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MINIMAL REGULAR GRAPHS WITH GIVEN GIRTHS AND CROSSING NUMBERS

G.L. Chia

Institute of Mathematical Sciences
University of Malaya
50603 Kuala Lumpur, Malaysia

AND

C.S. Gan

Faculty of Engineering and Technology
Multimedia University
75450 Malacca, Malaysia

Abstract

This paper investigates on those smallest regular graphs with given girths and having small crossing numbers.

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1. Introduction

Let G be a graph. The *girth* of G is the length of a smallest cycle in G. The *crossing number* of G, denoted $cr(G)$, is the minimum number of pairwise intersections of its edges when G is drawn in the plane.

An (r, g)-*graph* is an r-regular graph with girth g. Let $f(r, g)$ denote the minimum number of vertices in an (r, g)-graph. An (r, g)-graph with the minimum number of vertices is known as an (r, g)-*cage*. The problem of determining $f(r, g)$ or finding an (r, g)-cage is an old problem in graph theory. (See [11].) Most of the cages have high crossing numbers as is shown by inequality (2) below.

Let G be a graph. The *removal number* of G, denoted $rem(G)$, is defined to be the minimum number of edges in G whose removal results in a planar graph. Obviously $cr(G) \geq rem(G)$.

Recall Euler's formula for plane graphs. This states that $n + f \geq m + 2$ where n, m and f denote the number of vertices, edges and faces in the plane graph G. Equality holds if and only if G is a connected graph.

Let n and m denote the number of vertices and edges respectively in an (r, g)-graph G. Then $rn = 2m$. Let G^* denote the planar graph obtained from G by deleting $rem(G)$ edges. Let f denote the number of faces in G^*. Then from Euler's formula and from the Hand-Shaking Lemma for plane graphs, we have $n - (m - rem(G)) + f \geq 2$ and $2(m - rem(G)) \geq gf$ respectively. By getting rid of m and f, we have

$$rem(G) \geq \frac{4g + (r(g-2) - 2g)n}{2(g-2)} \tag{1}$$

Since $cr(G) \geq rem(G)$ and $n \geq f(r, g)$, we have

$$cr(G) \geq \frac{4g + (r(g-2) - 2g)f(r,g)}{2(g-2)}. \tag{2}$$

In this paper, we are interested in those (r, g)-graphs having small crossing numbers.

Call a graph an (r, g, c)-*graph* if it is an (r, g)-graph with crossing number c. Let $f(r, g, c)$ denote the minimum number of vertices in an (r, g, c)-graph. Our problem is to determine $f(r, g, c)$ and, if possible, a smallest (r, g, c)-graph. Note that for a given c, an (r, g, c)-graph may not exist.

Obviously, $f(2, g, 0) = g$ for any $g \geq 3$ and the g-cycle is the only smallest $(2, g, 0)$-graph. Note that $f(2, g, c)$ does not exist for any $c \geq 1$. In what follows, we shall assume that r, $g \geq 3$. There are three cases which we wish to consider. The case $c = 0$ is treated in Section 2 and the cases $c = 1$ and $c = 2$ in the subsequent sections.

Throughout this paper, we let K_s denote a complete graph on s vertices and $K_{s,t}$ the complete bipartite graphs whose two partite sets have s and t vertices.

2. The Case $c = 0$

In this case, all the graphs under consideration are planar graphs. Let G be an $(r, g, 0)$-graph. Then by counting the number of edges and the number of vertices in G, and by using Euler's formula for plane graphs, we have

$$\sum_{i\geq g}(2i-(i-2)r)f_i \quad = \quad 4r \tag{3}$$

where f_i is the number of i-sided faces in G. From this formula, it is easily deduced that $g \leq 5$ and that $r \leq 5$. A closer look at the above formula reveals that there are only five possible pairs of (r, g), namely $(3,3), (3,4), (3,5), (4,3)$ and $(5,3)$. For each pair of (r, g), we shall determine the value of $f(r, g, 0)$ and the corresponding possible smallest $(r, g, 0)$-graphs.

Note that in an $(r, g, 0)$-graph G with n vertices, m edges and f faces, we have $rn = 2m \geq gf$. Substituting these into Euler's formula for plane graphs, we have $m \leq \frac{g}{g-2}(n-2)$ where equality holds if and only if every face of G is a g-sided face. Replacing m by $\frac{rn}{2}$, we have

$$n \quad \geq \quad \frac{4g}{2r+2g-rg}. \tag{4}$$

Immediate from the above inequality, it is deduced that $f(3,3,0) \geq 4$, $f(3,4,0) \geq 8$, $f(3,5,0) \geq 20$, $f(4,3,0) \geq 6$ and $f(5,3,0) \geq 12$. It is easy to see that K_4 is the only smallest $(3,3,0)$-graph and so $f(3,3,0) = 4$. Also, it is easily shown that the cube is the only smallest $(3,4,0)$-graph and so $f(3,4,0) = 8$.

Likewise, it is readily verified that the octahedron is the only smallest $(4,3,0)$-graph and so $f(4,3,0) = 6$.

The icosahedron shows that $f(5,3,0) = 12$. To see that it is the only smallest $(5,3,0)$-graph, note that any $(5,3,0)$-graph on 12 vertices is maximal planar. This means that every face is a triangle and that $f_3 = 20$. The only graph that satisfies these conditions is the icosahedron. (See [1] pp. 159–161.)

The dodecahedron shows that $f(3,5,0) = 20$. To see that it is the only smallest $(3,5,0)$-graph, note that any $(3,5,0)$-graph G on 20 vertices has 30 edges. Since $m = \frac{g(n-2)}{g-2}$, every face in G is a 5-sided face. Hence $f_5 = 12$. The only graph that satisfies these conditions is the dodecahedron. (See [1] pp. 159–161.)

We shall summarize the above observations in the following theorem.

Theorem 1. *The five platonic solids are the only smallest* $(r, g, 0)$*-graphs.*

3. The Case $c = 1$

Let G be an $(r, g, 1)$-graph with n vertices and m edges. Then $cr(G) = 1 = rem(G)$ and so there exists an edge e in G such that $G - e$ is planar with degree sequence $(r-1, r-1, r, \ldots, r)$. Note that the girth of $G-e$ is at least g. Let f denote the number of faces in $G - e$ and let f_i denote the number of i-sided faces in $G - e$. Then $2(m-1) = \sum_{i \geq g} i f_i = 2(r-1) + r(n-2)$ and $f = \sum_{i \geq g} f_i$. Substituting these into Euler's formula for plane graphs, we have

$$\sum_{i \geq g} (2i - (i-2)r) f_i \quad \geq \quad 4r - 4. \tag{5}$$

Like the case of planar graphs, it is readily deduced that there are only five possible pairs of (r, g), namely $(3, 3), (3, 4), (3, 5), (4, 3)$ and $(5, 3)$.

Notice that the number of vertices in G satisfies the following inequality

$$n \quad \geq \quad \frac{2g + 4}{2r + 2g - rg}. \tag{6}$$

It is easy to see that $f(4, 3, 1) = 5$ because K_5 is the smallest $(4, 3, 1)$-graph. Also, $f(3, 4, 1) = 6$ because $K_{3,3}$ is the smallest $(3, 4, 1)$-graph.

Now $f(3, 3, 1) \geq 8$. To see this, let H be a $(3, 3, 1)$-graph on no more than 6 vertices. Then H must be isomorphic to the complete bipartite graph $K_{3,3}$. But this is not possible because $K_{3,3}$ contains no triangle.

A smallest $(3, 3, 1)$-graph is obtained from $K_{3,3}$ by replacing a vertex by a triangle with edges joining the triangle in a corresponding way (see Figure 1). To see the uniqueness of this graph, let H be a smallest $(3, 3, 1)$-graph.

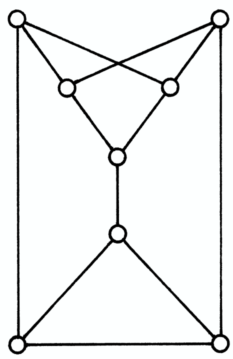


Figure 1. Smallest $(3, 3, 1)$-graph

Then H contains a triangle. Replace this triangle by a vertex of degree 3, the resulting graph is a non-planar cubic graph on 6 vertices which must be the complete bipartite graph $K_{3,3}$.

It follows from the inequality (6) that $f(3,5,1) \geq 14$. The graph depicted in Figure 2 is a $(3,5,1)$-graph and so $f(3,5,1) \leq 18$. In [10], Royle has listed all the $(3,5)$-graphs on 14 and 16 vertices. We check that all these graphs have crossing number at least 2. There are eight $(3,5)$-graphs with 14 vertices. Six of them contains a subdivision of the Petersen graph as a subgraph. One of the two remaining graphs in the list is the graph G_7 of Figure 6 (which has crossing number 2 as will be explained later in the last section). The last graph in the list has removal number at least 2 because the resulting graph obtained by deleting any edge from this graph contains a subdivision of $K_{3,3}$ as subgraph. There are 48 $(3,5)$-graphs on 16 vertices. Except for the last graph, all of them has crossing number at least 2 because they all contain a subdivision of the graph G_3 (which has crossing number 2 as will be explained later in the section) of Figure 6 as subgraph. The last graph in the list contains a subdivision of the Petersen graph as a subgraph. Hence all of these graphs have crossing number at least 2. Thus we may conclude that $f(3,5,1) = 18$. However we do not know whether or not the graph in Figure 2 is the only smallest $(3,5,1)$-graph.

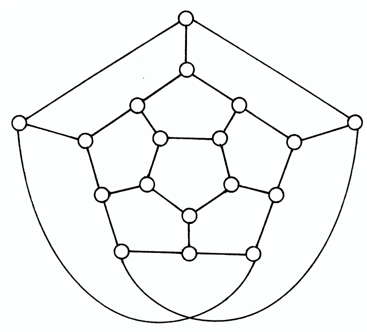


Figure 2. Smallest $(3,5,1)$-graph

The graph in Figure 3 is a $(5,3,1)$-graph and so $f(5,3,1) \leq 14$. Since $10 \leq f(5,3,1)$, there are only three possible values of $f(5,3,1)$. In the next section, we shall prove that $f(5,3,1) \geq 12$. We do not know whether or not a smallest $(5,3,1)$-graph is unique.

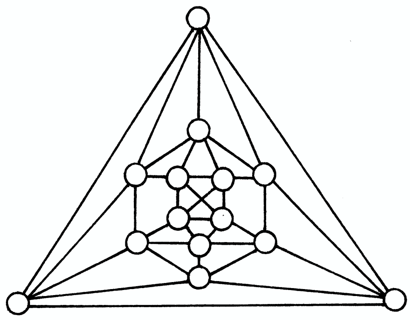


Figure 3. A $(5,3,1)$-graph

We shall summarize the above observations in the following theorem.

Theorem 2. *$f(3,3,1)=8$, $f(3,4,1)=6$, $f(3,5,1)=18$, $f(4,3,1)=5$ and $12 \le f(5,3,1) \le 14$.*

4. $(5,3,1)$-Graphs

In this section, we prove that $(5,3,1)$-graphs do not exist if the number of vertices is no more than 10.

Proposition 1. *Let G be a 5-regular graph on 10 vertices. Then $cr(G) \ge 2$.*

The proof of this proposition is by contradiction. It consists of a series of lemmas that we shall now prove. It is easy to see that if G is 5-regular and has 10 vertices, then it is non-planar and so $cr(G) \ge 1$. In this section, unless otherwsie stated, we shall assume that G is a 5-regular graph on 10 vertices and that $cr(G) = 1$ and obtain a contradiction.

Let H be a regular graph and let v be a vertex of H. Let A_v denote the subgraph of H induced by the set of vertices adjacent to v. Also, let B_v denote the subgraph obtained by deleting $A_v \cup \{v\}$ from H.

Lemma 1. *Let H be an r-regular graph on $2r$ vertices. Suppose $v \in V(H)$. Then A_v and B_v have the same number of edges.*

Proof. Note that A_v and B_v have r and $r-1$ vertices respectively. Let $(a_1, \ldots, a_r)$ and $(b_1, \ldots, b_{r-1})$ denote the degree sequences of A_v and B_v respectively.

Note that the number of edges in the subgraph $H - v$ is $r(r-1)$ and that it is also equal to $|E(A_v)| + |E(B_v)| +$ number of edges from A_v to B_v.

Therefore,

$$\frac{\sum_{i=1}^{r} a_i}{2} + \frac{\sum_{i=1}^{r-1} b_i}{2} + \sum_{i=1}^{r}(r - 1 - a_i) = r(r-1)$$

which, on simplification, leads to $\frac{\sum_{i=1}^{r} a_i}{2} = \frac{\sum_{i=1}^{r-1} b_i}{2}$. Thus A_v and B_v have the same number of edges. ■

Lemma 2. *If $rem(G) = 1$, then there is an edge e in G such that $G - e$ is a planar triangulation.*

Proof. Since $rem(G) = 1$, there is an edge e in G such that $G - e$ is planar. Since $G - e$ has 10 vertices and 24 edges, $G - e$ is a triangulation. ■

Lemma 3. *If G contains a 4-cycle $a_0a_1a_2a_3$, then either $a_ia_{i+2} \in E(G)$ for some i or else there is a vertex $u \in V(G)$ such that u is adjacent to a_i for all i. Here i is considered modulo 4.*

Proof. This is a consequence of Lemma 2 because otherwise $G - e$ is not a triangulation for any edge e in G. ■

Lemma 4. *A_v is connected for any vertex v in G.*

Proof. Suppose A_v is disconnected for some vertex v in G. Let w_1 and w_2 be two vertices in two different components of A_v, say G_1 and G_2 respectively.

Since $|V(G_1)| + |V(G_2)| \leq 5$, we may assume that w_1 is of degree d where $d \leq 1$ and w_2 is of degree $3 - d$ in A_v. Therefore w_1 and w_2 must be adjacent to a common vertex z in B_v. But then w_1zw_2v is a 4-cycle in G not satisfying Lemma 3. This contradiction proves that A_v is connected. ■

Lemma 4 implies that A_v has at least 4 edges because it has 5 vertices. By Lemma 1, because A_v and B_v have the same number of edges, it follows that A_v has at most 6 edges because B_v has 4 vertices. Next, we shall dispose of some degree sequences of A_v.

Lemma 5. *For any $v \in V(G)$, A_v contains no vertices of degree 4.*

Proof. Suppose on the contrary that A_v contains a vertex u of degree 4. Evidently there exist two non-adjacent vertices x and y in $A_v - u$.

If x and y are adjacent to a common vertex z in B_v (call this condition (*)), then $xzyu$ is a 4-cycle in G not satisfying Lemma 3.

Notice that the condition (*) is always satisfied if $|E(A_v)| \leq 5$.

So assume that $|E(A_v)| = 6$ and that the condition (*) is not satisfied. Then A_v is the graph of Figure 4(a). In this case, the only graph in which condition (*) is not satisfied is the graph of Figure 4(b). However, the removal number of this graph is at least 2 because it contains a subdivision of $K_{3,4}$ (the subgraph induced by the thick edges) as a subgraph. This contradiction proves the lemma. ∎

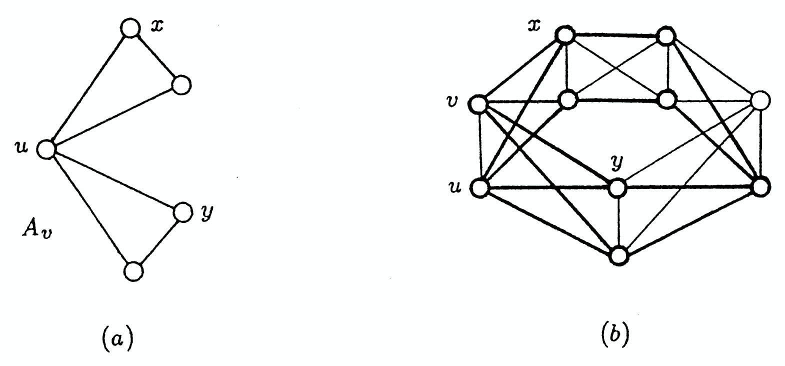


Figure 4. Graph not satisfying condition (*)

Lemma 6. *It is not possible that A_v contains two non-adjacent vertices v_1 and v_2 such that $d_{A_v}(v_1) = 1$ and $d_{A_v}(v_2) \leq 2$.*

Proof. If these conditions are satisfied, then v_1 must be adjacent to three vertices of B_v and v_2 must be adjacent to at least two vertices of B_v. This means that they must be adjacent to a common vertex z of B_v. But then G contains a 4-cycle vv_1zv_2 not satisfying Lemma 3. ∎

Lemmas 5 and 6 imply that the only possible degree sequences for A_v left to be considered are $(2, 2, 2, 2, 2)$ and $(2, 2, 2, 3, 3)$. There are only three graphs associated with these degree sequences. However, these three cases are disposed of in the next three lemmas.

Lemma 7. *A_v is not the graph $K_{2,3}$.*

Proof. In this case, B_v is the complete graph on 4 vertices. Let x_1, x_2 and x_3 denote the three vertices of degree 2 in $K_{2,3}$. Then each x_i must be adjacent to two vertices in B_v. But this means that there are x_i and x_j $(i \neq j)$ which are adjacent to a common vertex z in B_v giving rise to a 4-cycle vx_izx_j in G not satisfying Lemma 3. ∎

Lemma 8. *A_v is not a cycle on 5 vertices.*

Proof. Suppose A_v is a cycle on 5 vertices whose vertices are labelled as $v_1, v_2, \ldots, v_5$ in cyclic order.

Note that B_v is the graph obtained from the complete graph on 4 vertices by deleting an edge. So B_v contains two vertices x_1 and x_2 of degree 2, and two vertices y_1 and y_2 of degree 3. Clearly each x_i is adjacent to three vertices of A_v.

Now, x_1 and x_2 are adjacent to a common vertex w in A_v. But then $x_1wx_2y_1$ or $x_1wx_2y_2$ is a 4-cycle in G not satisfying Lemma 3. ∎

Lemma 9. *A_v is not the graph of Figure* 5(a).

Proof. Suppose A_v is the graph of Figure 5(a). In this case, B_v is the complete graph on 4 vertices. Let $V(B_v) = \{u_1, \ldots, u_4\}$.

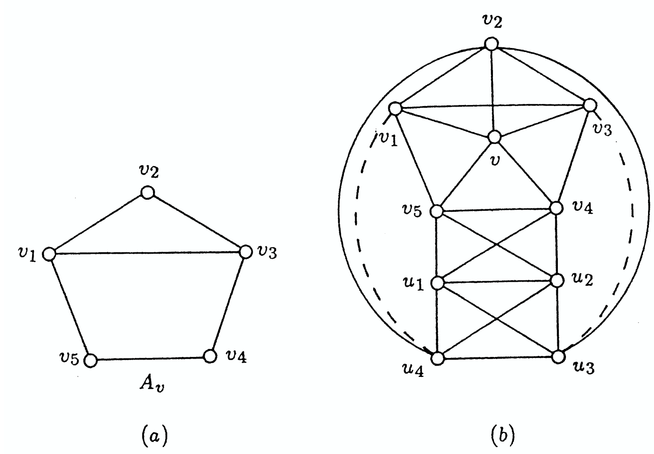


Figure 5

Note that any two non-adjacent vertices x and y in A_v must not be adjacent to a common vertex z in B_v. This is because otherwise we have a 4-cycle $vxzy$ in G not satisfying Lemma 3.

Hence we may assume without loss of generality that v_2 is adjacent to u_3 and u_4 and that v_4 is adjacent to u_1 and u_2. Since v_5 and v_2 are non-adjacent, v_5 is adjacent to u_1 and u_2 (see Figure 5(b)).

Now v_1 is adjacent either to u_3 or u_4. However in either case, we have a 4-cycle $v_1v_5u_1w$ not satisfying Lemma 3, where $w \in \{u_3, u_4\}$. This contradiction proves the lemma. ■

5. The Case $c = 2$

Let G be an $(r, g, 2)$-graph with n vertices. Then $1 \leq rem(G) \leq 2$. Let $f_i(r, g, c)$ denote the minimum number of vertices in an (r, g, c)-graph with removal number i, $i = 1, 2$.

The rest of this paper is to prove the following theorem.

Theorem 3. *$f_1(3,3,2) = 12$, $f_1(3,4,2) = 10$, $f_1(3,5,2) = 14$, $f_1(4,3,2) = 8$, $12 \leq f_1(5,3,2) \leq 16$, $f_2(3,3,2) = 12$, $f_2(3,4,2) = 12$, $f_2(3,5,2) = 10$, $f_2(4,3,2) = 7$ and $f_2(5,3,2) = 8$.*

Suppose $rem(G) = 1$. Then inequalities (5) and (6) also hold for G and so there are only five possible pairs of (r, g), namely $(3,3), (3,4), (3,5), (4,3)$ and $(5,3)$.

Suppose $rem(G) = 2$. Let x_1 and x_2 be two edges of G such that $G - \{x_1, x_2\}$ is planar. Then the degree sequence of $G - \{x_1, x_2\}$ is either $(r-2, r-1, r-1, r, \cdots, r)$ or $(r-1, r-1, r-1, r-1, r, \cdots, r)$. Following similar argument as was done for the case $c = 1$, we have

$$\sum_{i \geq g} (2i - (i-2)r) f_i \quad \geq \quad 4r - 8 \tag{7}$$

and the number of vertices in G satisfies the following inequality

$$n \quad \geq \quad \frac{8}{2r + 2g - rg}. \tag{8}$$

Again, only the same five pairs of (r, g) satisfy inequality (7).

Let G be a non-planar graph and let e be an edge in G. Then e is called a *p-critical edge* if $G - e$ is a planar graph.

Lemma 10. *Let G be a non-planar graph. If G contains a unique p-critical edge, then $cr(G) \geq 2$.*

Proof. If $cr(G) = 1$, then there exist two edges e_1 and e_2 in G which intersect each other and such that $G - \{e_i\}$ is planar for $i = 1, 2$. But this contradicts the uniqueness of the removal edge of G. ■

5..1 $(3, g, 2)$-graphs

In [9] (p. 647–648), Royle has listed all connected cubic graphs of order up to and including 10. It is easily seen that there are only two cubic graphs in the list with $c > 1$, the Petersen graph G_8 and the graph G_3 in Figure 6. Thus $f_i(3, g, 2) \geq 10$. Further, Royle's list also indicates that $f_2(3, 4, 2) \geq 12$ and that $f_i(3, 3, 2) \geq 12$.

Clearly, the Petersen graph is a $(3, 5, 2)$-graph. It has removal number 2 because removing any one of its edges yields a non-planar graph. Therefore $f_2(3, 5, 2) = 10$ and the Petersen graph is the only smallest $(3, 5, 2)$-graph with removal number 2.

Note that the graph G_3 in Figure 6 has crossing number 2. To see this, we note that G_3 contains two vertex-disjoint graphs $K_{2,3}$ which are not outerplanar. If $cr(G_3) = 1$, then at least one of these subgraphs has a planar drawing. This subgraph does not separate the vertices of the other $K_{2,3}$ subgraph and, because of outerplanarity, its edges are crossed by at least one of the edges e_1, e_2 and e_3. A similar argument can be used for the other $K_{2,3}$ subgraph. Hence G_3 is a $(3, 4, 2)$-graph. Moreover it has removal number 1 because $G_3 - e_i$ is planar for any $1 \leq i \leq 3$. Therefore $f_1(3, 4, 2) = 10$ and G_3 is the only smallest $(3, 4, 2)$-graph with removal number 1.

$f_1(3, 5, 2) \geq 14$ follows from inequality (1). In Royle's list of cubic graphs on 14 vertices [10], there are eight graphs with girth equal to 5. All but the graph G_7 contain a subdivision of the Petersen graph as a subgraph. This means that except for the graph G_7, they all have removal number at least 2. It is a routine exercise to verify that the edge e is the only p-critical edge in G_7. By Lemma 10, $cr(G_7) \geq 2$. Therefore $f_1(3, 5, 2) = 14$ and G_7 is the only smallest $(3, 5, 2)$-graph with removal number 1.

Let H be a smallest $(3, g, 2)$-graph with removal number i and $g \neq 3$. We may obtain a $(3, 3, 2)$-graph with removal number i by replacing a vertex of degree 3 from H by a triangle with edges joining the triangle in a corresponding way. The graph G_1 (respectively G_2) is obtained from G_3 (respectively the Petersen graph) in this way. Combining this with the previous observations, we have $f_1(3,3,2) = 12$ and $f_2(3,3,2) = 12$. The uniqueness of these graphs follows from that of the graph G_3 and the Petersen graph.

We now look at Royle's list of cubic graphs on 12 vertices with girth 4 (see [10]). There are twenty such cubic graphs. However there are only three graphs G_4, G_5 and G_6 from this list with removal number at least 2. Since these three graphs can all be drawn on the plane with only two crossings, they are the smallest $(3,4,2)$-graphs with removal number 2. Thus $f_2(3,4,2) = 12$.

5..2 $(4,3,2)$-graphs

Let G be a $(4,3,2)$-graph on n vertices. Then clearly $n \geq 6$.

If $n = 6$, then G is the complementary graph of $3K_2$ and is planar. Hence $n \geq 7$.

If $n = 7$, then $\overline{G}$ is a 2-regular graph which is either $C_3 \cup C_4$ or C_7. If $\overline{G}$ is $C_3 \cup C_4$, then G is the graph G_{10} and $rem(G_{10}) = 2$ because G_{10} contains $K_{3,4}$ as a subgraph. If $\overline{G}$ is C_7, then $cr(G) = 1$ (see [2]). Therefore $f_2(4,3,2) = 7$ and G_{10} is the only smallest $(4,3,2)$-graph with removal number 2.

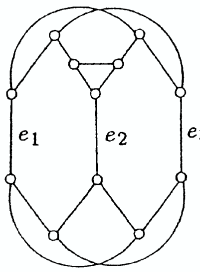
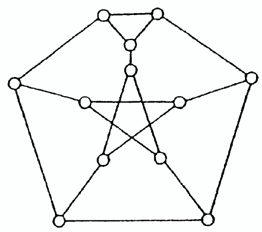
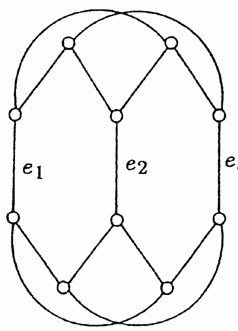
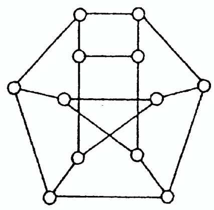
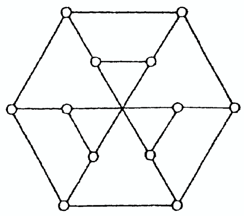
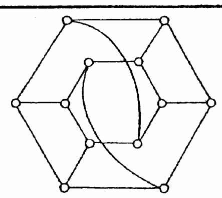
If $n = 8$, then $\overline{G}$ is a cubic graph on 8 vertices. There are precisely five cubic graphs on 8 vertices. If $\overline{G}$ is the cube, then G is the graph G_9 which is the cartesian product $K_4 \times K_2$ and has crossing number 2 (see [8]). Now, $rem(G_9) = 1$ because $G_9 - e_i$ is planar for each $i = 1,2,3$. If $\overline{G}$ is not the cube, we have checked, by direct verification that either $rem(G) \geq 2$ or else $cr(G) \leq 1$. Therefore $f_1(4,3,2) = 8$ and G_9 is the only smallest $(4,3,2)$-graph with removal number 1.

5..3 $(5,3,2)$-graphs

It follows from inequality (1) that $f_1(5,3,2) \geq 10$. However the proof of Proposition 1 implies that $f_1(5,3,2) \geq 12$. Clearly, the graph G_{11} in Figure 6 has removal number 1. We have checked that the edge e is the only removal edge. By Lemma 10, $cr(G_{11}) = 2$. Hence G_{11} is a $(5,3,2)$-graph with $rem(G_{11}) = 1$ and so $12 \leq f_1(5,3,2) \leq 16$.

Let G be a 5-regular graph on 8 vertices. Then $\overline{G}$ is a 2-regular graph. Hence there are only three 5-regular graphs on 8 vertices namely, $\overline{C_8}$, $\overline{C_3 \cup C_5}$ and $\overline{C_4 \cup C_4}$. Since $\overline{C_r \cup C_s}$ contains $K_{r,s}$ as a subgraph, it follows that $cr(\overline{C_3 \cup C_5}) \geq 4$ and $cr(\overline{C_4 \cup C_4}) \geq 4$.

Now it follows from inequality (1) that $rem(\overline{C_8}) \geq 2$. Figure 6 depicts a drawing of $\overline{C_8}$ ($\cong G_{12}$) with two crossings and we have $rem(G_{12}) = 2 = cr(G_{12})$. Thus $f_2(5,3,2) = 8$ and G_{12} is the only smallest $(5,3,2)$-graph with removal number 2.

$(r,g,2)$	$rem(G)=1$	$rem(G)=2$
$(3,3,2)$	G_1  $f_1(3,3,2)=12$	G_2  $f_2(3,3,2)=12$
$(3,4,2)$	G_3  $f_1(3,4,2)=10$	G_4  G_5  G_6  $f_2(3,4,2)=12$

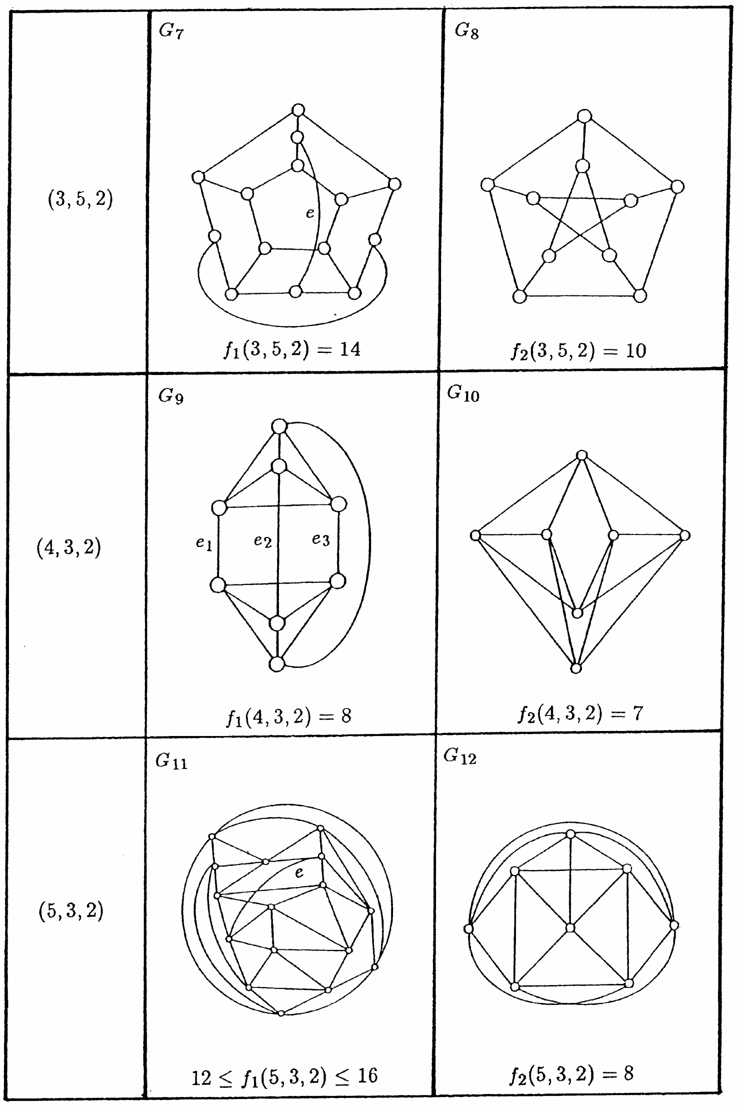


Figure 6

Acknowledgement

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