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## $P_m$ -SATURATED BIPARTITE GRAPHS WITH MINIMUM SIZE

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#### Abstract

A graph G is said to be *H*-saturated if G is *H*-free i.e., (G has no subgraph isomorphic to *H*) and adding any new edge to G creates a copy of *H* in G. In 1986 L. Kászonyi and Zs. Tuza considered the following problem: for given *m* and *n* find the minimum size  $sat(n; P_m)$ of  $P_m$ -saturated graph of order *n*. They gave the number  $sat(n; P_m)$ for *n* big enough. We deal with similar problem for bipartite graphs.

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## 1. Preliminaries

We deal with simple graphs without loops and multiple edges. As usual V(G) and E(G) denote the vertex set and the edge set, respectively, |G|, e(G) the order and the size of G, and  $d_G(v)$  the degree of  $v \in V(G)$ . By  $P_m$  we denote the path of order m, and by  $K_m$  the complete graph on m vertices. We define  $G_{a,b}$  to be a bipartite graph where a, b are the numbers of vertices in bipartition sets. Let us consider two graphs G and H. We say that G is H-free if it contains no copy of H, that is, no subgraph of G is isomorphic to H. A graph G is H-saturated if G is H-free and adding any new edge e to G creates a copy of H. In particular complete H-free graphs trivially satisfy this condition and therefore are H-saturated. We define also:

<sup>\*</sup>This work was carried out while the second author was visiting University of Orleans.

$$ex(n; F) = \max\{e(G) : |G| = n, G \text{ is } F\text{-saturated}\},\$$

$$Ex(n; F) = \{G : |G| = n, e(G) = ex(n; F), G \text{ is } F\text{-saturated}\},\$$

$$sat(n; F) = \min\{e(G) : |G| = n, G \text{ is } F\text{-saturated}\},\$$

$$Sat(n; F) = \{G : |G| = n, e(G) = sat(n; F), G \text{ is } F\text{-saturated}\}.$$

Observe that in the definitions of Ex(n; F) and ex(n; F) the word saturated may be replaced with *free*. The first results concerning saturated graphs were given by Turán [6] in 1941 who asked for  $ex(n; K_p)$  and  $Ex(n; K_p)$ . Later results were given by P. Erdös, A. Hajnal and J.W. Moon [3] (see also [2]) in 1964 who proved

$$sat(n; K_p) = {\binom{p-2}{2}} + (p-2)(n-p+2), \quad (n \ge p \ge 2)$$
$$Sat(n; K_p) = \{K_{p-2} * \bar{K}_{n-p+2}\}.$$

A corresponding theorem for bipartite graphs was given by N. Alon in 1983 (see [1]). The extremal problem for  $P_m$ -saturated bipartite graphs was solved by A. Gyárfás, C.C. Rousseau and R.H. Schelp [4]. We are interested in finding  $P_m$ -saturated bipartite graphs with minimum size. In Section 2 we present some results concerning  $P_m$ -saturated bipartite graphs. The proofs are given in Section 3.

In [5] L. Kászonyi and Zs. Tuza, gave the following results on  $Sat(n; P_m)$  and  $sat(n; P_m)$ .

## **Theorem 1** ([5]).

$$sat(n; P_3) = \left\lfloor \frac{n}{2} \right\rfloor,$$

$$Sat(n; P_3) = \begin{cases} kK_2 & \text{if } n = 2k, \\ kK_2 \cup K_1 & \text{if } n = 2k + 1, \end{cases}$$

$$sat(n; P_4) = \begin{cases} k & \text{if } n = 2k, \\ k + 2 & \text{if } n = 2k + 1, \end{cases}$$

$$Sat(n; P_4) = \begin{cases} kK_2 & \text{if } n = 2k, \\ (k - 1)K_2 \cup K_3 & \text{if } n = 2k + 1, \end{cases}$$

$$sat(n; P_5) = n - \left\lfloor \frac{n - 2}{6} \right\rfloor - 1 \quad \text{for } n \ge 6.$$

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Let

$$a_m = \begin{cases} 3 \cdot 2^{k-1} - 2 & \text{if } m = 2k, k > 2, \\ 2^{k+1} - 2 & \text{if } m = 2k + 1, k \ge 2. \end{cases}$$

Then  $sat(n; P_m) = n - \lfloor \frac{n}{a_m} \rfloor$  for  $n \ge a_m$ .



 $Sat(n; P_5)$  for  $n \ge 6$ .



# 2. $P_m$ -Saturated Bipartite Graphs with Minimum Size

Let G = (B, W; E) be a bipartite graph with vertex set  $V = B \cup W$ ,  $B \cap W = \emptyset$ . For convenience of the reader we call the set B the set of *black* vertices and the set W the set of *white* vertices. For bipartite graphs G = (B, W; E) and F = (B', W'; E') such that the sets B, W, B' and W' are mutually disjoint we define:  $G \cup F = (B \cup B', W \cup W'; E \cup E')$ .

**Definition 1.** Let G = (B, W; E) be a bipartite graph. Then G is called *F*-saturated if

- 1. G is F-free,
- 2.  $(x \in B, y \in W, xy \notin E) \Rightarrow G \cup xy \supseteq F.$

We denote also

$$sat_{bip}(p,q;F) = \min\{e(G) : |B| = p, |W| = q, G \text{ is } F\text{-saturated}\},\$$

 $\begin{aligned} Sat_{bip}(p,q;F) &= \{G = (B,W;E) : |B| = p, |W| = q, e(G) = sat_{bip}(p,q;F), \\ G \text{ is } F\text{-saturated} \}. \end{aligned}$ 

**Proposition 2.**  $sat_{bip}(p,q;P_3) = p, p \leq q.$ 



 $Sat_{bip}(p,q;P_3).$ 



**Proposition 3.**  $sat_{bip}(p,q;P_4) = p, 2 \le p \le q.$ 



 $Sat_{bip}(p,q;P_4).$ 



**Proposition 4.** Let  $p \ge 2, q \ge 3, p \le q$ . Then

$$sat_{bip}(p,q;P_5) = \begin{cases} 2p & \text{if } 2p \le q, \ p \ is \ even \ or \ q = 2p - 2, \\ q & \text{if } 2p \le q, \ p \ is \ odd, \\ p + \left\lceil \frac{q}{2} \right\rceil & \text{if } 3 < q < 2p, \ q \ne 2p - 2, \\ 5 & \text{if } p = q = 3, \\ 4 & \text{if } p = 2. \end{cases}$$

**Proposition 5.** Let  $3 \le p \le q$ . Then

$$sat_{bip}(p,q;P_{6}) = \begin{cases} p+q - \left\lfloor \frac{p}{3} \right\rfloor - 1 & if \quad p \equiv 2(\text{mod } 3) \text{ and } 3q \leq 4p - 2 \text{ or} \\ p \equiv 1(\text{mod } 3) \text{ and } 3q \leq 4p - 1, \\ p+q - \left\lfloor \frac{p}{3} \right\rfloor & if \quad p \equiv q \equiv 1(\text{mod } 3) \text{ or} \\ p \equiv 0(\text{mod } 3) \text{ and } 3q \leq 4p - 1, \\ 2p & otherwise. \end{cases}$$

**Definition 2.** Let us suppose that  $m \ge 7$  is an integer. Then  $A_m$  is the following tree. All penultimate vertices of  $A_m$  have degree two and all vertices of  $A_m$  which are neither penultimate nor pendant have their degree equal to three. If  $m = 2k, k \ge 4$  then  $A_m$  has two centers  $u \in B$  and  $w \in W$  and each component of G - uw has k - 1 levels (see Figure 4). If  $m = 2k + 1, k \ge 3$  then  $A_m$  has one center and k levels. The center is black when k is even (see Figure 5).



Figure 4

When  $m = 2k, k \ge 4$  we observe that  $|B| = |W| = 3 \cdot 2^{k-3} - 1$ .



Figure 5

If  $m = 2k + 1, k \ge 3$  and the center is white then in  $A_m$  we have  $|B| = 4 \cdot 2^{k-3} - 1$  and  $|W| = 5 \cdot 2^{k-3} - 1$  when k is odd or  $|W| = 4 \cdot 2^{k-3} - 1$  and  $|B| = 5 \cdot 2^{k-3} - 1$  when k is even. Denote by v the center of  $A_{2k+1}, k \ge 3$ .

Observe that if  $|B| \leq |W|$  then for  $m = 2k + 1, k \geq 3$  we obtain  $|B| = 4 \cdot 2^{k-3} - 1, |W| = 5 \cdot 2^{k-3} - 1$  and  $v \in B$  if k is even or  $v \in W$  if k is odd.

**Remark 1.** Observe that  $A_m$  is  $P_m$ -saturated and  $P_{m-1}$ -saturated for every  $m \ge 7$ .

**Remark 2.**  $lA_{2k}$  is  $P_{2k}$ -saturated for  $k \ge 4, l = 1, 2, ..., n$ .

**Remark 3.** The union of two copies of  $A_{2k+1}$  is  $P_{2k+1}$ -saturated,  $k \ge 3$ , if and only if their centers have the same colour (see Figure 6).

**Theorem 6.** Let  $k \ge 4$  and let G = (B, W; E) be a  $P_{2k}$ -saturated bipartite graph without isolated vertices and with the minimum size,  $|B| = p, |W| = q, 3 \cdot 2^{k-3} - 1 \le p \le q$ . Then

$$e(G) = p + q - \left\lfloor \begin{array}{c} \frac{p}{3 \cdot 2^{k-3} - 1} \end{array} \right\rfloor.$$

**Theorem 7.** Let  $k \geq 3$  and let G = (B, W; E) be a  $P_{2k+1}$ -saturated bipartite graph without isolated vertices and with the minimum size,  $|B| = p \leq |W| = q, 4 \cdot 2^{k-3} - 1 \leq p, 5 \cdot 2^{k-3} \leq q$ . Then

$$e(G) = \begin{cases} p + q - \frac{q}{5 \cdot 2^{k-3} - 1} + 1 & if \quad \frac{q}{5 \cdot 2^{k-3} - 1} = \left\lfloor \frac{q}{5 \cdot 2^{k-3} - 1} \right\rfloor < \frac{p}{4 \cdot 2^{k-3} - 1} \\ p + q - \min\left\{ \left\lfloor \frac{p}{4 \cdot 2^{k-3} - 1} \right\rfloor, \left\lfloor \frac{q}{5 \cdot 2^{k-3} - 1} \right\rfloor \right\} & otherwise. \end{cases}$$



Figure 6

Theorem 6 and 7 imply the following corollary.

**Corollary 8.** If |B| = p, |W| = q,  $3 \cdot 2^{k-3} - 1 \le p \le q$ ,  $k \ge 4$  then

$$sat_{bip}(p,q;P_{2k}) \le p + q - \left\lfloor \frac{p}{3 \cdot 2^{k-3} - 1} \right\rfloor.$$

If  $|B| = p, |W| = q, p \le q, 4 \cdot 2^{k-3} - 1 \le p, 5 \cdot 2^{k-3} \le q$ , then

 $sat_{bip}(p,q;P_{2k+1})$ 

$$= \left\{ \begin{array}{ll} p+q-\frac{q}{5\cdot 2^{k-3}-1}+1 & if \quad \frac{q}{5\cdot 2^{k-3}-1} = \left\lfloor \frac{q}{5\cdot 2^{k-3}-1} \right\rfloor < \frac{p}{4\cdot 2^{k-3}-1} \ , \\ p+q-\min\left\{ \left\lfloor \frac{p}{4\cdot 2^{k-3}-1} \right\rfloor, \left\lfloor \frac{q}{5\cdot 2^{k-3}-1} \right\rfloor \right\} & otherwise. \end{array} \right.$$

## 3. Proofs

We first give some definitions. The graphs  $K_{1,n}$  and  $K_{n,1}$  are called *stars* when  $n \ge 1$  and *non-trivial stars* if  $n \ge 2$ . Let  $K_{1,b}$  and  $K_{a,1}$  be two vertex disjoint stars. Then the tree obtained by join of their centers is called *double star*  $S_{a,b}^2$  (see Figure 7). A double star  $S_{a,b}^2$  is said to be *non-trivial* if a > 0, b > 0 and  $a + b \ge 3$ . Propositions 2 and 3 are evident. To prove Proposition 4 we give Lemmas 9–13 and Proposition 14 below.

**Lemma 9.** Let G = (B, W; E) be a connected bipartite  $P_5$ -saturated graph |B| = p, |W| = q. Then either

- 1. G is a star or
- 2. G is non-trivial double star  $S^2_{a,b}$  or else
- 3.  $G = K_{2,2}$ .

**Lemma 10.** Let G = (B, W; E) be a bipartite  $P_5$ -saturated graph,  $|B| = p, |W| = q, p \le q, p \ge 2, q \ge 3$ , such that there is at least one isolated vertex in W. Then  $p = 2k, k \ge 1$  and  $G = kK_{2,2} \cup K_{0,q-p}$ . In particular we have

1. p < q and 2. e(G) = 2p.

Lemma 9 is evidently true. Lemma 10 follows from Lemma 9 easily.

**Lemma 11.** Let G = (B, W; E) be a bipartite  $P_5$ -saturated graph without isolated vertices  $|B| = p, |W| = q, p \le q, p \ge 2, q \ge 3$ . Then G is vertex disjoint union of

- 1. complete graphs  $K_{2,2}$ ,
- 2. non-trivial double stars,
- 3. non-trivial stars,
- 4. at most one trivial star  $K_{1,1}$ .

If  $K_{1,1}$  is a component of G then no other star is a component of G.

Lemma 11 follows from Lemma 9.

If G = (B, W; E) is a bipartite  $P_5$ -saturated graph with |B| = p, |W| = qthen we have either

(1) 
$$G = nK_{2,2} \cup \bigcup_{i=1}^{k} S_{a_i,b_i}^2 \cup \bigcup_{j=1}^{l} K_{1,c_j} \cup \delta K_{1,1}$$

(2) 
$$G = nK_{2,2} \cup \bigcup_{i=1}^{k} S^2_{a_i,b_i} \cup \bigcup_{j=1}^{l} K_{d_j,1} \cup \delta K_{1,1}$$

where  $S_{a_i,b_i}^2$  are non-trivial double stars,  $K_{1,c_j}$  and  $K_{d_j,1}$  are non-trivial stars, and  $\delta \in \{0,1\}$ . We have  $p = 2n + \left(\sum_{i=1}^k a_i + k\right) + l + \delta$ ,  $q = 2n + \left(\sum_{i=1}^k b_i + k\right) + \sum_{j=1}^l c_j + \delta$  if G is given by (1), and  $p = 2n + \left(\sum_{i=1}^k a_i + k\right) + \sum_{j=1}^l d_j + \delta$ ,  $q = 2n + \left(\sum_{i=1}^k b_i + k\right) + l + \delta$  if G is given by (2), and  $\delta = 0$  if l > 0.

**Lemma 12.** Let G = (B, W; E) be a union of non-trivial double stars, such that  $|B| = p, |W| = q, p \le q, p \ge 2, q \ge 3$ . Then G has the minimum size if

$$e(G) = \begin{cases} p+q - \left\lfloor \frac{p}{2} \right\rfloor & \text{if } 3p \le 2q, \\ p+q - \left\lfloor \frac{1}{2} \left( p - \left\lceil \frac{3p-2q}{5} \right\rceil \right) \right\rfloor & \text{if } 2q \le 3p. \end{cases}$$

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**Proof.** Let G be a union of non-trivial double stars,  $G = \bigcup_{i=1}^{l} S_{a_i,b_i}^2$  where

$$S_{a_i,b_i}^2 = (B_i, W_i; E_i), |B_i| = a_i + 1, |W_i| = b_i + 1, i = 1, \dots, l$$
$$\bigcup_{i=1}^l B_i = B, \bigcup_{i=1}^l W_i = W,$$

such that for fixed p and q, G has the minimum size e(G). We observe that e(G) = p + q - c where c = c(G) is the number of components of G. So, e(G) is the minimum whenever c(G) is the maximum. Since every component of G has at least two vertices in B then  $c(G) \leq \lfloor \frac{p}{2} \rfloor$ . If  $3p \leq 2q$ , then  $c = \lfloor \frac{p}{2} \rfloor$  and c components of G are  $S_{1,b_i}^2$  stars with  $b_i \geq 2, i = 1, 2, \ldots, c - 1$  and  $a_c = 1, b_c \geq 2$  if p is even, and  $a_c = 2, b_c \geq 1$  when p is odd.

Therefore  $e(G) = p + q - \lfloor \frac{p}{2} \rfloor$  when  $3p \leq 2q$ . So we may assume from now that 3p > 2q. Since the lemma is easy to verify for  $p \leq 4$  we shall assume  $p \geq 5$ . Observe that there are two different components  $C_1$  and  $C_2$ of G such that  $C_1 = S_{a_1,b_1}^2$ ,  $C_2 = S_{a_2,b_2}^2$ ,  $b_1 \geq 2$  and  $a_2 \geq 2$ .

If  $p \leq 6$  or  $q \leq 7$  then c(G) = 2 and the proof is finished. So we suppose  $p \geq 7$  and  $q \geq 8$ . Then there is at least one component  $C, C \neq C_1, C \neq C_2$ . Let x, y be the centers of  $C, x_1, y_1$  be the centers of  $C_1, x_2, y_2$  be the centers of  $C_2$ , such that  $x, x_1, x_2 \in B, y, y_1, y_2 \in W$ . It is clear that the number of components of G will not change if we proceed the following operation:

- delete from  $C_1$  all but one black pendant vertices and all but two white pendant vertices (we denote then by  $C'_1$  the obtained component),
- delete from  $C_2$  all but two black pendant vertices and all but one white pendant vertices (we denote then by  $C'_2$  the obtained component),
- join x with all white vertices deleted from  $C_1$  and  $C_2$  and join y with all black vertices deleted from  $C_1$  and  $C_2$  (we denote then by C' the obtained component).

The new graph G' has exactly the same number of components as G and all the components of G' are non-trivial double stars. The number of components of G is equal to  $c = 2t + \lfloor (\frac{p-5t}{2}) \rfloor = \lfloor (\frac{p-t}{2}) \rfloor$  where t is the minimum integer verifying  $3(p-5t) \leq 2(q-5t), 3p-2q \leq 5t$  and by consequence  $t = \lceil (\frac{3p-2q}{5}) \rceil$  and Lemma 12 is proved.

**Lemma 13.** Let  $p \ge 4$  and let G = (B, W; E) be a bipartite  $P_5$ -saturated graph such that  $|B| = p \le q = |W|, K_{1,1}$  is a component of G and G has the

minimum size. Then

$$e(G) = \begin{cases} p+q - \left\lfloor \frac{p-1}{2} \right\rfloor - 2 & \text{if } 3p-1 \le 2q, \\ p+q - \left\lfloor \frac{1}{2} \left( p-1 - \left\lceil \frac{3p-2q-1}{5} \right\rceil \right) \right\rfloor - 2 & \text{if } 2q \le 3p-1. \end{cases}$$

**Proof.** By Lemma 11 each component of G is either complete graph  $K_{2,2}$  or non-trivial double star  $S_{a,b}^2$  and exactly one component is isomorphic to  $K_{1,1}$ . The size of G is equal to e(G) = p+q-c-1 where c is the number of double stars. We have  $e(2K_{2,2}) = 8 > e(S_{3,3}^2) = 7$  and  $e(K_{2,2} \cup S_{a,b}^2) = e(S_{a+2,b+2}^2)$ . So we may suppose that G has no components isomorphic to  $K_{2,2}$ . The lemma follows from Lemma 12.

**Proposition 14.** Let G = (B, W; E) be a bipartite  $P_5$ -saturated graph such that  $|B| = p \le q = |W|, 3 \le p \le q$  without isolated vertices and with the minimum size. Then

$$e(G) = \begin{cases} q & \text{if } 2p \le q, \\ p + \left\lceil \frac{q}{2} \right\rceil & \text{if } q < 2p, 2p - q \ne 2, \\ p + \left\lceil \frac{q}{2} \right\rceil + 1 & \text{if } 2p - q = 2. \end{cases}$$

**Proof.** The proof starts with the observation that by Lemma 11 G is a union of  $nK_{2,2}$  and  $S^2_{a_i,b_i}$ , i = 1, ..., k and some stars such that there is at most one  $K_{1,1}$  and the remaining stars have their centers in exactly one set of bipartition B or W. Now observe that if  $n \ge 2$  then  $S_{2n-1,2n-1}^2$ is non-trivial double star which has less edges than  $nK_{2,2}$  and the same number of vertices. Thus there is at most one  $K_{2,2}$ . But then there is at least one component C which is a star, or non-trivial double star. Then  $K_{2,2} \cup C$  may be replaced with a double star  $S^2_{a,b}$  with the same vertex set and with the size  $e(S_{a,b}^2) = e(K_{2,2} \cup C)$ . So we may suppose that no component of G is isomorphic to  $K_{2,2}$ . So G is a union of stars and double stars. We may check easily that if G has more then one double star then it is always possible to find a union of non-trivial stars and at most one double non-trivial star with the same size. Moreover all the stars may have their centers in a given set of bipartition. Hence we may suppose that either  $G = \bigcup_{i=1}^{k} K_{1,q_i} \cup S_{a,b}^2$ , k + a + 1 = p,  $\sum_{i=1}^{k} q_i + b + 1 = q$  or  $G = \bigcup_{i=1}^{l} K_{p_i,1} \cup S_{a,b}^2$ , l + b + 1 = q,  $\sum_{i=1}^{l} p_i + a + 1 = p$ . Similarly we may suppose that all non-trivial stars are isomorphic to  $K_{1,2}$  or  $K_{2,1}$  and we have 2k + b + 1 = q, k + a + 1 = p and 2l + a + 1 = p, l + b + 1 = q. Now the proof follows easily.

Clearly, Lemma 10 and Proposition 14 imply Proposition 4.

Proposition 5 follows from Lemma 18 and Corollary 17 given below. Let  $T_i, i \in \{1, 2\}$  be the tree defined in Figure 7.

**Lemma 15.** Let G = (B, W; E) be a connected bipartite  $P_6$ -saturated graph. Then either G contains one of graphs  $S_{2,2}^2$ ,  $T_i, i \in \{1,2\}$  or  $G = K_{r,s}$  with  $\min\{r,s\} \leq 2$ .

**Proof.** Let us denote |B| = p, |W| = q and let  $p \le q$ . For min $\{p,q\} \le 2$  the lemma is evident. So let us suppose that  $p,q \ge 3$ . It is easily seen that there exists at least one vertex  $x \in V(G)$  such that  $d_G(x) \ge 3$ . Let us suppose that  $x \in B$ . Denote by  $y_i, i = 1, 2, \ldots, n$  the neighbours of x. If there is a neighbour  $y_i, i = 1, 2, \ldots, n$  such that  $d_G(y_i) \ge 3$  then G contains  $S_{2,2}^2$ . So we may suppose that  $d_G(y_i) \le 2, i = 1, 2, \ldots, n$ . Since  $p \ge 3$  at least two of  $y_i, i = 1, 2, \ldots, n$  have their degrees equal to 2 and therefore G contains  $T_1$ .



Figure 7

**Lemma 16.** Let G = (B, W; E) be a bipartite  $P_6$ -saturated graph such that  $3 \leq |B| = p \leq q = |W|, p \geq 3$  and there is a vertex  $w \in W$  which is isolated in G. Then all the isolated vertices of G are in W and  $G = \bigcup_{i=1}^k K_{a_i,2} \cup \bigcup_{j=1}^l K_{0,1}^j$  where  $a_i \geq 3, i = 1, 2, \ldots, k, \sum_{i=1}^k a_i = p$  and q = 2k + l.

**Proof.** The fact that all isolated vertices are in W is evident. Let  $B = \{b_1, b_2, \ldots, b_p\}, W = \{w_1, w_2, \ldots, w_q\}$ . Denote by  $N_G(x)$  the set of the neighbours of the vertex  $x \in V(G)$ . It is clear that for every  $b \in B$  there is a path  $P_5$  starting from b. It is easy to check that every  $w \in N_G(b)$  belongs

to any path  $P_5$  starting from b. Thus  $d_G(b) \leq 2$ . Denote by  $b_1w_1b_2w_2b_3$ a path starting from  $b_1$ . It follows easily that for every  $x \in B$  such that  $w_i \in N_G(x), i \in \{1, 2\}$  we have  $N_G(x) \subseteq \{w_1, w_2\}$ . Therefore the component of G containing  $w_1$  and  $w_2$  is isomorphic to  $K_{a,2}, a \geq 3$ .

Corollary 17 follows immediately from Lemma 16.

**Corollary 17.** Let G = (B, W; E) be a bipartite  $P_6$ -saturated graph such that |B| = p and there is an isolated vertex in W. Then e(G) = 2p.

**Lemma 18.** If G = (B, W; E) is a bipartite  $P_6$ -saturated graph without isolated vertices and with the minimum size and  $3 \le |B| = p \le q = |W|$ , then

$$e(G) = \begin{cases} p+q - \left\lfloor \frac{p}{3} \right\rfloor & \text{if} \quad p \equiv 0 \pmod{3} \text{ or } \quad p = q \equiv 1 \pmod{3}, \\ p+q - \left\lfloor \frac{p}{3} \right\rfloor - 1 & \text{if} \quad p \equiv 1 \pmod{3} \text{ and } \quad p < q \text{ or} \\ & p \equiv 2 \pmod{3}. \end{cases}$$

**Proof.** For every graph G we have  $e(G) \ge |V(G)| - c$  where c is a number of components of G and equality holds if and only if G is a forest. The proof follows by Lemma 15.

Now, we turn to the case of  $m \ge 7$ .

**Lemma 19.** Let T = (B, W; E) be a  $P_m$ -saturated tree,  $m \ge 7$ ,  $x \in B \cup W$ , with  $d_T(x) > 1$  and let  $x_1, x_2, \ldots, x_k$  be the neighbors of x. For  $i = 1, 2, \ldots, k$  denote by  $l_i$  the maximum number of vertices in a path starting from x and containing  $x_i, i = 1, 2, \ldots, k, l_1 \ge l_2 \ge \ldots \ge l_k$ . The following holds:

- (i)  $m-1 \le l_1+l_i \le m, i=2,3,$
- (ii) if  $d_T(v) = 2$  then v is the neighbour of a pendant vertex (v is penultimate).

**Proof.** The inequality  $l_1 + l_i \leq m$  for i > 1 is evident. Let  $x_1^i, x_2^i, \ldots, x_{l_i}^i$  be a path of order  $l_i$  starting from  $x = x_1^i$  and containing  $x_i = x_2^i, i = 1, 2, \ldots, k$  (see Figure 8).

Suppose first that  $k \ge 3$  and x is not a penultimate. Then adding to T the edge  $x_2^1 x_3^2$  we create a path with m vertices. Thus  $l_1 - 1 + 2 + l_3 \ge m$  and therefore  $l_1 + l_3 \ge m - 1$ . So  $l_1 + l_2 \ge m - 1$  and (i) is proved.



Figure 8

Suppose that  $v \in B \cup W$  and  $d_T(v) = 2$  and v is not penultimate vertex. Denote by  $u_1, v_1$  the neighbours of  $v, P = v, u_1, \ldots, u_s$  and  $P' = v, v_1, \ldots, v_r$  the longest paths starting from v and passing by  $u_1, v_1$ , respectively. Then  $r, s \geq 2$ . The edge  $u_2v_1$  create a  $P_m$  contradicting the maximality of  $P = v, u_1, \ldots, u_s$ .



The  $P_7$ -saturated bipartite graphs with p = q = 5.

### Figure 9

The next lemma follows from Lemma 19.

**Lemma 20.** Let a tree T = (B, W; E) be a  $P_m$ -saturated bipartite graph  $m \ge 7$ . Then T contains  $A_m$ .

**Proofs of Theorem 6 and 7.** Like in the proof of Lemma 18 we use the fact that for every graph G we have  $e(G) \ge |V(G)| - c$  and equality holds if and only if G is a forest with exactly c components. Hence for given p, q and m, if there is a  $P_m$ -saturated forest F = (B, W; E) with |B| = p, |W| = q and the maximum number of components then F is a  $P_m$ -saturated bipartite graph with the minimum size. On the other hand it is clear that if the assumptions of Theorem 6 or 7 are verified then there exists such a forest F that each component of F contains  $A_m, m \ge 7$  (see Figure 9 and Figure 10).



The  $P_7$ -saturated graphs with p = 3, q = 6.

### Figure 10

Observe now that

- if  $m = 2k, k \ge 4$  and  $p = q = l(3 \cdot 2^{k-3} 1)$ , or
- if m = 2k + 1,  $k \ge 3$  and  $p = l(4 \cdot 2^{k-3} 1)$ ,  $q = l(5 \cdot 2^{k-3} 1)$ ,

then the  $P_m$ -saturated bipartite graph F = (B, W; E) without isolated vertices and with the minimum size and with |W| = q, |B| = p is the forest containing l trees  $A_m$ .

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