free, then the minimum degree is

$$\begin{aligned}&\frac{n-k+1}{l-2} \times (l-3) + k - 1 \\ &= (n-k+1)\frac{l-3}{l-2} + k - 1.\end{aligned}$$

Conjecture 1 is true for the case that $l = 3, 4$. (For $l = 3$, this case may follow from the result in [4] with some exceptional cases.) It seems that this conjecture is much more difficult than the complete graph K_l case. Note that Conjecture 1 is true for almost version, namely, if $\delta(G) \geq (n-k+1)\frac{l-3}{l-2} + k - 1$, then G contains $(1-o(1))k$ vertex disjoint copies of K_l^- when l is large. This result was proved by Komlós [9].

2. Preparation for the Proof of Theorem 2

The case of $n = 4k$ was already proved in [7], so we may assume $n > 4k$.

Let G be an edge-maximal counterexample. Since a complete graph of order $n > 4k$ has k vertex-disjoint D, so G is not a complete graph. Let u and v be nonadjacent vertices of G and define $G' = G + uv$, the graph obtained from G by adding the edge uv. Then G' is not a counterexample by the maximality of G and so G' has k vertex-disjoint D and that is, G' contains k vertex-disjoint subgraphs $D_1, \ldots, D_k$, where D_i is isomorphic to D or K_4. Since G is a counterexample, the edge uv lies in one of $D_1, \ldots, D_k$.

Without loss of generality, we may assume $uv \in E(D_k)$, that is, G has $k-1$ vertex-disjoint subgraph, $D_1, \ldots, D_{k-1}$ such that $\sum_{i=1}^{k-1} |D_i| = 4k-4$. Let H be the subgraph of G induced by $\bigcup_{i=1}^{k-1} V(D_i)$. Let M be the subgraph of G induced by $V(D_k)$. And also, let Z be the subgraph of G such that $Z := G - H - M$. Note that $< V(Z) \cup V(M) >$ does not contain D.

Since $uv \in E(D_k)$, M is obtained from D by removing just one edge. So there are two possibilities for M, namely S, C_4.

Now we choose $D_1, \ldots, D_{k-1}$ so that

(a) M is S or C_4.

(b) Subject to the condition (a), $\sum_{i=1}^{k-1} |E(D_i)|$ is as large as possible, that is, take K_4 instead of D as many as possible.

(c) Subject to the conditions (a) and (b), if there are still two possibilities for M, namely S, C_4, we choose S.

3. The Case where M is Isomorphic to C_4

We shall settle the case where M is isomorphic to C_4 by reducing the situation to the case where M is isomorphic to S.

With an additional notation, for each i, let a_i, b_i, c_i, d_i be the vertex in D_i such that $d_{D_i}(a_i) = d_{D_i}(c_i) = 3$, $d_{D_i}(b_i) \leq 3$ and $d_{D_i}(d_i) \leq 3$. (If D_i is D, $d_{D_i}(b_i) = d_{D_i}(d_i) = 2$ and b_i, d_i are nonadjacent. If D_i is K_4, $d_{D_i}(b_i) = d_{D_i}(d_i) = 3$.)

Suppose M is C_4. Let a, b, c, d be the vertices in C_4 with a and c being nonadjacent.

For a subgraph N of G, let $\mu_N = d_N(a) + d_N(b) + d_N(c) + d_N(d)$.

Claim 1. For any $z \in Z$, $\mu_z \leq 2$.

Proof. Assume the contrary. Since a, b, c, d are symmetric, without loss of generality, we may assume $az, bz, cz \in E(G)$. Then $< a, b, c, z >$ contains D, a contradiction. So, the result follows. ∎

For each D_i, $(i = 1, \ldots, k-1)$, we consider μ_{D_i}. If $\mu_{D_i} \leq 10$ for any D_i, $(i = 1, \ldots, k-1)$, then, since $d_M(a) + d_M(b) + d_M(c) + d_M(d) = 8$, we get the following:

$$\mu_G \leq 10(k-1) + 8 + 2(n-4k) = 2n + 2k - 2 < 2n + 2k$$

So, for some i, $\mu_{D_i} \geq 11$.

In the proof of Theorem 1 in [7], we have already proved the following lemma.

Lemma 1 ([7], Lemma 1)**.** *If $\mu_{D_i} \geq 11$, then the following hold:*

(1) *D_i is isomorphic to K_4,*

(2) *$\mu_{D_i} = 11$ and*

(3) *for each vertex $x \in V(M)$, $d_{D_i}(x) = 1, 3$ or 4 and for each edge $xy \in E(M)$, $d_{D_i}(x) + d_{D_i}(y) = 4$ or 7.*

So, without loss of generality, we may assume $\mu_{D_{k-1}} = 11$. We need some more definition.

Let e, f, g, h be the vertices in D_{k-1}. By (1), D_{k-1} is isomorphic to K_4. Hence e, f, g, h are symmetric. Also, by (2) and (3), we may assume $N_{D_{k-1}}(a) \cap N_{D_{k-1}}(c) = \{f, g, h\}$, $d_{D_{k-1}}(b) = 4$ and $de \in E(G)$. See in Figure 1.

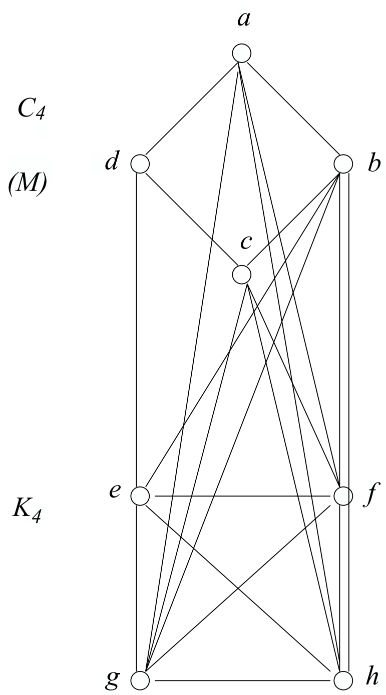

Figure 1

First, note that the following fact is easily observed.

Fact 1. $< a, d, e, f >$ is C_4 and $< b, c, g, h >$ is K_4. Also, $< c, d, e, f >$ is C_4 and $< a, b, g, h >$ is K_4, $< b, d, c, e >$ is C_4 and $< a, f, g, h >$ is K_4, and $< b, d, a, e >$ is C_4 and $< c, f, g, h >$ is K_4.

Proof. Trivial by Figure 1. ∎

For a subgraph N of G, let $\nu_N = d_N(a) + d_N(b) + d_N(c) + d_N(d) + d_N(e) + d_N(f)$. In the proof of Theorem 1 in [7], we proved the following claims.

Claim 2 ([7], Claim 3)**.** For $i = 1, \ldots, k-2$, if $d_{D_i}(d) = 4$, then $\nu_{D_i} \le 12$.

Claim 3 ([7], Claim 4)**.** For $i = 1, \ldots, k-2$, if $d_{D_i}(d) = 3$, then $\nu_{D_i} \le 13$.

Claim 4 ([7], Claim 5)**.** For $i = 1, \ldots, k-2$, if $d_{D_i}(d) = 2$, then $\nu_{D_i} \le 16$.

Claim 5 ([7], Claim 6)**.** For $i = 1, \ldots, k-2$, if $d_{D_i}(d) = 1$, then $\nu_{D_i} \le 18$.

Claim 6 ([7], Claim 7)**.** For $i = 1, \ldots, k-2$, if $d_{D_i}(d) = 0$, then $\nu_{D_i} \le 18$.

For $0 \le j \le 4$, let p_j denote the number of indices i such that $d_{D_i}(d) = j$.

By the definition, we get the following:

$$\sum_{j=0}^{4} p_j = k - 2 \,. \tag{1}$$

We prove the following claim.

Claim 7. For any $z \in Z$, $\nu_z \le 3$.

Proof. Assume the contrary. By Fact 1, $< c, d, e, f >$ contains C_4 and $< a, b, g, h >$ is K_4. So, by Claim 1, $d_{<a,b,c,d>}(z) \le 2$ and $d_{<c,d,e,f>}(z) \le 2$. Hence, we may assume $az, bz, ez, fz \in E(G)$. Then $< a, e, f, z >$ is D and $< b, c, g, h >$ is K_4, a contradiction. So, the result follows. ∎

Let $z \in Z$ be the vertex such that $\nu_z = 3$. By Claim 1 and Fact 1, we can get the following fact:
$|N_{<a,d,e,f>}(z)| \le 2$, $|N_{<c,d,e,f>}(z)| \le 2$, $|N_{<b,d,c,e>}(z)| \le 2$ and $|N_{<b,d,a,e>}(z)| \le 2$.

Suppose $dz \in E(G)$. Then, since $< a, b, c, d >$ is also C_4, the only possibility is $bz, dz, fz \in E(G)$. But in this case, $< z, b, f, c >$ contains a D, and $< a, e, g, h >$ contains a D, a contradiction. So, if $dz \in E(G)$, then

Theorem 3 holds. Therefore, we may assume $dz \notin E(G)$. This means, for any $z' \in Z$, if $dz' \in E(G)$, then $\nu_{z'} \leq 2$.

Let x be the number of the vertices $z \in Z$ such that $dz \in E(G)$ and let y be the number of the vertices $z \in Z$ such that $dz \notin E(G)$. For $0 \leq j \leq 4$, let p_j denote the number of indices i such that $d_{D_i}(d) = j$. By the definition, we get the following:

$$d_G(d) = p_1 + 2p_2 + 3p_3 + 4p_4 + 3 + x \geq \frac{n+k}{2}. \tag{2}$$

$$x + y = n - 4k. \tag{3}$$

We can easily get the facts that $d_M(a) = d_M(b) = d_M(c) = d_M(d) = 2$, $d_M(e) = 2$ and $d_M(f) = 3$. And $d_{D_{k-1}}(d) = 1$, $d_{D_{k-1}}(a) = d_{D_{k-1}}(c) = 3$, $d_{D_{k-1}}(b) = 4$, $d_{D_{k-1}}(e) = 3$ and $d_{D_{k-1}}(f) = 3$. So, by Claims 2-6, we get the following:

$$\begin{aligned} &\frac{n+k}{2} \times 6 = 3n + 3k \leq \nu_G \\ &\leq 18p_0 + 18p_1 + 16p_2 + 13p_3 + 12p_4 + 30 + 2x + 3y. \end{aligned} \tag{4}$$

From (2) $\times$ 4 + (4) $\times$ 2, we get the following:

$$36p_0 + 40p_1 + 40p_2 + 38p_3 + 40p_4 + 8x + 6y + 72 \geq 8n + 8k. \tag{5}$$

From (1) and (3), we get the following:

$$\begin{aligned} &36p_0 + 40p_1 + 40p_2 + 38p_3 + 40p_4 + 8x + 6y + 72 \\ &\leq 40(k-2) + 8(n-4k) + 72 = 8n + 8k - 8. \end{aligned} \tag{6}$$

But this contradicts (5). This completes the proof of the case that M is isomorphic to C_4.

4. The Case where M is Isomorphic to S

Finally, we consider the case that M is S.

Let a, b, c, d be the vertices of S such that $d_M(a) = 1$, $d_M(b) = 3$, $d_M(c) = d_M(d) = 2$. (c and d are symmetric.)

For a subgraph N of G, let $\mu_N = d_N(a) + d_N(b) + d_N(c) + d_N(d)$. Let B be the graph $< b, c, d >$.

Claim 8. For any $z \in Z$, $|N_B(z)| \leq 1$.

Proof. Assume, to the contrary. Then $< b, c, d, z >$ contains D, a contradiction. So, the result follows. ■

Note that, by Claim 8, we can get the fact that, for any $z \in Z$, $\mu_z \leq 2$.

In the proof of Theorem 1 in [7], we have already proved the following claims.

Claim 9 ([7], Claim 8)**.** For $i = 1, \ldots, k-1$, if $d_{D_i}(a) = 0$, then $\mu_{D_i} \leq 12$.

Claim 10 ([7], Claim 9)**.** For $i = 1, \ldots, k-1$, if $d_{D_i}(a) = 1$, then $\mu_{D_i} \leq 12$.

Claim 11 ([7], Claim 10)**.** For $i = 1, \ldots, k-1$, if $d_{D_i}(a) = 2$, then $\mu_{D_i} \leq 10$.

Claim 12 ([7], Claim 11)**.** For $i = 1, \ldots, k-1$, if $d_{D_i}(a) = 3$, then $\mu_{D_i} \leq 8$.

Claim 13 ([7], Claim 12)**.** For $i = 1, \ldots, k-1$, if $d_{D_i}(a) = 4$, then $\mu_{D_i} \leq 8$.

For $0 \leq j \leq 4$, let q_j denote the number of indices i such that $d_{D_i}(a) = j$.

By the definition, we get the following:

$$\sum_{j=0}^{4} q_j = k - 1. \tag{7}$$

We prove the following claim.

Claim 14. For some $z \in Z$, $\mu_z = 2$.

Proof. Assume, to the contrary. By Claim 8, we can easily get the fact that $\mu_z \leq 2$. So, we may assume that, for any $z \in Z$, $\mu_z \leq 1$. By the definition, we get the following:

$$q_1 + 2q_2 + 3q_3 + 4q_4 + 1 + n - 4k \geq d_G(a) \geq \frac{n+k}{2}. \tag{8}$$

And, since $d_M(a)+d_M(b)+d_M(c)+d_M(d)=8$, by Claims 9-13, we get the following:

$$\begin{aligned}(9)\qquad &\frac{n+k}{2}\times 4 = 2n+2k \le \mu_G\\ &\le 12q_0+12q_1+10q_2+8q_3+8q_4+8+n-4k.\end{aligned}$$

From (8) $\times 4+$ (9) $\times 3$, we can get the following:

$$(10)\qquad 36q_0+40q_1+38q_2+36q_3+40q_4+28 \ge n+36k > 40k.$$

From (7), we get the following:

$$(11)\quad 36q_0+40q_1+38q_2+36q_3+40q_4+28 \le 40(k-1)+28 = 40k-12.$$

But, this contradicts (10). So, the result follows. ■

We prove the following claim.

Claim 15. $< V(D)\cup V(M) >$ does not contain two vertex-disjoint triangles.

Proof. Assume, not. Note that we can assume $n \ge 4k+2$. Let T_1 and T_2 be two vertex-disjoint triangle induced by $\{v_1,v_2,v_3\}$ and induced by $\{v_4,v_5,v_6\}$, respectively. Let Z' be the subgraph of G such that $Z' := G-H-T_1-T_2$.

For a subgraph N of G, let $\nu_N = \sum_{i=1}^{6} d_N(v_i)$. We prove the following fact.

Fact 2. For any $z' \in Z'$, $|N_{T_1}(z')| \le 1$ and $|N_{T_2}(z')| \le 1$.

Proof. Assume, to the contrary. Without loss of generality, we may assume $|N_{T_1}(z')| \ge 2$. Then $< z', v_1, v_2, v_3 >$ contains D, a contradiction. So, the result follows. ■

Fact 3. For any $i=1,\dots,k-1$, $\nu_{D_i} \le 15$.

Proof. Assume, to the contrary. Since T_1 and T_2 are symmetric, without loss of generality, we may assume $d_{D_i}(v_1)+d_{D_i}(v_2)+d_{D_i}(v_3) \ge 8$. Then, there must exist at least two distinct vertices $v_i, u_i \in V(D_i)$ such that

$|N_{T_1}(v_i)| \geq 2$ and $|N_{T_1}(u_i)| \geq 2$. In this case, for any $s_i \in V(D_i)$, $|N_{T_2}(s_i)| \leq 1$. For otherwise, for some $s_i \in V(D_i)$, if $|N_{T_2}(s_i)| \geq 2$, then $< v_4, v_5, v_6, s_i >$ contains D and $< V(T_1) \cup (V(D_i) - \{s_i\}) >$ contains D, a contradiction. So, for any $s_i \in V(D_i)$, $|N_{T_2}(s_i)| \leq 1$. Therefore, $d_{D_i}(v_4) + d_{D_i}(v_5) + d_{D_i}(v_6) \leq 4$. So, we may assume $d_{D_i}(v_1) + d_{D_i}(v_2) + d_{D_i}(v_3) = 12$ and $d_{D_i}(v_4) + d_{D_i}(v_5) + d_{D_i}(v_6) = 4$. In this case, for some vertex $u \in V(T_2)$, say v_4, $d_{D_i}(v_4) \geq 2$.

We consider two cases for D.

Case 1. D_i is isomorphic to K_4.
Since a_i, b_i, c_i, d_i are symmetric, without loss of generality, we may assume $v_4a_i, v_4b_i \in E(G)$. Then, $< v_4, a_i, b_i, c_i >$ contains D and $< v_1, v_2, v_3, d_i >$ is K_4, a contradiction. So, the result follows.

Case 2. D_i is isomorphic to D.
Since a_i, c_i and b_i, d_i are symmetric, without loss of generality, we may assume $v_4a_i, v_4b_i \in E(G)$ or $v_4a_i, v_4c_i \in E(G)$ or $v_4b_i, v_4d_i \in E(G)$.

Suppose $v_4a_i, v_4b_i \in E(G)$ or $v_4a_i, v_4c_i \in E(G)$. Then $< v_4, a_i, b_i, c_i >$ contains D and $< v_1, v_2, v_3, d_i >$ is K_4, a contradiction. So, the result follows.

Suppose $v_4b_i, v_4d_i \in E(G)$. Then $< v_4, a_i, b_i, d_i >$ contains C_4 and $< v_1, v_2, v_3, c_i >$ is K_4, contrary to (b). So, the result follows. ■

Since there exist at most three edges connecting T_1 to T_2, by Facts 2 and 3, we get the following:

(12) $$\nu_G \leq 15(k-1) + 12 + 6 + 2(n - 4k - 2) = 2n + 7k - 1.$$

But, this contradicts the fact that $\nu_G \geq \frac{n+k}{2} \times 6 = 3n + 3k > 2n + 7k$. So, Claim 15 follows. ■

By Claim 14, there exists a vertex $z \in Z$ such that $\mu_z = 2$. For a subgraph N of G, let $\nu'_N = \mu_N + d_N(z)$. Let Z_1 be the subgraph of G that $Z_1 := G - H - M - \{z\}$. Since $< z, b, c, d >$ is S, by using Claims 9-12, the following fact is easily observed.

Fact 4.

(1) If $d_{D_i}(a) + d_{D_i}(z) = 0$, then $\nu'_{D_i} \leq 12$.
(2) If $d_{D_i}(a) + d_{D_i}(z) = 1$, then $\nu'_{D_i} \leq 12$.
(3) If $d_{D_i}(a) + d_{D_i}(z) = 2$, then $\nu'_{D_i} \leq 13$.
(4) If $d_{D_i}(a) + d_{D_i}(z) = 3$, then $\nu'_{D_i} \leq 11$.

(5) If $d_{D_i}(a) + d_{D_i}(z) = 4$, then $\nu'_{D_i} \leq 12$.
(6) If $d_{D_i}(a) + d_{D_i}(z) = 5$, then $\nu'_{D_i} \leq 10$.
(7) If $d_{D_i}(a) + d_{D_i}(z) = 6$, then $\nu'_{D_i} \leq 11$.
(8) If $d_{D_i}(a) + d_{D_i}(z) = 7$, then $\nu'_{D_i} \leq 11$.
(9) If $d_{D_i}(a) + d_{D_i}(z) = 8$, then $\nu'_{D_i} \leq 12$.

By Claim 8, we may assume $az \in E(G)$. Since c, d are symmetric, we only consider two cases that $bz \in E(G)$ or $cz \in E(G)$. We shall settle the case where $cz \in E(G)$ by reducing the situation to the case where $bz \in E(G)$.

Suppose $cz \in E(G)$. We prove the following fact.

Fact 5. For any $z_1 \in Z_1$, $\nu'_{z_1} \leq 2$.

Proof. Assume, to the contrary. Since $< z, a, b, c >$ is C_4, by Claim 1, $|N_{<z,a,b,c>}(z_1)| \leq 2$. So, we may assume $dz_1 \in E(G)$. So, by Claim 8, we may assume $dz_1, az_1, zz_1 \in E(G)$. Then $< z, z_1, a >$ is a triangle. As B is a triangle, this contradicts Claim 15. So the result follows. ■

We prove the following claim.

Claim 16. For any $i = 1, \ldots, k-1$, $\nu'_{D_i} \leq 12$.

Proof. Assume, to the contrary. To prove Claim 16, it is sufficient to consider only the case (3) of Fact 4.

Suppose $d_{D_i}(a)+d_{D_i}(z) = 2$ and $\nu'_{D_i} = 13$. First, we prove the following subclaim.

Subclaim 1. If $d_{D_i}(a) = 1$ and $\mu_{D_i} = 12$, then the followings hold:
(1) D_i is isomorphic to K_4.
(2) $d_{D_i}(b) = 3$ and $N_{D_i}(a) \cap N_{D_i}(b) = \emptyset$.

***Proof of* (1).** Assume that D_i is isomorphic to D. Since a_i, c_i and b_i, d_i are symmetric, without loss of generality, we may assume $aa_i \in E(G)$ or $ab_i \in E(G)$.

Suppose $aa_i \in E(G)$. Then $|N_B(b_i)| \leq 2$ and $|N_B(d_i)| \leq 2$. For otherwise, if $|N_B(b_i)| = 3$, then $< a, a_i, c_i, d_i >$ is S and $< b, c, d, b_i >$ is K_4, contrary to (b). Since b_i, d_i are symmetric, $|N_B(d_i)| \leq 2$. Therefore, $\mu_{D_i} \leq 1 + 10 = 11$, a contradiction.

Suppose $ab_i \in E(G)$. Then $|N_B(d_i)| \leq 2$. For otherwise, $< a, a_i, b_i, c_i >$ is S and $< b, c, d, d_i >$ is K_4, contrary to (b). Since $\mu_{D_i} - d_{D_i}(a) = 11$, we may assume that $|N_B(a_i)| = 3$, $|N_B(b_i)| = 3$, $|N_B(c_i)| = 3$ and $|N_B(d_i)| = 2$. Then $< a, b, a_i, b_i >$ contains D and $< c, d, c_i, d_i >$ contains D, a contradiction. So, the result follows. ∎

***Proof of* (2).** By (1), D_i is isomorphic to K_4. So, without loss of generality, we may assume $aa_i \in E(G)$. Suppose $d_{D_i}(b) \geq 3$ and $N_{D_i}(a) \cap N_{D_i}(b) = \{a_i\}$. Without loss of generality, we may assume $bb_i, bc_i \in E(G)$. Then $< a, b, a_i, b_i >$ is D and, since $|N_B(c_i)| = 3$ and $|N_B(d_i)| \geq 2$ or $|N_B(c_i)| \geq 2$ and $|N_B(d_i)| = 3$, $< c, d, c_i, d_i >$ contains D, a contradiction. So, the result follows. ∎

If $d_{D_i}(a) = 2$ or $d_{D_i}(z) = 2$, then, by Claim 11, $\mu_{D_i} \leq 10$, the result follows. So, we may assume $d_{D_i}(a) = d_{D_i}(z) = 1$. By (1), we may assume that D_i is isomorphic to K_4. By (2), $N_{D_i}(a) \cap N_{D_i}(b) = N_{D_i}(z) \cap N_{D_i}(c) = \emptyset$. Therefore, $d_{D_i}(b) \leq 3$ and $d_{D_i}(c) \leq 3$. Hence, $\nu'_{D_i} \leq 12$. So, Claim 16 follows. ∎

We can easily get the fact that $d_M(a) + d_M(b) + d_M(c) + d_M(d) = 8$ and $\mu_z = 2$, $|N_M(z)| = 2$. So, by Claim 16 and Fact 5, we get the following:

$$\nu'_G \leq 12(k-1) + 2(n-4k-1) + 12 = 2n + 4k - 2. \tag{13}$$

And also, we get the following:

$$\nu'_G \geq \frac{n+k}{2} \times 5 > 2n + \frac{9}{2}k. \tag{14}$$

But this contradicts (13). This proves the case $cz \in E(G)$.

Finally, suppose $bz \in E(G)$. By the same argument in the proof of Claim 8, we can easily get the following fact.

Fact 6. For any $z_1 \in Z_1$, $\nu'_{z_1} \leq 2$.

We prove the following claim.

Claim 17. For any $i = 1, \ldots, k-1$, $\nu'_{D_i} \leq 12$.

Proof. Assume, to the contrary. To prove Claim 17, it is sufficient to consider the case (3) of Fact 4.

Suppose $d_{D_i}(a) + d_{D_i}(z) = 2$ and $\nu_{D_i} = 13$. If $d_{D_i}(a) = 2$ or $d_{D_i}(z) = 2$, then, by Claim 11, $\mu_{D_i} \leq 10$, the result follows. So, we may assume $d_{D_i}(a) = d_{D_i}(z) = 1$. Since $< z, b, c, d >$ is S and $zb \in E(G)$, by the same proof of Subclaim 1, we may assume that D_i is isomorphic to K_4 and $N_{D_i}(a) \cap N_{D_i}(b) = N_{D_i}(z) \cap N_{D_i}(b) = \emptyset$. If $N_{D_i}(a) \cap N_{D_i}(z) = \emptyset$, then $d_{D_i}(b) \leq 2$, and hence $\nu'_{D_i} \leq 12$, and the result follows. So, we may assume that $aa_i, za_i \in E(G)$, $d_{D_i}(c) = d_{D_i}(d) = 4$, $d_{D_i}(b) = 3$ and $ba_i \notin E(G)$. Then $< a, b, z, a_i >$ contains D and $< c, b_i, c_i, d_i >$ is K_4, a contradiction. So, the result follows. ■

We can easily get the fact that $d_M(a) + d_M(b) + d_M(c) + d_M(d) = 8$ and $\mu_z = 2$, $|N_M(z)| = 2$. So, by Claim 17 and Fact 6, we get the followings:

$$\nu'_G \leq 12(k-1) + 2(n-4k-1) + 12 = 2n + 4k - 2. \tag{15}$$

And also, we get the following:

$$\nu'_G \geq \frac{n+k}{2} \times 5 > 2n + \frac{9}{2}k. \tag{16}$$

But this contradicts (15). So, Theorem 3 follows. ■

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