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BOUNDS FOR INDEX OF A MODIFIED GRAPH

Bo Zhou*

Department of Mathematics
South China Normal University
Guangzhou 510631, P.R. China

e-mail: zhoubo@scnu.edu.cn

Abstract

If a graph is connected then the largest eigenvalue (i.e., index) generally changes (decreases or increases) if some local modifications are performed. In this paper two types of modifications are considered:

(i) for a fixed vertex, t edges incident with it are deleted, while s new edges incident with it are inserted;

(ii) for two non-adjacent vertices, t edges incident with one vertex are deleted, while s new edges incident with the other vertex are inserted.

Within each case, we provide lower and upper bounds for the indices of the modified graphs, and then give some sufficient conditions for the index to decrease or increase when a graph is modified as above.

Keywords: graph, eigenvalue, principal eigenvector.

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1. Introduction and Preliminaries

We consider only finite undirected graphs without loops or multiple edges. The eigenvalues of a graph are the eigenvalues of the adjacency matrix of this graph. The largest eigenvalue of a graph G is called the index of G, denoted by $\mu_1(G)$. If a graph G is connected, then the unique positive unit eigenvector $\boldsymbol{x}$ corresponding to $\mu_1(G)$ is called the principal eigenvector

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of the labelled graph G. The study of graph perturbations is concerned primarily with changes in eigenvalues which result from local modifications of a graph. Maas [2], Rowlinson [3, 4] and Zhou [6] have investigated the indices of various modified graphs of a connected graph.

Let G be a graph with $u \in V(G)$. The neighborhood $N_G(u)$ of u is the set of vertices adjacent to u. The closed neighborhood of u is the set $N_G[u] = N_G(u) \cup \{u\}$.

In this paper, we provide lower and upper bounds for the indices of two types of modified graphs obtained from a connected graph G; one is obtained from G by deleting t edges incident with a given vertex u and adding s new edges between u and the vertices in $V(G) \setminus N[u]$; the other is obtained from G by deleting t edges incident with a given vertex v and adding s new edges between another given vertex u which is not adjacent to v and the vertices in $V(G) \setminus (N[u] \cup \{v\})$. These bounds are then used to derive sufficient conditions for the index to decrease or increase when G is modified as above. Some results of [2, 4] are generalized.

Our proof of the upper bounds for indices of modified graphs is based on intermediate eigenvalue problems of the second type. We outline the results required from [5] (also see [1]) in terms of an n-dimensional Euclidean space V in which $(\boldsymbol{u}, \boldsymbol{v})$ denotes the inner product of vectors $\boldsymbol{u}$ and $\boldsymbol{v}$. For a symmetric linear transformation T of V, let $\lambda_1(T), \ldots, \lambda_n(T)$ denote the eigenvalues of T in non-decreasing order.

Let $\tilde{A}$ be a symmetric linear transformation of V and let $\tilde{B}$ be a positive linear transformation of V. A second inner product is defined on V by $[\boldsymbol{u}, \boldsymbol{v}] = (\tilde{B}\boldsymbol{u}, \boldsymbol{v})$. Choose any basis $\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n\}$ for V and, using the second inner product, let P_r be the orthogonal projection onto the subspace spanned by $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_r$ $(1 \le r \le n)$. Thus $P_n = I$ and if P_0 is the zero transformation of V then $[P_{r-1}\boldsymbol{v}, \boldsymbol{v}] \le [P_r\boldsymbol{v}, \boldsymbol{v}]$, whence $((\tilde{A}+\tilde{B}P_{r-1})\boldsymbol{v}, \boldsymbol{v}) \le ((\tilde{A}+\tilde{B}P_r)\boldsymbol{v}, \boldsymbol{v})$ for all $\boldsymbol{v} \in V$ and $1 \le r \le n$. Note that for $0 \le r \le n$, $\tilde{B}P_r$ is a symmetric linear transformation of the original inner product space V, and that for any symmetric linear transformation T of V, $\lambda_i(T)$ is the minimum of $\max\{(T\boldsymbol{v}, \boldsymbol{v}) : \|\boldsymbol{v}\| = 1, \boldsymbol{v} \in U\}$ taken over all i-dimensional subspaces U of V. It follows that $\lambda_i(\tilde{A}) \le \lambda_i(\tilde{A} + \tilde{B}P_1) \le \lambda_i(\tilde{A} + \tilde{B}P_2) \le \cdots \le \lambda_i(\tilde{A} + \tilde{B}P_{n-1}) \le \lambda_i(\tilde{A} + \tilde{B})$ $(1 \le i \le n)$. The problem of determining the eigenvalues of $\tilde{A} + \tilde{B}P_r$ for some r is called an intermediate eigenvalue problem of the second type.

Let $\tilde{A}\boldsymbol{u}_i = \tilde{\lambda}_i\boldsymbol{u}_i$ where $\tilde{\lambda}_i = \lambda_i(\tilde{A})$ $(i = 1, \ldots, n)$ and $\boldsymbol{u}_1, \ldots, \boldsymbol{u}_n$ are orthonormal. Choose $\boldsymbol{v}_i = \tilde{B}^{-1}\boldsymbol{u}_i$ $(i = 1, \ldots, n)$. Then $P_r\boldsymbol{u}_j = \sum_{i=1}^{r} \gamma_{ij}\boldsymbol{v}_i$, where $(\gamma_{ij})^{-1}$ is the $r \times r$ Gram matrix T_r with (i,j)-entry $[\boldsymbol{v}_i, \boldsymbol{v}_j]$ $(i,j = 1,\ldots,r)$; moreover the matrix of $\tilde{A} + \tilde{B}P_r$ with respect to the basis $\{\boldsymbol{u}_1,\ldots,\boldsymbol{u}_n\}$ is

$$\operatorname{diag}(\tilde{\lambda}_1, \tilde{\lambda}_2, \ldots, \tilde{\lambda}_n) + \begin{pmatrix} T_r^{-1} & O \\ O & O \end{pmatrix}.$$

In what follows we take $V = \mathbb{R}^n$, $(\boldsymbol{u},\boldsymbol{v}) = \boldsymbol{u}^T\boldsymbol{v}$ and identify a linear transformation on $\mathbb{R}^n$ and its matrix with respect to the standard basis of $\mathbb{R}^n$. As usual, $J_{n\times k}$ denotes the $n \times k$ matrix with all entries equal to 1, $J_n = J_{n\times n}$, I_n denotes the $n \times n$ identity matrix. If A and $A + B$ are the adjacency matrices of a graph G and its modified graph G', then we take

$$\tilde{A} = -A - (\lambda_n(B) + \delta)I \text{ and } \tilde{B} = (\lambda_n(B) + \delta)I - B, \text{ where } \delta > 0.$$

Thus $\tilde{B}$ is positive, $\tilde{A} + \tilde{B} = -A - B$ and $\mu_1(G') = -\lambda_1(\tilde{A} + \tilde{B}) \le -\lambda_1(\tilde{A} + \tilde{B}P_r)$. For a given r, we can choose δ to optimize this upper bound for $\mu_1(G')$.

2. The Results

Let G be a connected graph, and let $u \in V(G)$, $S \subseteq V(G) \setminus N_G[u]$, $T \subseteq N_G(u)$ with $|S| = s$, $|T| = t$ and $s + t \ge 1$. The graph $G(u;S,T)$ is obtained from G by replacing edges uj, $j \in T$ with edges ui, $i \in S$.

Maas [2] obtained upper bounds for the index of $G(u;S,T)$ with $s = 1$, $t = 0$, while Rowlinson [4, Theorems 3.2 and 5.1] investigated the cases $s = t = 1$ and $s = 0$, $t = 1$, respectively.

Theorem 1. *Let G be a connected graph with eigenvalues $\mu_1, \mu_2, \ldots, \mu_n$ (in non-increasing order), and $(x_1, x_2, \ldots, x_n)^T$ be the principal eigenvector of G. Write $G' = G(u;S,T)$. Then*

$$\mu_1(G') \ge \mu_1 + 2x_u\Big(\sum_{v\in S} x_v - \sum_{v\in T} x_v\Big),$$

and

$$\mu_1(G') \le \mu_1 + \sqrt{s+t} + \delta - \gamma$$

where $\delta > 0$, and

$$\gamma = \frac{\delta(\sqrt{s+t} + \delta)(2\sqrt{s+t} + \delta)}{\alpha - \delta\beta + \delta(2\sqrt{s+t} + \delta)} = \mu_1 - \mu_2 \tag{1}$$

with

$$\alpha = (s+t)x_u^2 + \Big(\sum_{v\in S} x_v\Big)^2 + \Big(\sum_{v\in T} x_v\Big)^2$$
$$-2\sum_{v\in S} x_v \sum_{v\in T} x_v - 2\sqrt{s+t}\,x_u\Big(\sum_{v\in T} x_v - \sum_{v\in S} x_v\Big),$$

and

$$\beta = 2x_u\Big(\sum_{v\in T} x_v - \sum_{v\in S} x_v\Big).$$

Proof. Suppose without loss of generality that $u=1$, $S=\{2,\dots,s+1\}$ and $T=\{s+2,\dots,s+t+1\}$. Let A, $A+B$ be the adjacency matrices of G and G' respectively so that

$$B = \left(\begin{array}{c|cc|c} 0 & J_{1\times s} & -J_{1\times t} & \\ \hline J_{s\times 1} & & & O \\ -J_{t\times 1} & & O & \\ \hline & O & & O \end{array}\right).$$

Since $\mu_1(G') = \max\{\boldsymbol{y}^T(A+B)\boldsymbol{y} : \|\boldsymbol{y}\|=1\}$, by taking $\boldsymbol{y}$ to be the principal eigenvector of G, we get

$$\mu_1(G') \ge \mu_1 + 2x_1\Big(\sum_{i=2}^{s+1} x_i - \sum_{i=s+2}^{s+t+1} x_i\Big).$$

In what follows we will consider the upper bound for $\mu_1(G')$. It is easy to see that the largest eigenvalue of B is $\sqrt{s+t}$. Using the notation as in Section 1, we have $\tilde{A} = -A - (\sqrt{s+t}+\delta)I_n$, $\tilde{B} = (\sqrt{s+t}+\delta)I_n - B$, and hence $\tilde{\lambda}_i = -\mu_i - \sqrt{s+t} - \delta$ $(i=1,\dots,n)$. Write $a = \sqrt{s+t}+\delta$ and $b = a^2 - (s+t)$. Then we have

$$\frac{1}{b}\tilde{B}^{-1} = \left(\begin{array}{ccc|c} a & J_{1\times s} & -J_{1\times t} & \\ J_{s\times 1} & \frac{b}{a}I_s + \frac{1}{a}J_s & -\frac{1}{a}J_{s\times t} & O \\ -J_{t\times 1} & -\frac{1}{a}J_{t\times s} & \frac{b}{a}I_t + \frac{1}{a}J_t & \\ \hline & O & & \frac{b}{a}I \end{array}\right).$$

We use now the results from Section 1 with $r=1$ and with $\boldsymbol{u}_1 = (x_1, x_2,\dots,x_n)^T$. The matrix of $\tilde{A}+\tilde{B}P_1$ is similar to $\mathrm{diag}(\tilde{\lambda}_1+\gamma, \tilde{\lambda}_2,\dots,\tilde{\lambda}_n)$ where

$$\gamma = \gamma_{11}^{-1} = [\boldsymbol{v}_1, \boldsymbol{v}_1]^{-1} = [\tilde{B}^{-1}\boldsymbol{u}_1, \tilde{B}^{-1}\boldsymbol{u}_1]^{-1} = (\boldsymbol{u}_1, \tilde{B}^{-1}\boldsymbol{u}_1)^{-1}$$

$$= \frac{\delta(\sqrt{s+t}+\delta)(2\sqrt{s+t}+\delta)}{\alpha - \delta\beta + \delta(2\sqrt{s+t}+\delta)}.$$

Now $\lambda_1(\tilde{A} + \tilde{B}P_1) = \min\{\tilde{\lambda}_1 + \gamma, \tilde{\lambda}_2\}$ and so $\mu_1(G') = -\lambda_1(\tilde{A} + \tilde{B}) \leq -\lambda_1(\tilde{A} + \tilde{B}P_1) = \max\{\mu_1 + \sqrt{s+t} + \delta - \gamma, \mu_2 + \sqrt{s+t} + \delta\}$. As function of δ ($\delta > 0$), γ has range $(0, \infty)$ and so we may choose $\delta > 0$ such that $\gamma = \mu_1 - \mu_2$. Hence (1) holds, and $\mu_1(G') \leq \mu_1 + \sqrt{s+t} + \delta - \gamma$. ∎

As a direct application, we use Theorem 1 to obtain sufficient conditions in terms of $\mu_1, \mu_2, x_i (i \in \{u\} \cup S \cup T)$ for the index to decrease or increase when replacing edges uj, $j \in T$ with edges ui, $i \in S$.

Theorem 2. *Let G be a connected graph with eigenvalues $\mu_1, \mu_2, \ldots, \mu_n$ (in non-increasing order), and $(x_1, x_2, \ldots, x_n)^T$ be the principal eigenvector of G. Write $G' = G(u; S, T)$. If $\sum_{v\in S} x_v \geq \sum_{v\in T} x_v$, then $\mu_1(G') > \mu_1$. If $\sum_{v\in S} x_v < \sum_{v\in T} x_v$ and*

$$\mu_1 - \mu_2 > \frac{(s+t)x_u^2 + (\sum\limits_{v\in S} x_v)^2 + (\sum\limits_{v\in T} x_v)^2 - 2\sum\limits_{v\in S} x_v \sum\limits_{v\in T} x_v}{2x_u(\sum\limits_{v\in T} x_v - \sum\limits_{v\in S} x_v)}, \tag{2}$$

then $\mu_1(G') < \mu_1$.

Proof. We use the notation in the proof of Theorem 1. If $\sum_{v\in S} x_v \geq \sum_{v\in T} x_v$, then by Theorem 1, $\mu_1(G') \geq \mu_1 + 2x_u(\sum_{v\in S} x_v - \sum_{v\in T} x_v) \geq \mu_1$. Moreover, we have $\mu_1(G') > \mu_1$. Otherwise, $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)^T$ is an eigenvector of $A + B$ corresponding to the eigenvalue $\mu_1(G') = \mu_1$; hence $Bx = 0$. But then $x_v = \pm(Bx)_v = 0$ for $v \in S \cup T$, which is a contradiction.

Now suppose that $\sum_{v\in S} x_v < \sum_{v\in T} x_v$. Then $\beta = 2x_u(\sum_{v\in T} x_v - \sum_{v\in S} x_v) > 0$. By Theorem 1, $\mu_1(G') \leq \mu_1 + \sqrt{s+t} + \delta - \gamma$ where γ is given by (1). Note that (2) is equivalent to $\gamma > \sqrt{s+t} + \alpha\beta^{-1}$. We have $\alpha - \delta\beta < 0$. Otherwise, by the expression of γ, $\sqrt{s+t} + \alpha\beta^{-1} < \gamma \leq \sqrt{s+t} + \delta$, and hence $\alpha\beta^{-1} < \delta$, a contradiction. It follows that $\gamma > \sqrt{s+t} + \delta$ and we have $\mu_1(G') < \mu_1$. ∎

Let G be a connected graph, let $u,\ v \in V(G)$ such that u and v are not adjacent, and let $S \subseteq V(G)\setminus(N_G[u]\cup\{v\})$, $T \subseteq N_G(v)$ with $|S| = s$, $|T| = t$ and $s+t \geq 1$. The graph $G(u,v;S,T)$ is obtained from G by replacing edges vj, $j \in T$ with ui, $i \in S$. Note that the case $s = t = 1$ has been considered in [4].

Theorem 3. *Let G be a connected graph with eigenvalues $\mu_1, \mu_2, \ldots, \mu_n$ (in non-increasing order), and let $(x_1, x_2, \ldots, x_n)^T$ be the principal eigenvector of G. Write $G' = G(u,v;S,T)$ where $S \cap T = \emptyset$. Then*

$$\mu_1(G') \geq \mu_1 + 2x_u \sum_{i\in S} x_i - 2x_v \sum_{i\in T} x_i,$$

and

$$\mu_1(G') \leq \mu_1 + \max\{\sqrt{s}, \sqrt{t}\} + \delta - \gamma$$

where $\delta > 0$, $\gamma = \mu_1 - \mu_2$, and

$$\text{(3)}\ \gamma^{-1} = \frac{sx_u^2 + (\sum\limits_{i\in S} x_i)^2 + 2ax_u \sum\limits_{i\in S} x_i}{a(a^2-s)} + \frac{tx_v^2 + (\sum\limits_{i\in T} x_i)^2 - 2ax_v \sum\limits_{i\in T} x_i}{a(a^2-t)} + \frac{1}{a}$$

with $a = \max\{\sqrt{s}, \sqrt{t}\} + \delta$.

Proof. Suppose without loss of generality that $u = 1$, $S = \{2, \ldots, s+1\}$, $v = s+2$, $T = \{s+3, \ldots, s+t+2\}$. Let A, $A+B$ be the adjacency matrices of G and G' respectively so that

$$B = \begin{pmatrix} C_{s+1} & O & O \\ O & -C_{t+1} & O \\ O & O & O \end{pmatrix} \quad \text{where} \quad C_k = \begin{pmatrix} 0 & J_{1\times k} \\ J_{k\times 1} & O_k \end{pmatrix}.$$

Since $\mu_1(G') = \max\{\boldsymbol{y}^T(A+B)\boldsymbol{y} : \|\boldsymbol{y}\| = 1\}$, by taking $\boldsymbol{y}$ to be the principal eigenvector of G, we get

$$\mu_1(G') \geq \mu_1 + 2x_1 \sum_{i=2}^{s+1} x_i - 2x_{s+2} \sum_{i=s+3}^{s+t+2} x_i.$$

Note that the largest eigenvalue of B is $\max\{\sqrt{s}, \sqrt{t}\}$. Using the notation in Section 1, we have $\tilde{A} = -A - aI_n$, $\tilde{B} = aI_n - B$, and hence $\tilde{\lambda}_i = -\mu_i - a$

$(i = 1, \ldots, n)$. It can be checked readily that

$$\tilde{B}^{-1} = \left(\begin{array}{cc|cc|c} \frac{a}{a^2-s} & \frac{1}{a^2-s}J_{1\times s} & \multicolumn{2}{c|}{O} & \\ \frac{1}{a^2-s}J_{s\times 1} & \frac{1}{a}I_s + \frac{1}{a(a^2-s)}J_s & & & O \\ \cline{1-4} \multicolumn{2}{c|}{\multirow{2}{*}{O}} & \frac{a}{a^2-t} & -\frac{1}{a^2-t}J_{1\times t} & \\ & & -\frac{1}{a^2-t}J_{t\times 1} & \frac{1}{a}I_t + \frac{1}{a(a^2-t)}J_t & \\ \hline \multicolumn{4}{c|}{O} & \frac{1}{a}I \end{array}\right).$$

Using the results from Section 1 with $r = 1$ and $\boldsymbol{u}_1 = (x_1, x_2, \ldots, x_n)^T$, we know that the matrix $\tilde{A} + \tilde{B}P_1$ is similar to $\operatorname{diag}(\tilde{\lambda}_1 + \gamma, \tilde{\lambda}_2, \ldots, \tilde{\lambda}_n)$ where

$$\gamma^{-1} = \gamma_{11} = [\boldsymbol{v}_1, \boldsymbol{v}_1] = (\boldsymbol{u}_1, \tilde{B}^{-1}\boldsymbol{u}_1)$$

is given by (3). Now $\lambda_1(\tilde{A} + \tilde{B}P_1) = \min\{\tilde{\lambda}_1 + \gamma, \tilde{\lambda}_2\}$ and so $\mu_1(G') = -\lambda_1(\tilde{A} + \tilde{B}) \leq -\lambda_1(\tilde{A} + \tilde{B}P_1) = \max\{\mu_1 + a - \gamma, \mu_2 + a\}$. As function of δ $(\delta > 0)$, γ has range $(0, \infty)$ and so we may choose $\delta > 0$ such that $\gamma = \mu_1 - \mu_2$. ■

Remark. One can consider in a similar way the general case $|S \cap T| = r > 0$ for $G' = G(u, v; S, T)$. It can be showed that

$$\mu_1(G') \geq \mu_1 + 2x_u \sum_{i\in S} x_i - 2x_v \sum_{i\in T} x_i,$$

and

$$\mu_1(G') \leq \mu_1 + a - \gamma$$

where $\delta > 0$, $\gamma = \mu_1 - \mu_2$, and

$$\begin{aligned} \gamma^{-1} \;=\; & \frac{(a^2-t)s + r^2}{a\Delta}x_u^2 + \frac{(a^2-s)t + r^2}{a\Delta}x_v^2 + \frac{a^2-t}{a\Delta}\Big(\sum_{i\in S\setminus T} x_i\Big)^2 \\ & + \frac{a^2-s}{a\Delta}\Big(\sum_{i\in T\setminus S} x_i\Big)^2 + \frac{2a^2 - s - t + 2r^2}{a\Delta}\Big(\sum_{i\in S\cap T} x_i\Big)^2 \\ & + \Big(2\frac{a^2-t}{\Delta}x_u - 2\frac{r}{\Delta}x_v\Big)\sum_{i\in S} x_i - \Big(2\frac{a^2-s}{\Delta}x_v - 2\frac{r}{\Delta}x_u\Big)\sum_{i\in T} x_i \end{aligned}$$

$$+\frac{2}{a\Delta}\sum_{i\in S\cap T}x_i\Big((a^2-t+r)\sum_{i\in S\setminus T}x_i-(a^2-s+r)\sum_{i\in T\setminus S}x_i\Big)$$
$$+2\frac{r}{a\Delta}\sum_{i\in S\setminus T}x_i\sum_{i\in T\setminus S}x_i-2\frac{ar}{\Delta}x_ux_v+\frac{1}{a}$$

with $a=\sqrt{\frac{s+t+\sqrt{(s-t)^2+4r^2}}{2}}+\delta$, and $\Delta=(a^2-s)(a^2-t)-r^2$.

We may use Theorem 3 to derive sufficient conditions in terms of μ_1,μ_2,x_i ($i\in\{u,v\}\cup S\cup T$) for the index to decrease or increase when replacing edges vj, $j\in T$ with ui, $i\in S$ (provided $s=t$).

Theorem 4. *Let G be a connected graph with eigenvalues $\mu_1,\mu_2,\dots,\mu_n$ (in non-increasing order), and let $(x_1,x_2,\dots,x_n)^T$ be the principal eigenvector of G. Write $G'=G(u,v;S,T)$ where $S\cap T=\emptyset$ and $s=t$. If $x_u\sum_{i\in S}x_i\geq x_v\sum_{i\in T}x_i$, then $\mu_1(G')>\mu_1$. If $x_u\sum_{i\in S}x_i<x_v\sum_{i\in T}x_i$ and*

$$\mu_1-\mu_2>\frac{s(x_u^2+x_v^2)+(\sum\limits_{i\in S}x_i)^2+(\sum\limits_{i\in T}x_i)^2}{2(x_v\sum\limits_{i\in T}x_i-x_u\sum\limits_{i\in S}x_i)},\tag{4}$$

then $\mu_1(G')<\mu_1$.

Proof. We use the notation in the proof of Theorem 3. If $x_u\sum_{i\in S}x_i\geq x_v\sum_{i\in T}x_i$, then by Theorem 3, $\mu_1(G')\geq 2x_u\sum_{i\in S}x_i-2x_v\sum_{i\in T}x_i\geq\mu_1$. Moreover, we have $\mu_1(G')>\mu_1$. Otherwise, $\boldsymbol{x}=(x_1,x_2,\dots,x_n)^T$ is an eigenvector of $A+B$ corresponding to the eigenvalue $\mu_1(G')=\mu_1$; hence $Bx=0$. But then $x_u=(Bx)_i=0$ for $i\in S$, which is a contradiction.

Now suppose that $x_u\sum_{i\in S}x_i<x_v\sum_{i\in T}x_i$. By Theorem 3, $\mu_1(G')\leq\mu_1+\sqrt{s}+\delta-\gamma$ with

$$\gamma=\frac{\delta(\sqrt{s}+\delta)(2\sqrt{s}+\delta)}{\alpha-\delta\beta+\delta(2\sqrt{s}+\delta)}$$

where $\alpha=s(x_u^2+x_v^2)+(\sum_{i\in S}x_i)^2+(\sum_{i\in T}x_i)^2-2\sqrt{s}(x_v\sum_{i\in T}x_i-x_u\sum_{i\in S}x_i)$ and $\beta=2(x_v\sum_{i\in T}x_i-x_u\sum_{i\in S}x_i)>0$. Note that (4) is equivalent to $\gamma>\sqrt{s}+\alpha\beta^{-1}$ and we have $\alpha-\delta\beta<0$. It follows that $\gamma>\sqrt{s}+\delta$ and we have $\mu_1(G')<\mu_1$. ■

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