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LOWER BOUND ON THE DOMINATION NUMBER OF A TREE

Magdalena Lemańska

Department of Mathematics
Gdańsk University of Technology
Narutowicza 11/12, 80–952 Gdańsk, Poland

e-mail: magda@mif.pg.gda.pl

Abstract

We prove that the domination number $\gamma(T)$ of a tree T on $n \geq 3$ vertices and with n_1 endvertices satisfies inequality $\gamma(T) \geq \frac{n+2-n_1}{3}$ and we characterize the extremal graphs.

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1. Introduction

In a simple undirected graph $G = (V, E)$ a subset D of V is *dominating* if every vertex of $V - D$ has at least one neighbour in D and D is *independent* if no two vertices of D are adjacent. A set is *independent dominating* if it is independent and dominating. Let $\gamma(G)$ be the minimum cardinality of a dominating set and let $i(G)$ denotes the minimum cardinality of an independent dominating set of G. The *neighbourhood* $N_G(v)$ of a vertex v is the set of all vertices adjacent to v. For a set $X \subseteq V$, the *neighbourhood* $N_G(X)$ is defined to be $\bigcup_{v \in X} N_G(v)$. The degree of a vertex v is $d_G(v) = |N_G(v)|$. For unexplained terms and symbols see [2].

Here we consider trees on at least three vertices. If T is a tree, let $n = n(T)$ be the order of T and let $n_1 = n_1(T)$ denote the number of endvertices of T. The set of endvertices of T is denoted by $\Omega(T)$.

Let D be a dominating set of a tree T. We say that D has the property $\mathcal{F}$ if D contains no endvertex of T. It is obvious that in every tree on at least 3 vertices exists a minimum dominating set having property $\mathcal{F}$.

Favaron [1] has proved that $i(T) \leq \frac{n+n_1}{3}$ for a tree T. The number $\frac{n+n_1}{3}$ is also an upper bound on the domination number, because $\gamma(T) \leq i(T)$. In this paper we give a lower bound on the domination number of a tree in terms of n and n_1. Precisely, we prove that $\gamma(T) \geq \frac{n+2-n_1}{3}$ for a tree T on $n \geq 3$ vertices and we characterise all trees T for which $\gamma(T) = \frac{n+2-n_1}{3}$.

2. Results

Theorem 1. *If T is a tree of order at least 3, then $n_1(T) \geq n(T)+2-3\gamma(T)$.*

Proof. We use induction on n, the order of a tree. The result is trivial for a tree of order 3. Let T be a tree of order $n > 3$ and assume that $n_1(T') \geq n(T') + 2 - 3\gamma(T')$ for each tree T' with $3 < n(T') \leq n-1$. Let D be a minimum dominating set of T having property $\mathcal{F}$, let $P = (v_0, v_1, \ldots, v_l)$ be a longest path in T and let $T' = T - \{v_0\}$ be the subtree of T. Without loss of generality we may assume that P is chosen in such a way that $d_T(v_1)$ is as large as possible. We consider two cases: $d_T(v_1) > 2$ or $d_T(v_1) = 2$.

Case 1. $d_T(v_1) > 2$. In T' we have $n_1(T') \geq n(T') + 2 - 3\gamma(T')$ (by induction), and therefore $n_1(T) \geq n(T) + 2 - 3\gamma(T)$ as $n_1(T') = n_1(T) - 1$, $\gamma(T') = \gamma(T)$ and $n(T') = n(T) - 1$.

Case 2. If $d_T(v_1) = 2$, we consider two subcases: $\gamma(T') < \gamma(T)$ or $\gamma(T') = \gamma(T)$.

Subcase 2.1. If $\gamma(T') < \gamma(T)$, then it is easy to observe, that $\gamma(T') = \gamma(T) - 1$. By induction, $n_1(T') \geq n(T') + 2 - 3\gamma(T')$ and consequently $n_1(T) \geq n(T) + 2 - 3\gamma(T)$ as $n_1(T') = n_1(T)$, $n(T') = n(T) - 1$.

Subcase 2.2. If $\gamma(T') = \gamma(T)$, then $v_2 \notin N_T(\Omega(T))$ (otherwise $D - \{v_1\}$ would be a dominating set of T' and $\gamma(T') = \gamma(T - v_0) < \gamma(T)$) and therefore $l \geq 4$. By T_1 and T_2 we denote the subtrees of $T - v_2v_3$ to which belong vertices v_3 and v_2, respectively. If $n(T_1) = 2$, then certainly $n_1(T_1) \geq n_1(T_1) - 2 + 3\gamma(T_1)$. Thus assume that $n(T_1) \geq 3$.

Let Ω_2 denotes the set $\Omega(T_2) \cap \Omega(T)$ and let D_2 be a minimum dominating set of T_2 which does not contain v_2. Since $d_T(v_1) = 2$, from the choice of

P it follows that all neighbours of v_2 in T_2 are of degree two and this implies $|\Omega_2| = |D_2|$.

It is no problem to observe, that $\gamma(T) = \gamma(T_1) + \gamma(T_2) = \gamma(T_1) + |D_2|$ and $n(T) = n(T_1)+|\Omega_2|+|D_2|+1$. If v_3 is an endvertex of T_1 we have $n_1(T) = n_1(T_1)+|\Omega_2|-1$, otherwise $n_1(T) = n_1(T_1)+|\Omega_2| \geq n_1(T_1)+|\Omega_2|-1$ as well. Now, since $n(T_1) \geq 3$, we have by induction $n_1(T_1) \geq n(T_1) + 2 - 3\gamma(T_1)$. In both cases, for $n(T_1) = 2$ and for $n(T_1) \geq 3$ we get $n(T_1) + 2 - 3\gamma(T_1) \leq n_1(T_1) \leq n_1(T) - |\Omega_2| + 1$. Thus $n(T) - |\Omega_2| - |D_2| - 1 + 2 - 3(\gamma(T) - |D_2|) \leq n_1(T) - |\Omega_2| + 1$ and $n_1(T) \geq n(T) + 2|D_2| - 3\gamma(T) \geq n(T) + 2 - 3\gamma(T)$. ■

By $\mathcal{R}$ we denote the family of all trees in which the distance between any two distinct endvertices is congruent to 2 modulo 3, i.e., a tree $T \in \mathcal{R}$ if $d(x, y) \equiv 2 \pmod 3$ for distinct $x, y \in \Omega(T)$. The next lemma describes main properties of trees belonging to $\mathcal{R}$.

Lemma 2. *Let T be a tree belonging to $\mathcal{R}$ and let D be a minimum dominating set having property $\mathcal{F}$ in T. Then $d(u, v) \equiv 0 \pmod 3$ for every two vertices $u, v \in D$. In addition, $n_1(T) = n(T) + 2 - 3\gamma(T)$.*

Proof. We use induction on n, the order of a tree. The result is obvious for stars $K_{1,n-1}, n \geq 3$. Thus, let $T \in \mathcal{R}$ be a tree of order $n > 3$ which is not a star, and let D be a minimum dominating set with property $\mathcal{F}$ in T. Let $P = (v_0, v_1, \ldots, v_l)$ be a longest path in T. Since T is not a star and $T \in \mathcal{R}$ we certainly have $l \geq 5$ and $l \equiv 2 \pmod 3$. We consider two cases.

Case 1. At least one of the vertices v_1, v_{l-1} is of degree at least three, say $d_T(v_1) \geq 3$. Then $T' = T - v_0$ belongs to $\mathcal{R}$, the set D is a minimum dominating set with property $\mathcal{F}$ in T' and by induction $d(u, v) \equiv 0 \pmod 3$ for every two vertices $u, v \in D$. Consequently, D has the same property in T. By induction, $n_1(T') = n(T') + 2 - 3\gamma(T')$ and therefore $n_1(T) = n(T)+2-3\gamma(T)$ as $n_1(T') = n_1(T)-1$, $n(T') = n(T)-1$ and $\gamma(T') = \gamma(T)$.

Case 2. $d_T(v_1) = d_T(v_{l-1}) = 2$. Since D is a minimum dominating set having property $\mathcal{F}$ in T, vertices v_1 and v_{l-1} belong to D. Because $T \in \mathcal{R}$, $d_T(v_2) = d_T(v_3) = 2$ and it is possible to choose D containing v_4 and not v_3. In this case, the subgraph $T' = T - v_0 - v_1 - v_2$ is a tree belonging to $\mathcal{R}$ and v_3 is an end vertex of T'. The set $D' = D - \{v_1\}$ is a minimum dominating set with property $\mathcal{F}$ in T'. Since $v_3 \notin D'$, it follows that $v_4 \in D'$. By induction, $d(u, v) \equiv 0 \pmod 3$ if $u, v \in D'$. From this property and from the fact that

$T^{'} \in \mathcal{R}$ it follows that all vertices belonging to $V(T^{'}) - (D^{'} \cup \Omega(T^{'}))$ are of degree two in $T^{'}$. Since $d(u,v) \equiv 0 \,(\mathrm{mod}\, 3)$ for every two vertices $u, v \in D^{'}$, $d(v_1, v) = d(v_1, v_4) + d(v_4, v)$ is a multiple of 3 for every $v \in D$ and therefore the distance between any two vertices from D is a multiplicity of 3. This easily implies that each vertex belonging to $V(T) - (D \cup \Omega(T))$ is of degree two and this forces $|V(T) - (D \cup \Omega(T))| = 2(\gamma(T) - 1)$. Thus $n(T) = |V(T)| = |\Omega(T) \cup D \cup (V(T) - (D \cup \Omega(T)))| = n_1(T) + \gamma(T) + 2(\gamma(T) - 1)$ and so $n_1(T) = n(T) + 2 - 3\gamma(T)$. ■

Now we characterise trees T for which the following equality $n_1(T) = n(T) + 2 - 3\gamma(T)$ holds.

Theorem 3. *If T is a tree, then $n_1(T) = n(T) + 2 - 3\gamma(T)$ if and only if T belongs to $\mathcal{R}$.*

Proof. If the tree T belongs to $\mathcal{R}$ then $n_1(T) = n(T) + 2 - 3\gamma(T)$ by Lemma 1. Now assume that T does not belong to $\mathcal{R}$. Then T has at least four vertices and it suffices to show that $n_1(T) > n(T) + 2 - 3\gamma(T)$.

If T is of order four, then $T = P_4$ and certainly $n_1(P_4) > n(P_4) + 2 - 3\gamma(P_4)$. Assume that T has at least five vertices and let $P = (v_0, v_1, v_2, \ldots, v_l)$ be a longest path in T and let D be a minimum dominating set satisfying property $\mathcal{F}$ in T. We consider three cases.

Case 1. If $d_T(v_1) > 2$, then the tree $T^{'} = T - v_0$ does not belong to $\mathcal{R}$ and $n_1(T^{'}) > n(T^{'}) + 2 - 3\gamma(T^{'})$ (by induction), which implies $n_1(T) > n(T) + 2 - 3\gamma(T)$ as $n_1(T^{'}) = n_1(T) - 1, n(T^{'}) = n(T) - 1, \gamma(T^{'}) = \gamma(T)$.

Case 2. If $d_T(v_1) = 2$ and $d_T(v_2) \geq 3$ then we consider $T^{'} = T - v_0 - v_1$. Notice, that $v_1 \in D$, since D safisfies property $\mathcal{F}$, $D^{'} = D - \{v_1\}$ is a dominating set of $T^{'}$ and certainly it is the smallest. Thus $\gamma(T^{'}) = \gamma(T) - 1$. For a tree $T^{'}$ we have also $n_1(T^{'}) = n_1(T) - 1$ and $n(T^{'}) = n(T) - 2$. Then $n_1(T) - n(T) + 3\gamma(T) = n_1(T^{'}) + 1 - n(T^{'}) - 2 + 3(\gamma(T^{'}) + 1) = n_1(T^{'}) - n(T^{'}) + 3\gamma(T^{'}) + 2 \geq 2 + 2 > 2$ by Theorem 1 applied to $T^{'}$.

Case 3. If $d_T(v_1) = 2$ and $d_T(v_2) = 2$, then we consider $T^{'} = ((T - v_0) - v_1) - v_2$. Like in *Case* 2, $v_1 \in D$, since D safisfies property $\mathcal{F}$, and $D^{'} = D - \{v_1\}$ is a minimum dominating set of $T^{'}$. Thus $\gamma(T^{'}) = \gamma(T) - 1$.
If $d_T(v_3) > 2$, then $n_1(T^{'}) = n_1(T) - 1, n(T^{'}) = n(T) - 3$ and $n_1(T) - n(T) + 3\gamma(T) = n_1(T^{'}) + 1 - n(T^{'}) - 3 + 3\gamma(T^{'}) + 3 = n_1(T^{'}) - n(T^{'}) + 3\gamma(T^{'}) + 1 \geq 2 + 1 > 2$ by Theorem 1 applied to $T^{'}$. If $d_T(v_3) = 2$ then notice, that $T^{'} \notin \mathcal{R}$

(since $T \notin \mathcal{R}$). Hence $n_1(T^{'}) > n(T^{'}) + 2 - 3\gamma(T^{'})$ by induction and finally we have $n_1(T) > n(T) + 2 - 3\gamma(T)$ as $n_1(T^{'}) = n_1(T), n(T^{'}) = n(T) - 3$. ■

3. Concluding Remarks

From [1] and above results it follows that $\frac{n(T)+2-n_1(T)}{3} \leq \gamma(T) \leq \frac{n(T)+n_1(T)}{3}$ for every tree T on at least 3 vertices. The example of caterpillar given in Figure 1 proves that the difference between $\gamma(T)$ and $\frac{n(T)+2-n_1(T)}{3}$ can be arbitrarily large. It is no problem to observe that $\gamma(T_l) - \frac{n(T_l)+2-n_1(T_l)}{3} = \frac{2l-2}{3}$ for any integer $l \geq 3$.

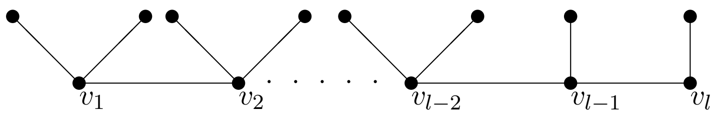

Figure 1. Caterpillar

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