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P_m-SATURATED BIPARTITE GRAPHS WITH MINIMUM SIZE

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Abstract

A graph G is said to be *H-saturated* if G is *H-free* i.e., (G has no subgraph isomorphic to H) and adding any new edge to G creates a copy of H in G. In 1986 L. Kászonyi and Zs. Tuza considered the following problem: for given m and n find the minimum size $sat(n; P_m)$ of *P_m-saturated* graph of order n. They gave the number $sat(n; P_m)$ for n big enough. We deal with similar problem for bipartite graphs.

Keywords: graph, saturated graph, extremal graph, bipartite graph.

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1. Preliminaries

We deal with simple graphs without loops and multiple edges. As usual $V(G)$ and $E(G)$ denote the vertex set and the edge set, respectively, $|G|, e(G)$ the order and the size of G, and $d_G(v)$ the degree of $v \in V(G)$. By P_m we denote the path of order m, and by K_m the complete graph on m vertices. We define $G_{a,b}$ to be a bipartite graph where a, b are the numbers of vertices in bipartition sets. Let us consider two graphs G and H. We say that G is *H-free* if it contains no copy of H, that is, no subgraph of G is isomorphic to H. A graph G is *H-saturated* if G is *H-free* and adding any new edge e to G creates a copy of H. In particular complete *H-free* graphs trivially satisfy this condition and therefore are *H-saturated*. We define also:

*This work was carried out while the second author was visiting University of Orleans.

$$ex(n;F) = \max\{e(G) : |G| = n, G \text{ is } F\text{-saturated}\},$$

$$Ex(n;F) = \{G : |G| = n, e(G) = ex(n;F), G \text{ is } F\text{-saturated}\},$$

$$sat(n;F) = \min\{e(G) : |G| = n, G \text{ is } F\text{-saturated}\},$$

$$Sat(n;F) = \{G : |G| = n, e(G) = sat(n;F), G \text{ is } F\text{-saturated}\}.$$

Observe that in the definitions of $Ex(n;F)$ and $ex(n;F)$ the word *saturated* may be replaced with *free*. The first results concerning saturated graphs were given by Turán [6] in 1941 who asked for $ex(n;K_p)$ and $Ex(n;K_p)$. Later results were given by P. Erdös, A. Hajnal and J.W. Moon [3] (see also [2]) in 1964 who proved

$$sat(n;K_p) = \binom{p-2}{2} + (p-2)(n-p+2), \quad (n \geq p \geq 2)$$

$$Sat(n;K_p) = \{K_{p-2} * \bar{K}_{n-p+2}\}.$$

A corresponding theorem for bipartite graphs was given by N. Alon in 1983 (see [1]). The extremal problem for P_m-*saturated* bipartite graphs was solved by A. Gyárfás, C.C. Rousseau and R.H. Schelp [4]. We are interested in finding P_m-*saturated* bipartite graphs with minimum size. In Section 2 we present some results concerning P_m-*saturated* bipartite graphs. The proofs are given in Section 3.

In [5] L. Kászonyi and Zs. Tuza, gave the following results on $Sat(n;P_m)$ and $sat(n;P_m)$.

Theorem 1 ([5]).

$$sat(n;P_3) = \left\lfloor \frac{n}{2} \right\rfloor,$$

$$Sat(n;P_3) = \begin{cases} kK_2 & \text{if} \quad n = 2k, \\ kK_2 \cup K_1 & \text{if} \quad n = 2k+1, \end{cases}$$

$$sat(n;P_4) = \begin{cases} k & \text{if} \quad n = 2k, \\ k+2 & \text{if} \quad n = 2k+1, \end{cases}$$

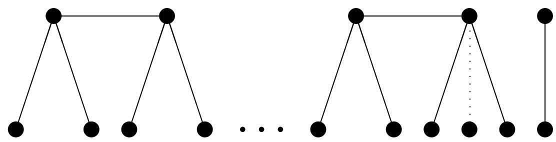
$$Sat(n;P_4) = \begin{cases} kK_2 & \text{if} \quad n = 2k, \\ (k-1)K_2 \cup K_3 & \text{if} \quad n = 2k+1, \end{cases}$$

$$sat(n;P_5) = n - \left\lfloor \frac{n-2}{6} \right\rfloor - 1 \quad \text{for } n \geq 6.$$

Let

$$a_m = \begin{cases} 3 \cdot 2^{k-1} - 2 & \text{if} \quad m = 2k, k > 2, \\ 2^{k+1} - 2 & \text{if} \quad m = 2k+1, k \geq 2. \end{cases}$$

Then $sat(n; P_m) = n - \lfloor \frac{n}{a_m} \rfloor$ *for* $n \geq a_m$.



$Sat(n; P_5)$ for $n \geq 6$.

Figure 1

2. P_m-Saturated Bipartite Graphs with Minimum Size

Let $G = (B, W; E)$ be a bipartite graph with vertex set $V = B \cup W$, $B \cap W = \emptyset$. For convenience of the reader we call the set B the set of *black* vertices and the set W the set of *white* vertices. For bipartite graphs $G = (B, W; E)$ and $F = (B', W'; E')$ such that the sets B, W, B' and W' are mutually disjoint we define: $G \cup F = (B \cup B', W \cup W'; E \cup E')$.

Definition 1. Let $G = (B, W; E)$ be a bipartite graph. Then G is called *F-saturated* if

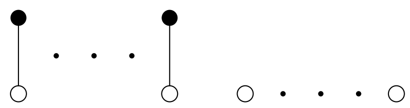
1. G is *F-free*,
2. $(x \in B, y \in W, xy \notin E) \Rightarrow G \cup xy \supseteq F$.

We denote also

$$sat_{bip}(p, q; F) = \min\{e(G) : |B| = p, |W| = q, G \text{ is } F\text{-saturated}\},$$

$$Sat_{bip}(p, q; F) = \{G = (B, W; E) : |B| = p, |W| = q, e(G) = sat_{bip}(p, q; F), \\ G \text{ is } F\text{-saturated}\}.$$

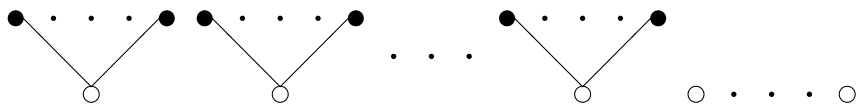
Proposition 2. $sat_{bip}(p, q; P_3) = p$, $p \leq q$. ■



$Sat_{bip}(p, q; P_3)$.

Figure 2

Proposition 3. $sat_{bip}(p, q; P_4) = p, \quad 2 \le p \le q.$ ■



$Sat_{bip}(p, q; P_4)$.

Figure 3

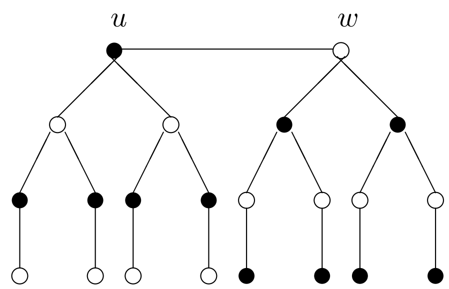
Proposition 4. *Let* $p \ge 2, q \ge 3, p \le q$. *Then*

$$sat_{bip}(p, q; P_5) = \begin{cases} 2p & \text{if } 2p \le q, \ p \text{ is even or } q = 2p-2, \\ q & \text{if } 2p \le q, \ p \text{ is odd}, \\ p + \left\lceil \frac{q}{2} \right\rceil & \text{if } 3 < q < 2p, \ q \ne 2p-2, \\ 5 & \text{if } p = q = 3, \\ 4 & \text{if } p = 2. \end{cases}$$

Proposition 5. *Let* $3 \le p \le q$. *Then*

$$sat_{bip}(p, q; P_6) = \begin{cases} p + q - \left\lfloor \frac{p}{3} \right\rfloor - 1 & \text{if } \begin{array}{l} p \equiv 2 \pmod 3 \text{ and } 3q \le 4p-2 \text{ or} \\ p \equiv 1 \pmod 3 \text{ and } 3q \le 4p-1, \end{array} \\ p + q - \left\lfloor \frac{p}{3} \right\rfloor & \text{if } \begin{array}{l} p = q \equiv 1 \pmod 3 \text{ or} \\ p \equiv 0 \pmod 3 \text{ and } 3q \le 4p-1, \end{array} \\ 2p & \text{otherwise}. \end{cases}$$

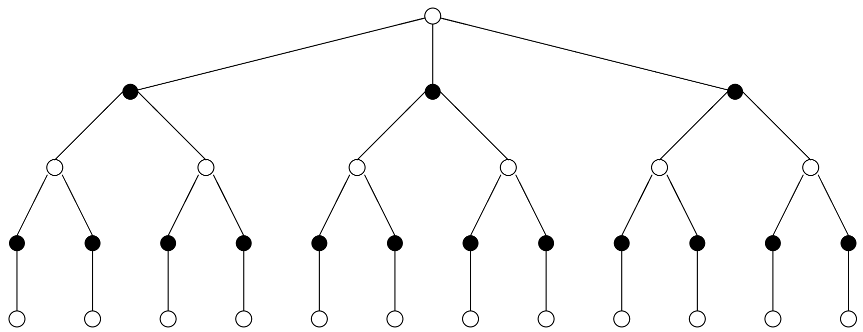
Definition 2. Let us suppose that $m \geq 7$ is an integer. Then A_m is the following tree. All penultimate vertices of A_m have degree two and all vertices of A_m which are neither penultimate nor pendant have their degree equal to three. If $m = 2k, k \geq 4$ then A_m has two centers $u \in B$ and $w \in W$ and each component of $G - uw$ has $k - 1$ levels (see Figure 4). If $m = 2k + 1, k \geq 3$ then A_m has one center and k levels. The center is black when k is even (see Figure 5).



A_{10}

Figure 4

When $m = 2k, k \geq 4$ we observe that $|B| = |W| = 3 \cdot 2^{k-3} - 1$.



A_{11}

Figure 5

If $m = 2k + 1, k \geq 3$ and the center is white then in A_m we have $|B| = 4 \cdot 2^{k-3} - 1$ and $|W| = 5 \cdot 2^{k-3} - 1$ when k is odd or $|W| = 4 \cdot 2^{k-3} - 1$ and $|B| = 5 \cdot 2^{k-3} - 1$ when k is even. Denote by v the center of $A_{2k+1}, k \geq 3$.

Observe that if $|B| \leq |W|$ then for $m = 2k+1, k \geq 3$ we obtain $|B| = 4 \cdot 2^{k-3} - 1$, $|W| = 5 \cdot 2^{k-3} - 1$ and $v \in B$ if k is even or $v \in W$ if k is odd.

Remark 1. Observe that A_m is P_m*-saturated* and P_{m-1}*-saturated* for every $m \geq 7$.

Remark 2. lA_{2k} is P_{2k}*-saturated* for $k \geq 4, l = 1, 2, \ldots, n$.

Remark 3. The union of two copies of A_{2k+1} is P_{2k+1}*-saturated*, $k \geq 3$, if and only if their centers have the same colour (see Figure 6).

Theorem 6. *Let $k \geq 4$ and let $G = (B, W; E)$ be a P_{2k}-saturated bipartite graph without isolated vertices and with the minimum size, $|B| = p, |W| = q, 3 \cdot 2^{k-3} - 1 \leq p \leq q$. Then*

$$e(G) = p + q - \left\lfloor \frac{p}{3 \cdot 2^{k-3} - 1} \right\rfloor .$$

Theorem 7. *Let $k \geq 3$ and let $G = (B, W; E)$ be a P_{2k+1}-saturated bipartite graph without isolated vertices and with the minimum size, $|B| = p \leq |W| = q, 4 \cdot 2^{k-3} - 1 \leq p,\ 5 \cdot 2^{k-3} \leq q$. Then*

$$e(G) = \begin{cases} p + q - \frac{q}{5 \cdot 2^{k-3} - 1} + 1 & \text{if} \quad \frac{q}{5 \cdot 2^{k-3} - 1} = \left\lfloor \frac{q}{5 \cdot 2^{k-3} - 1} \right\rfloor < \frac{p}{4 \cdot 2^{k-3} - 1}, \\ p + q - \min\left\{ \left\lfloor \frac{p}{4 \cdot 2^{k-3} - 1} \right\rfloor, \left\lfloor \frac{q}{5 \cdot 2^{k-3} - 1} \right\rfloor \right\} & \text{otherwise.} \end{cases}$$

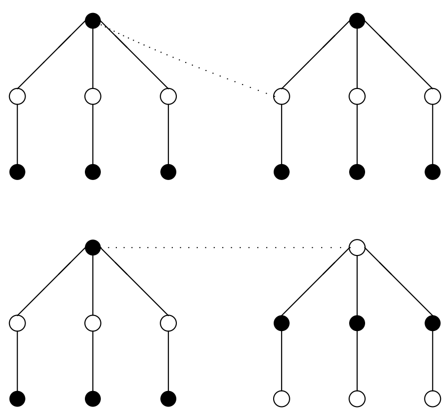


Figure 6

Theorem 6 and 7 imply the following corollary.

Corollary 8. *If* $|B| = p, |W| = q, 3 \cdot 2^{k-3} - 1 \leq p \leq q, k \geq 4$ *then*

$$sat_{bip}(p, q; P_{2k}) \leq p + q - \left\lfloor \frac{p}{3 \cdot 2^{k-3} - 1} \right\rfloor.$$

If $|B| = p, |W| = q,\ p \leq q, 4 \cdot 2^{k-3} - 1 \leq p,\ 5 \cdot 2^{k-3} \leq q,$ *then*

$$sat_{bip}(p, q; P_{2k+1})$$
$$= \begin{cases} p + q - \frac{q}{5 \cdot 2^{k-3}-1} + 1 & if \quad \frac{q}{5 \cdot 2^{k-3}-1} = \left\lfloor \frac{q}{5 \cdot 2^{k-3}-1} \right\rfloor < \frac{p}{4 \cdot 2^{k-3}-1}, \\ p + q - \min\left\{ \left\lfloor \frac{p}{4 \cdot 2^{k-3}-1} \right\rfloor, \left\lfloor \frac{q}{5 \cdot 2^{k-3}-1} \right\rfloor \right\} & otherwise. \end{cases}$$ ■

3. Proofs

We first give some definitions. The graphs $K_{1,n}$ and $K_{n,1}$ are called *stars* when $n \geq 1$ and *non-trivial stars* if $n \geq 2$. Let $K_{1,b}$ and $K_{a,1}$ be two vertex disjoint stars. Then the tree obtained by join of their centers is called *double star* $S^2_{a,b}$ (see Figure 7). A double star $S^2_{a,b}$ is said to be *non-trivial* if $a > 0, b > 0$ and $a + b \geq 3$. Propositions 2 and 3 are evident. To prove Proposition 4 we give Lemmas 9–13 and Proposition 14 below.

Lemma 9. *Let* $G = (B, W; E)$ *be a connected bipartite* P_5*-saturated graph* $|B| = p, |W| = q$*. Then either*

1. *G is a star or*
2. *G is non-trivial double star* $S^2_{a,b}$ *or else*
3. $G = K_{2,2}$.

Lemma 10. *Let* $G = (B, W; E)$ *be a bipartite* P_5*-saturated graph,* $|B| = p, |W| = q,\ p \leq q, p \geq 2, q \geq 3,$ *such that there is at least one isolated vertex in* W*. Then* $p = 2k, k \geq 1$ *and* $G = kK_{2,2} \cup K_{0,q-p}$*. In particular we have*

1. $p < q$ *and*
2. $e(G) = 2p$.

Lemma 9 is evidently true. Lemma 10 follows from Lemma 9 easily. ■

Lemma 11. *Let $G = (B, W; E)$ be a bipartite P_5-saturated graph without isolated vertices $|B| = p, |W| = q$, $p \le q, p \ge 2, q \ge 3$. Then G is vertex disjoint union of*

1. *complete graphs $K_{2,2}$,*
2. *non-trivial double stars,*
3. *non-trivial stars,*
4. *at most one trivial star $K_{1,1}$.*

If $K_{1,1}$ is a component of G then no other star is a component of G.

Lemma 11 follows from Lemma 9. ∎

If $G = (B, W; E)$ is a bipartite P_5-*saturated* graph with $|B| = p, |W| = q$ then we have either

$$G = nK_{2,2} \cup \bigcup_{i=1}^{k} S^2_{a_i,b_i} \cup \bigcup_{j=1}^{l} K_{1,c_j} \cup \delta K_{1,1} \tag{1}$$

$$G = nK_{2,2} \cup \bigcup_{i=1}^{k} S^2_{a_i,b_i} \cup \bigcup_{j=1}^{l} K_{d_j,1} \cup \delta K_{1,1} \tag{2}$$

where $S^2_{a_i,b_i}$ are non-trivial double stars, K_{1,c_j} and $K_{d_j,1}$ are non-trivial stars, and $\delta \in \{0, 1\}$. We have $p = 2n + \left(\sum_{i=1}^{k} a_i + k\right) + l + \delta$,
$q = 2n + \left(\sum_{i=1}^{k} b_i + k\right) + \sum_{j=1}^{l} c_j + \delta$ if G is given by (1), and
$p = 2n + \left(\sum_{i=1}^{k} a_i + k\right) + \sum_{j=1}^{l} d_j + \delta$, $q = 2n + \left(\sum_{i=1}^{k} b_i + k\right) + l + \delta$ if G is given by (2), and $\delta = 0$ if $l > 0$.

Lemma 12. *Let $G = (B, W; E)$ be a union of non-trivial double stars, such that $|B| = p, |W| = q$, $p \le q, p \ge 2, q \ge 3$. Then G has the minimum size if*

$$e(G) = \begin{cases} p + q - \left\lfloor \frac{p}{2} \right\rfloor & \textit{if} \quad 3p \le 2q, \\ p + q - \left\lfloor \frac{1}{2}\left(p - \left\lceil \frac{3p-2q}{5} \right\rceil\right) \right\rfloor & \textit{if} \quad 2q \le 3p. \end{cases}$$

Proof. Let G be a union of non-trivial double stars, $G = \bigcup_{i=1}^{l} S^2_{a_i,b_i}$ where

$$S^2_{a_i,b_i} = (B_i, W_i; E_i), |B_i| = a_i + 1, |W_i| = b_i + 1, i = 1, \dots, l,$$

$$\bigcup_{i=1}^{l} B_i = B, \bigcup_{i=1}^{l} W_i = W,$$

such that for fixed p and q, G has the minimum size $e(G)$. We observe that $e(G) = p+q-c$ where $c = c(G)$ is the number of components of G. So, $e(G)$ is the minimum whenever $c(G)$ is the maximum. Since every component of G has at least two vertices in B then $c(G) \leq \lfloor \frac{p}{2} \rfloor$. If $3p \leq 2q$, then $c = \lfloor \frac{p}{2} \rfloor$ and c components of G are S^2_{1,b_i} stars with $b_i \geq 2, i = 1, 2, \dots, c-1$ and $a_c = 1, b_c \geq 2$ if p is even, and $a_c = 2, b_c \geq 1$ when p is odd.

Therefore $e(G) = p + q - \lfloor \frac{p}{2} \rfloor$ when $3p \leq 2q$. So we may assume from now that $3p > 2q$. Since the lemma is easy to verify for $p \leq 4$ we shall assume $p \geq 5$. Observe that there are two different components C_1 and C_2 of G such that $C_1 = S^2_{a_1,b_1}$, $C_2 = S^2_{a_2,b_2}$, $b_1 \geq 2$ and $a_2 \geq 2$.

If $p \leq 6$ or $q \leq 7$ then $c(G) = 2$ and the proof is finished. So we suppose $p \geq 7$ and $q \geq 8$. Then there is at least one component C, $C \neq C_1$, $C \neq C_2$. Let x, y be the centers of C, x_1, y_1 be the centers of C_1, x_2, y_2 be the centers of C_2, such that $x, x_1, x_2 \in B, y, y_1, y_2 \in W$. It is clear that the number of components of G will not change if we proceed the following operation:

- delete from C_1 all but one black pendant vertices and all but two white pendant vertices (we denote then by C_1' the obtained component),
- delete from C_2 all but two black pendant vertices and all but one white pendant vertices (we denote then by C_2' the obtained component),
- join x with all white vertices deleted from C_1 and C_2 and join y with all black vertices deleted from C_1 and C_2 (we denote then by C' the obtained component).

The new graph G' has exactly the same number of components as G and all the components of G' are non-trivial double stars. The number of components of G is equal to $c = 2t + \lfloor (\frac{p-5t}{2}) \rfloor = \lfloor (\frac{p-t}{2}) \rfloor$ where t is the minimum integer verifying $3(p-5t) \leq 2(q-5t), 3p-2q \leq 5t$ and by consequence $t = \lceil (\frac{3p-2q}{5}) \rceil$ and Lemma 12 is proved. ■

Lemma 13. *Let $p \geq 4$ and let $G = (B, W; E)$ be a bipartite P_5-saturated graph such that $|B| = p \leq q = |W|, K_{1,1}$ is a component of G and G has the* *minimum size. Then*

$$e(G) = \begin{cases} p+q-\left\lfloor \frac{p-1}{2} \right\rfloor - 2 & \text{if} \quad 3p-1 \le 2q, \\ p+q-\left\lfloor \frac{1}{2}\left(p-1-\left\lceil \frac{3p-2q-1}{5} \right\rceil\right) \right\rfloor - 2 & \text{if} \quad 2q \le 3p-1. \end{cases}$$

Proof. By Lemma 11 each component of G is either complete graph $K_{2,2}$ or non-trivial double star $S^2_{a,b}$ and exactly one component is isomorphic to $K_{1,1}$. The size of G is equal to $e(G) = p+q-c-1$ where c is the number of double stars. We have $e(2K_{2,2}) = 8 > e(S^2_{3,3}) = 7$ and $e(K_{2,2} \cup S^2_{a,b}) = e(S^2_{a+2,b+2})$. So we may suppose that G has no components isomorphic to $K_{2,2}$. The lemma follows from Lemma 12. ■

Proposition 14. *Let $G = (B, W; E)$ be a bipartite P_5-saturated graph such that $|B| = p \le q = |W|, 3 \le p \le q$ without isolated vertices and with the minimum size. Then*

$$e(G) = \begin{cases} q & \text{if} \quad 2p \le q, \\ p + \left\lceil \frac{q}{2} \right\rceil & \text{if} \quad q < 2p, 2p-q \ne 2, \\ p + \left\lceil \frac{q}{2} \right\rceil + 1 & \text{if} \quad 2p - q = 2. \end{cases}$$

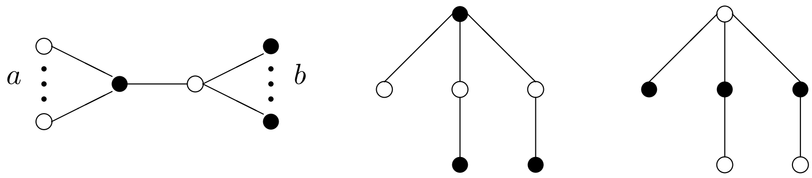
Proof. The proof starts with the observation that by Lemma 11 G is a union of $nK_{2,2}$ and $S^2_{a_i,b_i}, i = 1, \ldots, k$ and some stars such that there is at most one $K_{1,1}$ and the remaining stars have their centers in exactly one set of bipartition B or W. Now observe that if $n \ge 2$ then $S^2_{2n-1,2n-1}$ is non-trivial double star which has less edges than $nK_{2,2}$ and the same number of vertices. Thus there is at most one $K_{2,2}$. But then there is at least one component C which is a star, or non-trivial double star. Then $K_{2,2} \cup C$ may be replaced with a double star $S^2_{a,b}$ with the same vertex set and with the size $e(S^2_{a,b}) = e(K_{2,2} \cup C)$. So we may suppose that no component of G is isomorphic to $K_{2,2}$. So G is a union of stars and double stars. We may check easily that if G has more then one double star then it is always possible to find a union of non-trivial stars and at most one double non-trivial star with the same size. Moreover all the stars may have their centers in a given set of bipartition. Hence we may suppose that either $G = \bigcup_{i=1}^{k} K_{1,q_i} \cup S^2_{a,b},\ k+a+1 = p,\ \sum_{i=1}^{k} q_i + b + 1 = q$ or $G = \bigcup_{i=1}^{l} K_{p_i,1} \cup S^2_{a,b},\ l+b+1 = q,\ \sum_{i=1}^{l} p_i + a + 1 = p$. Similarly we may suppose that all non-trivial stars are isomorphic to $K_{1,2}$ or $K_{2,1}$ and we have $2k+b+1=q,\ k+a+1=p$ and $2l+a+1=p,\ l+b+1=q$. Now the proof follows easily. ■

Clearly, Lemma 10 and Proposition 14 imply Proposition 4.

Proposition 5 follows from Lemma 18 and Corollary 17 given below. Let $T_i, i \in \{1,2\}$ be the tree defined in Figure 7.

Lemma 15. *Let $G = (B,W;E)$ be a connected bipartite P_6-saturated graph. Then either G contains one of graphs $S^2_{2,2}$, $T_i, i \in \{1,2\}$ or $G = K_{r,s}$ with* $\min\{r,s\} \le 2$.

Proof. Let us denote $|B| = p, |W| = q$ and let $p \le q$. For $\min\{p,q\} \le 2$ the lemma is evident. So let us suppose that $p,q \ge 3$. It is easily seen that there exists at least one vertex $x \in V(G)$ such that $d_G(x) \ge 3$. Let us suppose that $x \in B$. Denote by $y_i, i = 1,2,\dots,n$ the neighbours of x. If there is a neighbour $y_i, i = 1,2,\dots,n$ such that $d_G(y_i) \ge 3$ then G contains $S^2_{2,2}$. So we may suppose that $d_G(y_i) \le 2, i = 1,2,\dots,n$. Since $p \ge 3$ at least two of $y_i, i = 1,2,\dots,n$ have their degrees equal to 2 and therefore G contains T_1. ■



$S^2_{a,b}$ T_1 T_2

Figure 7

Lemma 16. *Let $G = (B,W;E)$ be a bipartite P_6-saturated graph such that $3 \le |B| = p \le q = |W|$, $p \ge 3$ and there is a vertex $w \in W$ which is isolated in G. Then all the isolated vertices of G are in W and $G = \bigcup_{i=1}^{k} K_{a_i,2} \cup \bigcup_{j=1}^{l} K^j_{0,1}$ where $a_i \ge 3, i = 1,2,\dots,k$, $\sum_{i=1}^{k} a_i = p$ and $q = 2k+l$.* ■

Proof. The fact that all isolated vertices are in W is evident. Let $B = \{b_1,b_2,\dots,b_p\}$, $W = \{w_1,w_2,\dots,w_q\}$. Denote by $N_G(x)$ the set of the neighbours of the vertex $x \in V(G)$. It is clear that for every $b \in B$ there is a path P_5 starting from b. It is easy to check that every $w \in N_G(b)$ belongs to any path P_5 starting from b. Thus $d_G(b) \leq 2$. Denote by $b_1w_1b_2w_2b_3$ a path starting from b_1. It follows easily that for every $x \in B$ such that $w_i \in N_G(x), i \in \{1,2\}$ we have $N_G(x) \subseteq \{w_1, w_2\}$. Therefore the component of G containing w_1 and w_2 is isomorphic to $K_{a,2}, a \geq 3$. ■

Corollary 17 follows immediately from Lemma 16.

Corollary 17. *Let $G = (B, W; E)$ be a bipartite P_6-saturated graph such that $|B| = p$ and there is an isolated vertex in W. Then $e(G) = 2p$.* ■

Lemma 18. *If $G = (B, W; E)$ is a bipartite P_6-saturated graph without isolated vertices and with the minimum size and $3 \leq |B| = p \leq q = |W|$, then*

$$e(G) = \begin{cases} p + q - \left\lfloor \frac{p}{3} \right\rfloor & \text{if} \quad p \equiv 0 (\text{mod } 3) \text{ or } \quad p = q \equiv 1 (\text{mod } 3), \\ p + q - \left\lfloor \frac{p}{3} \right\rfloor - 1 & \text{if} \quad p \equiv 1 (\text{mod } 3) \text{ and } \quad p < q \text{ or} \\ & \qquad p \equiv 2 (\text{mod } 3). \end{cases}$$

Proof. For every graph G we have $e(G) \geq |V(G)| - c$ where c is a number of components of G and equality holds if and only if G is a forest. The proof follows by Lemma 15. ■

Now, we turn to the case of $m \geq 7$.

Lemma 19. *Let $T = (B, W; E)$ be a P_m-saturated tree, $m \geq 7$, $x \in B \cup W$, with $d_T(x) > 1$ and let $x_1, x_2, \ldots, x_k$ be the neighbors of x. For $i = 1, 2, \ldots, k$ denote by l_i the maximum number of vertices in a path starting from x and containing $x_i, i = 1, 2, \ldots, k, l_1 \geq l_2 \geq \ldots \geq l_k$. The following holds:*

(i) $m - 1 \leq l_1 + l_i \leq m,\ i = 2, 3,$

(ii) *if $d_T(v) = 2$ then v is the neighbour of a pendant vertex (v is penultimate).*

Proof. The inequality $l_1 + l_i \leq m$ for $i > 1$ is evident. Let $x_1^i, x_2^i, \ldots, x_{l_i}^i$ be a path of order l_i starting from $x = x_1^i$ and containing $x_i = x_2^i$, $i = 1, 2, \ldots, k$ (see Figure 8).

Suppose first that $k \geq 3$ and x is not a penultimate. Then adding to T the edge $x_2^1 x_3^2$ we create a path with m vertices. Thus $l_1 - 1 + 2 + l_3 \geq m$ and therefore $l_1 + l_3 \geq m - 1$. So $l_1 + l_2 \geq m - 1$ and (i) is proved.

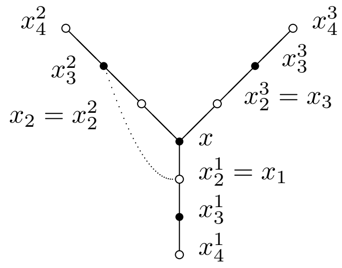
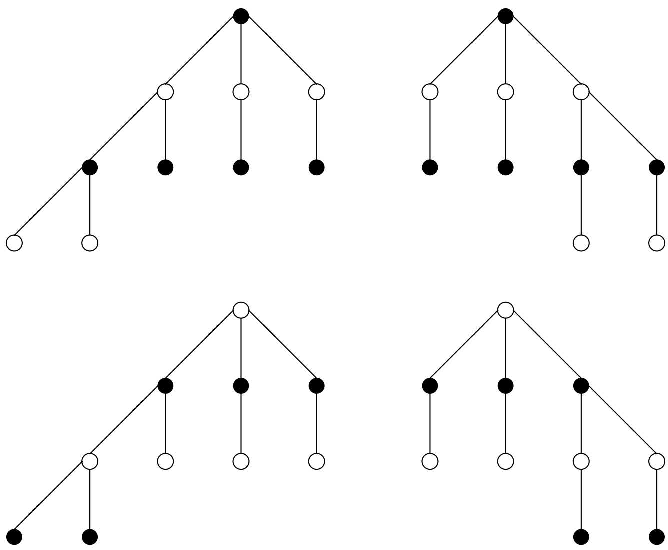


Figure 8

Suppose that $v \in B \cup W$ and $d_T(v) = 2$ and v is not penultimate vertex. Denote by u_1, v_1 the neighbours of v, $P = v, u_1, \ldots, u_s$ and $P' = v, v_1, \ldots, v_r$ the longest paths starting from v and passing by u_1, v_1, respectively. Then $r, s \geq 2$. The edge $u_2 v_1$ create a P_m contradicting the maximality of $P = v, u_1, \ldots, u_s$. ■



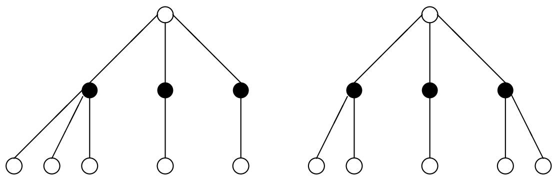
The P_7-saturated bipartite graphs with $p = q = 5$.

Figure 9

The next lemma follows from Lemma 19.

Lemma 20. *Let a tree $T = (B, W; E)$ be a P_m-saturated bipartite graph $m \geq 7$. Then T contains A_m.* ∎

***Proofs of Theorem* 6 *and* 7.** Like in the proof of Lemma 18 we use the fact that for every graph G we have $e(G) \geq |V(G)| - c$ and equality holds if and only if G is a forest with exacly c components. Hence for given p, q and m, if there is a P_m-*saturated* forest $F = (B, W; E)$ with $|B| = p, |W| = q$ and the maximum number of components then F is a P_m-*saturated* bipartite graph with the minimum size. On the other hand it is clear that if the assumptions of Theorem 6 or 7 are verified then there exists such a forest F that each component of F contains $A_m, m \geq 7$ (see Figure 9 and Figure 10). ∎



The P_7-*saturated* graphs with $p = 3, q = 6$.

Figure 10

Observe now that

- if $m = 2k$, $k \geq 4$ and $p = q = l(3 \cdot 2^{k-3} - 1)$, or
- if $m = 2k + 1$, $k \geq 3$ and $p = l(4 \cdot 2^{k-3} - 1)$, $q = l(5 \cdot 2^{k-3} - 1)$,

then the P_m-saturated bipartite graph $F = (B, W; E)$ without isolated vertices and with the minimum size and with $|W| = q$, $|B| = p$ is the forest containing l trees A_m.

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