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SOME SUFFICIENT CONDITIONS ON ODD DIRECTED CYCLES OF BOUNDED LENGTH FOR THE EXISTENCE OF A KERNEL

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Abstract

A kernel N of a digraph D is an independent set of vertices of D such that for every $w \in V(D) - N$ there exists an arc from w to N. If every induced subdigraph of D has a kernel, D is said to be a kernel-perfect digraph. In this paper I investigate some sufficient conditions for a digraph to have a kernel by asking for the existence of certain diagonals or symmetrical arcs in each odd directed cycle whose length is at most $2\alpha(D) + 1$, where $\alpha(D)$ is the maximum cardinality of an independent vertex set of D. Previous results are generalized.

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1. Introduction

For general concepts we refer the reader to [2]. Let D be a digraph, $V(D)$ and $A(D)$ will denote the sets of vertices and arcs of D, respectively. A kernel of a digraph D is a subset $K \subseteq V(D)$ such that K is independent ($K \cap \Gamma_D^{-1}(K) = \emptyset$) and absorbing ($K \cup \Gamma_D^{-1}(K) = V(D)$). When every induced subdigraph of D has a kernel, D is said to be kernel-perfect or a KP-digraph. We say that D is a critical kernel-imperfect digraph or a CKI-digraph if D does not have a kernel but every proper induced subdigraph of D does have at least one.

The concept of kernel was introduced by von Neumann and Morgenstern [12] in the context of Game Theory. They also proved that any finite acyclic digraph has a (unique) kernel. The problem of the existence of a kernel in a given digraph has been studied by several authors, in particular by Richardson [13, 14], Duchet and Meyniel [6], Duchet [4, 5], Galeana-Sánchez and Neumann-Lara [8]. This concept has found applications, for instance in cooperative n-person games, in Nim-type games [2], in logic [3], etc.. Since not all digraphs has a kernel, the natural problem is: which structural properties of a graph imply the existence of a kernel?

In this paper I investigate some sufficient conditions for a digraph to have a kernel by asking for the existence of certain diagonals or symmetrical arcs in each odd directed cycle whose length is at most $2\alpha(D)+1$, where $\alpha(D)$ is the maximum cardinality of an independent vertex set of D. Previous results are generalized.

Sometimes a digraph D will be viewed as an orientation of its underlying undirected graph that we denote by G_D. The edges of G will be denoted by $E(G_D)$ and we write u_1u_2 for an edge and (u_1, u_2) for an arc.

If D_0 is a subdigraph (resp. induced subdigraph) of D we write $D_0 \subset D$ (resp. $D_0 \subseteq^* D$). Let S_1, S_2 be two subsets of $V(D)$, the arc (u_1, u_2) of D will be called an S_1S_2-arc whenever $u_1 \in S_1$ and $u_2 \in S_2$; $D[S_1]$ will denote the subdigraph of D induced by S_1. An arc $(u_1, u_2) \in A(D)$ is called asymmetrical (resp. symmetrical) if $(u_2, u_1) \notin A(D)$ (resp. $(u_2, u_1) \in A(D)$). The asymmetrical part of D (resp. the symmetrical part of D) which is denoted by Asym (D) (resp. Sym (D)) is the spanning subdigraph of D whose arcs are the asymmetrical (resp. symmetrical) arcs of D.

Let $C = (0, 1, \ldots, m-1, 0)$ be a directed cycle of D, we denote by $\ell(C)$ its length. For $i \neq j$, $i, j \in V(C)$ we denote by (i, C, j) the ij-directed path contained in C and by $\ell(i, C, j)$ its length; $f = (i, j) \in (A(D) - A(C))$ is a diagonal of C iff $i \neq j$, $i, j \in V(C)$ and $\ell(i, C, j) < \ell(C) - 1$, $f = (i, j) \in (A(D) - A(C))$ is a pseudodiagonal of C iff, $i \neq j, i, j \in V(C)$ and $\ell(i, C, j) \leq \ell(C) - 1$. A short chord of C is a diagonal of C with length two ($\ell(i, C, j) = 2$). We will denote by $t(C) = \{z \in V(C) \mid$ there exists a pseudodiagonal (w, z) of $C\}$ the set of terminal endpoints of pseudodiagonals of C, also they will be named the poles of C. Two consecutive poles of C are two poles which are consecutive in C.

We will denote: $C_0^1 = \{i \in C \,|\, i \equiv 1 (\mathrm{mod}\, 2)\}$. For instance if $C = (0, 1, 2, 0)$, $C_0^1 = \{1\}$.

Theorem 1.1 [8]. *Let D be a CKI-digraph, $u \in V(D)$. Then there exists a directed cycle C of odd length passing by u and having no $V(C)(C_u^1 \cup \{u\})$-pseudodiagonals (We are considering $C = (u = 0, 1, 2, \ldots, 2n, 0)$, $C_u^1 = \{i \in V(C) \mid i \equiv 1(mod 2)\}$. In particular C_u^1 is an independent set.*

Theorem 1.2 [8]. *Let D be a digraph. If for each odd directed cycle C contained in D and $u \in V(C)$, there exists a $V(C)(C_u^1 \cup \{u\})$-pseudodiagonal, then D is a KP-digraph.*

Theorem 1.3 [8]. *Let D be a digraph. If each odd directed cycle C of D has two consecutive poles in C then D is a kernel-perfect digraph.*

Theorem 1.4 [5]. *Let D be a digraph in which each 3-circuit is symmetrical. If every odd directed cycle possesses two short chords, then every odd directed cycle has at least two consecutive poles.*

Theorem 1.5 [4]. *Let D be a digraph such that each odd directed cycle has at least two symmetrical arcs, then D is a kernel-perfect digraph.*

A digraph D is called a *fraternal orientation* of G_D if $(u, w) \in A(D)$ and $(v, w) \in A(D)$ implies $(u, v) \in A(D)$ or $(v, u) \in A(D)$. If D is a fraternal orientation of G_D we will say that D is a *fraternally oriented digraph*, G is *fraternally orientable* if it admits a fraternal orientation. The concept of fraternal orientation was introduced by D.J. Skrien [16] and a characterization in case that D has no symmetrical arcs has been obtained by F. Gavril and J. Urrutia [11], they also proved that triangulated graphs and circular arc graphs are all fraternally orientable graphs. Many properties of fraternally orientable graphs have been obtained by J. Bang-Jensen, J. Huang and E. Prisner in [1]. A class of normal fraternally orientable graphs has been studied by F. Gavril, V. Toledano Laredo and D. de Werra in [10]. They described a polynomial time algorithm to find a kernel in such a class of normal fraternally orientable graphs.

We recall that every triangulated graph has an acyclical fraternal-orientation (D.J. Rose [15]).

Normal-fraternally orientable graphs are an interesting class of graphs as we can conclude from the following result:

Theorem 1.6 [7]. *The Strong Perfect Graph Conjecture is true if and only if every critically imperfect graph not isomorphic to $\overline{C}_{2n+1}$ $n \geq 4$ is a normal fraternally orientable graph.*

Theorem 1.7 [9]. *Let D be a digraph, and suppose that every odd directed cycle C of D satisfies the following properties:*

(i) *$D[V(C)]$ is a fraternally oriented digraph and*

(ii) *C has at least two pseudodiagonals.*

Then each odd directed cycle of D has at least two consecutive poles.

2. Some Sufficient Conditions on odd Directed Cycles of Bounded Length for the Existence of a Kernel

In this section I show some sufficient conditions for a digraph to have a kernel by asking for the existence of certain diagonals or symmetrical arcs in each odd directed cycle whose length is at most $2\alpha(D)+1$, generalyzing some previous results.

Theorem 2.1. *Let D be a digraph. If for each odd directed cycle C whose length is at most $2\alpha(D)+1$, contained in D and for each vertex $u \in V(C)$, there exists $V(C)(C_u^1 \cup \{u\})$-pseudodiagonal. (Considering $C = (u = 0, 1, \ldots, 2n, 0)$), then D is a kernel-perfect digraph.*

Proof. By contradiction, suppose that D is not a KP-digraph. Then there exists a CKI-digraph H, $H \subseteq^* D$. It follows from Theorem 1.1 that for $u \in V(H)$ there exists a directed cycle C of odd length passing by u, contained in H and having no $V(C)(C_u^1 \cup \{u\})$-pseudodiagonals. The hypothesis implies $\ell(C) > 2\alpha(D)+1$, hence $|C_u^1| > \alpha(D)$ and therefore there exists an arc of D with both terminal endpoints in C_u^1 which clearly is a $V(C)(C_u^1 \cup \{u\})$-pseudodiagonal. This contradicts the choice of C. ■

Clearly Theorem 2.1 is an equivalent version of Theorem 1.1 and as we will see it will be useful in order to obtain some generalizations of previous results.

Theorem 2.2. *Let D be a digraph. If each odd directed cycle C contained in D whose length is at most $2\alpha(D)+1$, has at least two consecutive poles in C, then D is a kernel-perfect digraph.*

Proof. Let C be any odd directed cycle contained in D whose length is at most $2\alpha(D)+1$ and $u \in V(C)$. Since C has two consecutive poles it follows that C has a pole in $C_u^1 \cup \{u\}$ and then C has a $V(C)(C_u^1 \cup \{u\})$-pseudodiagonal. It follows from Theorem 2.1 that D is a kernel-perfect digraph. ■

Theorem 2.3. *Let D be a digraph such that each 3-circuit is symmetrical. If each odd directed cycle contained in D whose length is at most $2\alpha(D)+1$ has two short chords, then D is a kernel-perfect digraph.*

Proof. Let C be an odd directed cycle whose length is at most $2\alpha(D)+1$, and denote $H = D[V(C)]$; the hypothesis implies that every 3-circuit contained in H is symmetrical and each odd directed cycle contained in H has two short chords; so it follows from Theorem 1.4 that C has two consecutive poles. We conclude that each odd directed cycle contained in D whose length is at most $2\alpha(D)+1$ has at least two consecutive poles. Hence Theorem 2.2 implies that D is a kernel-perfect digraph. ■

Theorem 2.4. *Let D be a digraph such that every 3-circuit has two symmetrical arcs. If for every odd directed cycle $C = (u = 0, 1, 2, \ldots, 2n, 0)$ such that $(1, 2, \ldots, 2n, 0)$ is an asymmetrical directed path, (i.e., C has at most one symmetrical arc and in case is the arc $(0, 1)$), $5 \leq \ell(C) \leq 2\alpha(D)+1$ there exists a $V(C)\,(C_u^1 \cup \{u\})$-asymmetrical diagonal in D, then D is a kernel-perfect digraph.*

Proof. First we will prove: (1) $\mathrm{Asym}\,(D)$ is a kernel-perfect digraph. By contradiction assume that $\mathrm{Asym}\,(D)$ is not a kernel-perfect digraph. Then, there exists a subdigraph $H \subseteq^* \mathrm{Asym}\,(D)$ which is a critical kernel-imperfect digraph. Let $u \in V(H)$; it follows from Theorem 1.1 that there exists an odd directed cycle C contained in H, passing by u and having no $V(C)\,(C_u^1 \cup \{u\})$-pseudodiagonals in H, since $H \subseteq^* \mathrm{Asym}\,(D)$ it follows that C has no $V(C)(C_u^1 \cup \{u\})$-asymmetrical diagonal in D. It follows from the hypothesis that $\ell(C) > 2\alpha(D)+1$ and hence $|C_u^1| > \alpha(D)$.

Since C_u^1 is independent in H and $H \subseteq^* \mathrm{Asym}\,D$ it follows that C_u^1 is independent in $\mathrm{Asym}\,(D)$; and the fact $|C_u^1| > \alpha(D)$ implies that there exists at least one symmetrical arc of D with both terminal endpoints in C_u^1.

Denote $C = (u = 0, 1, 2, \ldots, 2n, 0)$, $n > \alpha(D)$; so $C_u^1 = \{1, 3, 5, \ldots, 2n-1\}$, and let $(i, j) \in \mathrm{Sym}\,(D)$ such that $i < j$, $\{i, j\} \subseteq C_u^1$ and

$$A\left(D\left[\{i, i+2, i+4, \ldots, j\}\right]\right) = A\left(D\left[C_u^1 \cap V(i, C, j)\right]\right) = \{(i, j), (j, i)\}\,.$$

Now define $\gamma = (i, C, j) \cup (j, i)$; clearly γ is an odd directed cycle contained in D with exactly one symmetrical arc (j, i). The election of (i, j) implies that γ_j^1 is an indepedent set of D; (Considering

$$\gamma = (j = 0, i = 1, i + 1 = 2, \ldots, i + (j - 2) = j - 1, j = 0)$$

$$\gamma_j^1 = \{i, i + 2, i + 4, \ldots, j - 2\})$$

hence $\ell(\gamma) \leq 2\alpha(D)+1$ and the hypothesis imply that γ has a $V(\gamma)(\gamma_j^1 \cup \{j\})$-asymmetrical diagonal. Notice that $\gamma_j^1 \cup \{j\} \subseteq C_u^1 \cup \{u\}$ and then C has an asymmetrical $V(C)$ $(C_u^1 \cup \{u\})$-diagonal contradicting the selection of C. We conclude that $\mathrm{Asym}\,(D)$ is a kernel-perfect digraph.

Now we will prove by induction on $|A(\mathrm{Sym}\,(D))|$:

(2) There exists an independent set $N \subseteq V(D)$ such that for each $z \in (V(D) - N)$ if there exists an Nz-asymmetrical arc then also exists a zN-asymmetrical arc.

When $|A(\mathrm{Sym}\,(D))| = 0$ we have $D = \mathrm{Asym}\,(D)$ and it follows directly from the fact that $\mathrm{Asym}\,(D)$ is kernel-perfect that $\mathrm{Asym}\,(D)$ has a kernel N. Clearly N satisfies the required properties.

Suppose that (2) holds for any digraph D' which satisfies the hypothesis of Theorem 2.4, and with $|A(\mathrm{Sym}\,(D'))| = n \geq 0$. Let D be any digraph satisfying the hypothesis of Theorem 2.4 with $|A(\mathrm{Sym}\,(D))| = n + 1$, and take any $(u, v) \in A(\mathrm{Sym}\,(D))$. Denote $D' = D - \{(u, v), (v, u)\}$. First we will show that D' satisfies the hypothesis of Theorem 2.4. When $\alpha(D') = \alpha(D)$ it is clear that D' satisfies the hypothesis. Suppose that $\alpha(D') = \alpha(D) + 1$ and let C be any odd directed cycle of length $2\alpha(D') + 1$ with at most one symmetrical arc. Denote $C = (0, 1, \ldots, 2\alpha(D'), 0)$ where $(1, 2, \ldots, 2\alpha(D'), 0)$ is an asymmetrical directed path (we order the vertices of C in such a way that the path $(1, 2, \ldots, 2\alpha(D'), 0)$ be an asymmetrical directed path; that is possible because C has at most one symmetrical arc). Hence $C_0^1 = \{1, 3, 5, \ldots, 2\alpha(D') - 1\}$ and $|C_0^1| = \alpha(D') = \alpha(D) + 1$ and there exists an arc of D with both terminal endpoints in C_0^1; let (i, j) be such an arc. When (i, j) is asymmetrical it is clear that (i, j) is a $V(C)(C_0^1 \cup \{0\})$-asymmetrical diagonal. When (i, j) is symmetrical and say $i < j$ then $\gamma = (i, C, j) \cup (j, i)$ is an odd directed cycle whose length is at most $2\alpha(D)+1$ and with at most one symmetrical arc; it follows from the hypothesis that it has a $V(\gamma)(\gamma_j^1 \cup \{j\})$-asymmetrical diagonal and hence C has a $V(C)(C_0^1 \cup \{0\})$-asymmetrical diagonal. So we have proved that D' also satisfies the hypothesis of Theorem

2.4 and the inductive hypothesis implies that there exists an independent vertex set of D' (N' is independent in D')such that for every $z \in (V(D') - N')$, if there exists a zN-asymmetrical arc in D' then also there exists an Nz-asymmetrical arc in D'.

Now we will prove that (2) holds in D.
Take the independent set N' of D' described above. When N' is independent in D it is clear that $N = N'$ satisfies the required properties. So we can assume that N' is not independent in D and the definition of $D' = D - \{(u,v),(v,u)\}$ implies that $\{u,v\} \subseteq N'$, and in fact $A(D[N']) = \{(u,v),(v,u)\}$.

Denote by:

$E_0 = \{u\}$.

$S_1 = \{x \in V(D) - N' \mid$ there exists an E_0x-asymmetrical arc $\}$.

$E_1 = \{x \in N' - E_0 \mid$ there exists a S_1x-asymmetrical arc $\}$.

$S_2 = \{x \in V(D) - (N' \cup S_1) \mid$ there exists an E_1x-asymmetrical arc and there is no $x(E_0 \cup E_1)$-asymmetrical arc $\}$.

$E_2 = \{x \in N' - (E_0 \cup E_1) \mid$ there exists an S_2x-asymmetrical arc $\}$.

$S_3 = \{x \in V(D) - (N' \cup S_1 \cup S_2) \mid$ there exists an E_2x-asymmetrical arc and there is no $x(E_0 \cup E_1 \cup E_2)$-asymmetrical arc $\}$.

$E_3 = \{x \in N' - (E_0 \cup E_1 \cup E_2) \mid$ there exists an S_3x-asymmetrical arc $\}$.

$S_4 = \{x \in V(D) - (N' \cup S_1 \cup S_2 \cup S_3 \mid$ there exists an E_3x-asymmetrical arc and there is no $x(E_0 \cup E_1 \cup E_2 \cup E_3)$-asymmetrical arc $\}$.

If $E_0, \ldots, E_i$ and $S_0, \ldots, S_i$ are defined, we define:
$S_{i+1} = \{x \in V(D) - (N' \cup \bigcup_{j=1}^{i} S_j) \mid$ there exists an E_ix-asymmetrical arc and there is no $x(\bigcup_{j=0}^{i} E_j)$-asymmetrical arc $\}$.
$E_{i+1} = \{x \in N' - \bigcup_{j=0}^{i} E_j \mid$ there exists an $S_{i+1}x$-asymmetrical arc $\}$.

Clearly $E_i \cap E_j = \emptyset$, $S_i \cap S_j = \emptyset$ for $i \neq j$ and $E_i \cap S_j = \emptyset$ for any i, j. And since $V(D)$ is a finite set it follows that there exists a smallest integer k such that $S_k = \emptyset$.

Considering $E = \bigcup_{i=0}^{k} E_i$ we have that the definition of E_i implies that for any $x \in V(D) - E$ for which there exists an Ex-asymmetrical arc, also there exists an xE-asymmetrical arc. Hence; if E is an independent set of D then $N = E$ satisfies the required properties. So we can assume that E is not an independent set of D'; now the fact that $E \subseteq N'$ and $A(D[N']) = \{(u,v),(v,u)\}$ implies that there exists some $r \in \{1, \ldots, k\}$ with $v \in E_r$.

Denote $v = e_r$; the definition of E_r implies that there exists $s_r \in S_r$ such that $(s_r, e_r) \in \mathrm{Asym}\,(D)$, the definition of S_r implies that there exists $e_{r-1} \in E_{r-1}$ such that $(e_{r-1}, s_r) \in \mathrm{Asym}\,(D)$. Proceeding in this way we obtain an odd directed cycle

$$C = (u = e_0, s_1, e_1, s_2, e_2, \ldots, e_{r-1}, s_r, e_r = v)$$

with $(e_0, s_1, e_1, s_2, e_2, \ldots, s_r, e_r)$ an asymmetrical directed path, the definition of E_i and S_i imply that the only $V(C)(C_u^1 \cup \{u\})$-asymmetrical diagonals in D are of the form (s_i, e_j) where $j > i$. Now we take the minimum asymmetrical directed path

$$T = (e_0 = u, s_{n_1}, e_{m_1}, s_{n_2}, e_{m_2}, s_{n_3}, e_{m_3}, \ldots, e_r = v)$$

such that $e_{m_i} \in E_{m_i}$, $s_{n_i} \in S_{n_i}$ and $n_i < m_i < n_{i+1}$. Clearly T is an even directed path; $\gamma = T \cup (v, u)$ is an odd directed cycle of length greather than 3 whose only symmetrical arc is (v, u) and without $V(\gamma)(\gamma_v^1 \cup \{v\})$-asymmetrical diagonals. Moreover since γ_v^1 is an independent set of D we have $\ell(\gamma) \leq 2\alpha(D) + 1$. This contradicts the hypothesis of Theorem 2.4.

We conclude that N' is an independent set of D' and (2) is proved. Finally we will prove that:

(3) D is a kernel-perfect digraph.
By contradiction, suppose that D is not a KP-digraph, then there exists a CKI-digraph H, $H \subseteq^* D$. Since $H \subseteq^* D$ we have that $\alpha(H) \leq \alpha(D)$ and hence H satisfies the hypothesis of Theorem 2.4; it follows from (2) that there exists a set S independent in H (and hence independent in D) such that for every $z \in (V(H) - S)$ for which there exists an Sz-asymmetrical arc, also there exists a zS-asymmetrical arc. Now consider

$$W = V(H) - S \cup \{x \in V(H) - S \mid \text{there exists an } xS\text{-arc}\}\,.$$

If $W = \emptyset$ then S is a kernel of H which is a contradiction.

If $W \neq \emptyset$, taking a kernel X of $H[W]$ it is easy to see that $S \cup X$ is a kernel of H contradicting that H has no kernel.

We conclude that D is a kernel-perfect digraph. ■

Theorem 2.5. *Let D be a digraph. If every odd directed cycle C whose length is at most $2\alpha(D) + 1$ has two symmetrical arcs then D is a kernel-perfect digraph.*

Theorem 2.5 is a direct consequence of Theorem 2.4.

Theorem 2.6. *Let D be a digraph such that $Asym\,(D)$ is a kernel-perfect digraph and each 3-circuit has two symmetrical arcs. If each odd directed cycle C, $5 \leq \ell(C) \leq 2\alpha(D)+1$ whose only symmetrical arc is (u,v) has a $V(C)(C_u^1 \cup \{u\})$-diagonal in $Asym\,(D)$, then D is a kernel-perfect digraph.*

I omit the proof of Theorem 2.6 which is completly analogous of those of Theorem 2.4.

Theorem 2.7. *Let D be a digraph such that $Asym\,(D)$ is a kernel-perfect digraph and every 3-circuit has two symmetrical arcs. If no odd directed cycle C with $5 \leq \ell(C) \leq 2\alpha(D)+1$ has exactly one symmetrical arc then D is a kernel-perfect digraph.*

It is a direct consequence of Theorem 2.6.

Theorem 2.8. *Let D be a digraph such that every 3-circuit is symmetrical. If every odd directed cycle C, $5 \leq \ell(C) \leq 2\alpha(D)+1$ with at most one symmetrical arc has two short chords, then D is a kernel-perfect digraph.*

Proof. Let $C = (u = 0, 1, 2, \ldots, 2n, 0)$ be an odd directed cycle where $(1, 2, \ldots, 2n, 0)$ is an even asymmetrical directed path i.e., C has at most one symmetrical arc and in case it is the arc $(0,1)$; $5 \leq \ell(C) \leq 2\alpha(D)+1$. Denote by H the digraph defined as follows: $V(H) = V(C)$

$$A(H) = \begin{cases} A(\mathrm{Asym}\,(D[V(C)])) & \text{if } (0,1) \in \mathrm{Asym}\,(D)\,, \\ A(\mathrm{Asym}\,(D[V(C)])) \cup \{(0,1)\} & \text{if } (0,1) \in \mathrm{Sym}(D)\,. \end{cases}$$

We claim that:

(1) H is an asymmetrical digraph,

(2) Every odd directed cycle contained in H has two short chords in H.

Let γ be an odd directed cycle contained in H.

If $(0,1) \notin A(\gamma)$ then $\gamma \subseteq \mathrm{Asym}\,(D)$. Let $\gamma = (i_0, i_1, \ldots, i_{2k}, i_0)$ and suppose that (i_j, i_{j+2}) is a short chord of γ contained in D when $(i_j, i_{j+2}) \in \mathrm{Asym}\,(D)$ it follows that $(i_j, i_{j+2}) \in A(H)$. If $(i_j, i_{j+2}) \in \mathrm{Sym}\,(D)$ then $(i_j, i_{j+1}, i_{j+2}, i_j)$ is a 3-circuit contained in D with exactly one symmetrical arc contradicting the hypothesis.

We conclude that γ has two short chords in H.

Now, assume that $(0,1) \in A(\gamma)$ and denote $\gamma = (i_0 = 0, i_1 = 1, i_2, \ldots, i_{2k}, i_0 = 0)$ the definition of H implies that $(i_1, i_2, i_3, \ldots, i_{2k}, i_0)$ is an even asymmetrical directed path. Let (i_j, i_{j+2}) a short chord of γ contained in D (notation $\bmod 2k+1$). If (i_j, i_{j+2}) is asymmetrical we have that $(i_j, i_{j+2}) \in A(H)$. If (i_j, i_{j+2}) is symmetrical then $\vec{C}_3 = (i_j, i_{j+1}, i_{j+2}, i_j)$ is a 3-circuit with at most two symmetrical arcs (if $(0,1) \in \{(i_j, i_{j+1}), (i_{j+1}, i_{j+2})\}$ then $\vec{C}_3$ has two symmetrical arcs; in other case $\vec{C}_3$ has only one symmetrical arc) contradicting the hypothesis. We conclude that in any case γ has two short chords in H.

(3) H has no 3-circuits.

It follows directly from the definition of H and from the hypothesis (every 3-circuit contained in D is symmetrical).

It follows from (2) and (3) by applying Theorem 1.4 that every odd directed cycle contained in H has two pseudodiagonals in H whose terminal endpoints are consecutive. In particular C has two diagonals in H whose terminal endpoints are consecutive in C; hence C has a $V(C)(C_u^1 \cup \{u\})$-asymmetrical diagonal.

We conclude that every odd directed cycle $C = (u = 0, 1, 2, \ldots, 2n, 0)$ where $(1, 2, 3, \ldots, 2n, 0)$ is an even asymmetrical directed path (i.e., C has at most one symmetrical arc in case is the arc $(0,1)$) with $5 \leq \ell(C) \leq 2\alpha(D)+1$ has a $V(C)(C_u^1 \cup \{u\})$-asymmetrical diagonal. And it follows from Theorem 2.4 that D is a kernel-perfect digraph. ■

As a direct consequence of Theorem 2.8 we obtain:

Theorem 2.9. *Let D be a digraph such that every 3-circuit is symmetrical. If every odd directed cycle C with $5 \leq \ell(C) \leq 2\alpha(D)+1$ satisfies at least one of the two following properties:*

(1) *C has two symmetrical arcs,*

(2) *C has two short chords. then D is a kernel-perfect digraph.*

Corollary 2.1. *Let D be a digraph such that every 3-circuit is symmetrical. If every odd directed cycle C satisfies at least one of the two following properties:*

(1) *C has two symmetrical arcs,*

(2) *C has two short chords, then D is a kernel-perfect digraph.*

Theorem 2.10. *Let D be a digraph, and suppose that every odd directed cycle C with $\ell(C) \leq 2\alpha(D) + 1$ satisfies the following properties:*

(1) *$D[V(C)]$ is a fraternally oriented digraph and*

(2) *C has a least two pseudodiagonals, then D is a kernel-perfect diagrph.*

Proof. Let C be any odd directed cycle with $\ell(C) \leq 2\alpha(D)+1$ and denote by $H = D[V(C)]$. Clearly H satisfies the hypothesis of Theorem 1.7 and hence every odd directed cycle contained in H has two consecutive poles, in particular C has two consecutive poles.

We conclude that every odd directed cycle C with $\ell(C) \leq 2\alpha(D) + 1$ has two consecutive poles, and then Theorem 2.2 implies that D is a kernel-perfect digraph. ■

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