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GENERALISED IRREDUNDANCE IN GRAPHS: NORDHAUS-GADDUM BOUNDS

Ernest J. Cockayne and Stephen Finbow

University of Victoria
B.C., Canada V8W 3P4

e-mail: cockayne@math.uvic.ca

Abstract

For each vertex s of the vertex subset S of a simple graph G, we define Boolean variables $p = p(s, S)$, $q = q(s, S)$ and $r = r(s, S)$ which measure existence of three kinds of S-private neighbours (S-*pns*) of s. A 3-variable Boolean function $f = f(p, q, r)$ may be considered as a compound existence property of S-*pns*. The subset S is called an f-set of G if $f = 1$ for all $s \in S$ and the class of f-sets of G is denoted by $\Omega_f(G)$. Only 64 Boolean functions f can produce different classes $\Omega_f(G)$, special cases of which include the independent sets, irredundant sets, open irredundant sets and CO-irredundant sets of G.

Let $Q_f(G)$ be the maximum cardinality of an f-set of G. For each of the 64 functions f, we establish sharp upper bounds for the sum $Q_f(G) + Q_f(\overline{G})$ and the product $Q_f(G)Q_f(\overline{G})$ in terms of n, the order of G.

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1. Introduction

Generalised irredundant sets were defined in [2]. We repeat the definition here for completeness but omit motivation which may be found in [2]. The open (closed) neighbourhood of the vertex subset S of a simple graph $G = (V, E)$ is denoted by $N(S)$ ($N[S]$) and as usual, for $s \in V$, $N(\{s\})$ and $N[\{s\}]$ are abbreviated to $N(s)$ and $N[s]$.

The basic ingredients of the definition of generalised irredundant sets are three properties which make a vertex s (informally) important in a vertex subset S of a graph G. It will also help the intuition to replace the word "important" by "essential" or "non-redundant." Each property depends on the existence of one of the three types of *S-private neighbour* (*S-pn*) t for s, which we now formally define.

For $s \in S$, vertex t is an:

(i) *S-self private neighbour* (*S-spn*) of s if $t = s$ and s is an isolated vertex of $G[S]$,

(ii) *S-internal private neighbour* (*S-ipn*) of s if $t \in S - \{s\}$ and $N(t) \cap S = \{s\}$,

(iii) *S-external private neighbour* (*S-epn*) of s if $t \in V - S$ and $N(t) \cap S = \{s\}$.

Observe that each such t is an element of $N[s] - N(S - \{s\})$ and that no $s \in S$ may have *S-pns* of both type (i) and type (ii).

For $s \in S$ let $p(s, S)$, $q(s, S)$, $r(s, S)$ be Boolean Variables which take the value 1 if and only if s has an *S-pn* of type (i), (ii), (iii) respectively. Whenever possible we use the abbreviations p, q, r for these variables. Further let $S(s) = (p(s, S), q(s, S), r(s, S))$. Observe that for all s and S, $p(s, S) \cap q(s, S) = 0$, i.e., the three Boolean variables are not independent and $S(s)$ is never $(1, 1, 0)$ or $(1, 1, 1)$.

Example 1. Consider the vertex subset $S = \{a, b, c, d\}$ of the graph G depicted in Figure 1. The *S-pns* of vertices of S are tabulated in Table 1 and we observe

$$S(a) = (0, 1, 1), \quad S(b) = (0, 0, 0), \quad S(c) = (0, 0, 1), \quad S(d) = (1, 0, 1).$$

We are now ready to define generalised irredundant sets. Let f be a Boolean function of the three variables $p(s, S)$, $q(s, S)$, $r(s, S)$.

Definition. The vertex subset S of G is an *f-set* of G if for each $s \in S$

$$f(S(s)) = f(p(s, S), q(s, S), r(s, S)) = 1.$$

The function f may be viewed as a compound existence/non-existence property of the three types of *S-pn*. The class of all f-sets of G will be denoted by $\Omega_f(G)$ (abbreviated to Ω_f whenever possible).

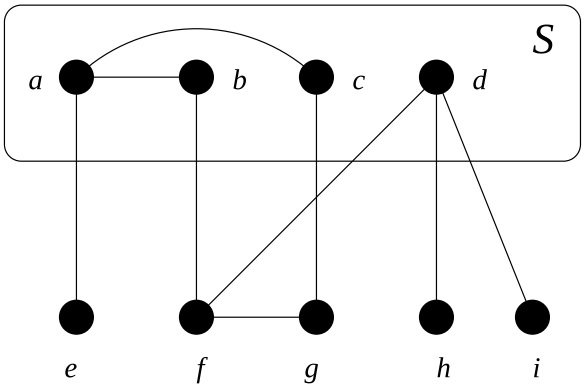

Figure 1. Graph G for Example 1

	type(i)	type(ii)	type(iii)
a		b, c	e
b			
c			g
d	d		h, i

Table 1. S-*pns* of vertices of S for graph G for Example 1.

The rows of the truth table of f will be labelled $0, \ldots, 7$, so that the entry in row i is $f(p, q, r)$, where pqr is the binary representation of the integer i (e.g., $f(1, 0, 1)$ is the fifth entry in the table). Recall that for each $s \in S$, $S(s)$ is never equal to $(1, 1, 0)$ or $(1, 1, 1)$. We deduce:

(a) If the only 1's in the truth table for f occur in rows 6 or 7, then $\Omega_f = \emptyset$.

(b) If f' is formed from f by replacing the values in rows 6 and 7 by 0's, then $\Omega_{f'} = \Omega_f$.

Thus we will only be concerned with the set F of 64 functions with 0's in rows 6 and 7. Two of these are in fact rather uninteresting since $f = 0$ gives $\Omega_f = \emptyset$ and the function g with 1's in all rows $0, 1, \ldots, 5$ has Ω_g equal to the class of all subsets of V.

The functions of F will be numbered (as in [4]) as follows. Let $a_0a_1a_2$ $a_3a_4a_5$ be the binary representation of i. Then f_i is defined to be the function with entries $a_0a_1a_2a_3a_4a_5$ in rows 0 through 5, respectively. Note that $F = \{f_0, \dots, f_{63}\}$.

We now list four special classes of f-sets. Additional examples may be found in [2, 4].

Example 2.

(i) The function p.
The truth table column is $0,0,0,0,1,1,0,0$. Since 3 (decimal) = 00011 (binary), $p = f_3$. The subset S of $V(G)$ is an f-set of G if and only if each $s \in S$ is isolated in $G[S]$, i.e., S is independent in G. Thus $\Omega_p = \Omega_{f_3}$ is precisely the class of independent sets of G.

(ii) The function $p \vee r$.
The truth table column is $0,1,0,1,1,1,0,0$. Since 010111 (binary) = 23 (decimal), $p \vee r = f_{23}$. Then $S \subseteq V(G)$ is an f_{23}-set of G if and only if each $s \in S$ is isolated in $G[S]$ or has an S-*epn*, i.e., S is an irredundant set of G (originally defined in [7]). Hence $\Omega_{f_{23}}$ is precisely the class of irredundant sets of G. See [18] for a bibliography of over 100 papers concerning irredundance.

(iii) The function $p \vee q \vee r$,
The truth table column is $0,1,1,1,1,1,0,0$. So that $p \vee q \vee r = f_{31}$. Each vertex of an f_{31}-set S has at least one S-*pn*, i.e., Ω_{31} is the class of CO-irredundant sets which are defined in [14] and studied in [8, 9, 12, 21].

(iv) The function r.
The truth table column is $0,1,0,1,0,1,0,0$. Since (010101) binary = 21 (decimal), $r = f_{21}$. The subset S is an f_{21}-set if each $s \in S$ has an S-*epn*. Such sets (called *open irredundant*) were introduced in [14] and applied to broadcast networks. They are also known as *OC-irredundant sets* and have been studied in [1, 2, 3, 5, 13, 15, 16, 17, 19].

In view of Example 2, we regard each Ω_f as a class of generalised irredundant sets.

In [2, 4] the hereditary classes among the Ω_f's were determined and Ramsey properties of the classes were investigated.

Let $Q_i(G)$ be the maximum cardinality of an f_i-set of G. Wherever possible we abbreviate $Q_i(G)$, $Q_i(\overline{G})$ to Q_i, $\overline{Q}_i$ respectively. In this paper we determine Nordhaus-Gaddum type bounds (see [20]) for these parameters.

More specifically for each $i = 1, \ldots, 63$ we find upper bounds for

$$\max_G \left(Q_i + \overline{Q}_i\right) \qquad \text{and} \qquad \max_G \left(Q_i \overline{Q}_i\right)$$

where the maximum is taken over all n vertex graphs G. The bounds are attained for an infinite number of values of n.

2. The Bounds

The Nordhaus-Gaddum bounds for the 63 non-zero values of i, will be given in Theorems 3, 5 and 11. We first state an obvious Lemma.

Lemma 1. *If $f_i \Longrightarrow f_j$, then for any graph G, $Q_i \leq Q_j$.*

Theorem 1. *If $i \geq 32$ and $n \geq 5$, then*

$$\max_G \left(Q_i + \overline{Q}_i\right) = 2n \qquad \textit{and} \qquad \max_G \left(Q_i \overline{Q}_i\right) = n^2.$$

Proof. If $i \geq 32$, then $f_{32} \Longrightarrow f_i$, so that for all G (using Lemma 1) $Q_{32} \leq Q_i \leq n$ and $\overline{Q}_{32} \leq \overline{Q}_i \leq n$. Hence

$$Q_{32} + \overline{Q}_{32} \leq Q_i + \overline{Q}_i \leq 2n$$

and

$$Q_{32}\overline{Q}_{32} \leq Q_i\overline{Q}_i \leq n^2.$$

However for $n \geq 5$, $Q_{32}(C_n) = Q_{32}(\overline{C}_n) = n$ and the result follows. ■

We next use the Nordhaus-Gaddum bounds for standard irredundant (i.e., f_{23}-) sets obtained by Cockayne and Mynhardt [10] to deduce the same bounds for other values of i.

Theorem 2 ([10])**.** *If $n \geq 3$, then for any graph G*

$$Q_{23} + \overline{Q}_{23} \leq n + 1 \qquad \textit{and} \qquad Q_{23}\overline{Q}_{23} \leq \left\lceil \frac{n^2 + 2n}{4} \right\rceil.$$

Theorem 3. *If $n \geq 5$ and $i \in \{2, 3, 6, 7, 18, 19, 22, 23\}$, then*

$$\max_G \left(Q_i + \overline{Q}_i\right) = n + 1 \qquad \textit{and} \qquad \max_G \left(Q_i\overline{Q}_i\right) = \left\lceil \frac{n^2 + 2n}{4} \right\rceil.$$

Proof. If $i \in \{2,3,6,7,18,19,22,23\}$, then $f_2 \implies f_i \implies f_{23}$ hence by Lemma 1 and Theorem 3

$$Q_2 + \overline{Q}_2 \le Q_i + \overline{Q}_i \le Q_{23} + \overline{Q}_{23} \le n+1$$

and

$$Q_2\overline{Q}_2 \le Q_i\overline{Q}_i \le Q_{23}\overline{Q}_{23} \le \left\lceil \frac{n^2+2n}{4} \right\rceil.$$

Consider the graph H which consists of a set X of $\lfloor \frac{n+1}{2} \rfloor$ vertices, a set Y of $\lceil \frac{n+1}{2} \rceil$ vertices (where $X \cap Y = \{x\}$), the edges to make $H[Y]$ complete and a matching joining the vertices of $X - \{x\}$ to $Y - \{x\}$. In the case where n is even, an edge is added between the vertex of Y which was not previously matched and any vertex of $X - \{x\}$.

Since each vertex of an f_2-set S is a S-*spn* and has no S-*epn*, it is easily seen that X, Y are f_2-sets of H, $\overline{H}$ respectively and so $Q_2(H) \ge |X|$ and $Q_2(\overline{H}) \ge |Y|$. Hence for H all of the above inequalities are equalities and the result follows. ■

We now proceed in a similar manner using the bounds for CO-irredundant (i.e., f_{31}-) sets established by Cockayne, McCrea and Mynhardt [9].

Theorem 4 ([9])**.** *For any graph G,*

$$Q_{31} + \overline{Q}_{31} \le n+2 \qquad \text{and} \qquad Q_{31}\overline{Q}_{31} \le \left\lfloor \frac{(n+2)^2}{4} \right\rfloor.$$

Theorem 5. *If $8 \le i \le 15$ or $24 \le i \le 31$, then*

$$\max_G \left(Q_i + \overline{Q}_i\right) \le n+2, \qquad \max_G \left(Q_i\overline{Q}_i\right) \le \left\lfloor \frac{(n+2)^2}{4} \right\rfloor$$

and these bounds are attained for $n \equiv 2 \pmod 4$, $n \ge 6$.

Proof. For any i satisfying $8 \le i \le 15$ or $24 \le i \le 31$, $f_8 \implies f_i \implies f_{31}$. Thus, by Lemma 1, for any G,

$$Q_8\overline{Q}_8 \le Q_i\overline{Q}_i \le Q_{31}\overline{Q}_{31} \le \left\lfloor \frac{(n+2)^2}{4} \right\rfloor$$

and

$$Q_8 + \overline{Q}_8 \le Q_i + \overline{Q}_i \le Q_{31} + \overline{Q}_{31} \le n+2.$$

Thus the bounds of the theorem are established. Now let $n \equiv 2 \pmod 4$ and $n \geq 6$. Let the graph H consist of vertex sets X and Y where $|X| = |Y| = (n+2)/2$ and $|X \cap Y| = 2$. Add edges so that $H[X]$ and $\overline{H}[Y]$ are both isomorphic to $(\frac{n+2}{4})K_2$ and add a matching from $X - Y$ to $Y - X$.

Since a subset S is an f_8-set if each vertex has an S-*ipn* and no S-*epn*, it is easily seen that X, Y are f_8-sets of H, $\overline{H}$ respectively. Therefore H attains the bounds. ∎

In order to find the bounds for the remaining values of i, it will be necessary to improve the following result of Cockayne [3] concerning open irredundant (i.e., f_{21}-) sets. A set S is an f_{21}-set if each $s \in S$ has an S-*epn*.

Theorem 6 ([3])**.** *For any graph G with $n \geq 16$,*

$$Q_{21} + \overline{Q}_{21} \leq \left\lfloor \frac{3n}{4} \right\rfloor.$$

Further if $n \geq 17$, then

$$Q_{21}\overline{Q}_{21} < \frac{9n^2}{64}.$$

We show that for larger n, the second bound of Theorem 6 can be improved to $n^2/8$. This will be accomplished by more detailed analysis of the various cases used in the proof of Theorem 6 given in [3]. Some of the details of our proof may be found in [3] but must be repeated here for completeness.

Let $X(Y)$ be open irredundant sets of $G(\overline{G})$, $|X| = x$ and $|Y| = y$. Each $u \in X(v \in Y)$ has an at least one X-*epn* in G (Y-*epn* in $\overline{G}$). Let $u_r(v_b)$ be any X-*epn* of u in G (Y-*epn* of v in $\overline{G}$). The edges of G (resp. $\overline{G}$) will be coloured red (blue). Occasionally u_r (v_b) will be called a *red epn* of u (*blue epn* of v). Let $X' = \{u_r | u \in X\}$. Then each edge of $\{uu_r | u \in X\}$ is red while all other edges joining X to X' are blue. Hence the set $\{uu_r | u \in X\}$ induces a matching in G. Similarly, it can be seen that, the set $\{vv_b | v \in Y\}$ induces a matching in $\overline{G}$. Note that the set X' is also an open irredundant set of G and u is an X'-*epn* of u_r in G. Let $Z = V - (X \cup X')$.

The principal result will follow immediately from three propositions which are broken down into cases depending on the distribution of vertices of Y and blue epns among the three sets X, X', Z.

The open irredundance property implies that both x and y are at most $n/2$. From this we deduce that $xy \leq \frac{n^2}{8}$ if x (or y) $\leq \frac{n}{4}$. Hence it is sufficient to establish the propositions under the assumption $x, y > \frac{n}{4}$ and we use this hypothesis in the proofs without further emphasis. We also repeatedly use the following obvious fact.

Lemma 2. *Let A be an open irredundant set in a graph F and $B \subseteq V(F)$. If each $u \in A \cap B$ has A-epn in B, then $|A \cap B| \leq |B|/2$.*

Proposition 7. *If $n \geq 32$ and $|Y \cap X| \geq 3$, then $xy \leq n^2/8$.*

Proof. Since $|Y \cap X| \geq 3$, for each $u \in Y \cap X$, $u_b \notin X'$. Hence $u_b \in X \cup Z$. Define

$$X_1 = \{u \in Y \cap X | u_b \in X\},$$

$$X_2 = \{u \in Y \cap X | u_b \in Z\},$$

$$X_3 = X - (X_1 \cup X_2)$$

and for $i = 1, 2, 3$, let $|X_i| = x_i$.

For $w \in Y \cap Z$, $w_b \notin X_1 \cup X_2 \cup X'$, hence $w_b \in X_3 \cup Z$.

Case 1. $Y \cap X' = \phi$.

Let $t = |\{w \in Y \cap Z | w_b \in X_3\}|$. Then by Lemma 2

$$|\{w \in Y \cap Z | w_b \in Z\} \leq (n - 2x - x_2 - t)/2. \tag{1}$$

We will now give more detailed justification for (1). Similar explanations will be omitted in future cases of the propositions. Define

$$B = Z - (\{w \in Y \cap Z | w_b \in X_3\} \cup \{w_b \in Z | w \in X_2\})$$

(disjoint Union).

Note that $|B| = (n - 2x - x_2 - t)$ and

$$\{w \in Y \cap Z | w_b \in Z\} = \{w \in Y \cap B | w_b \in B\}.$$

Then (1) follows by applying Lemma 2 with $A = Y$.

Now

$$x + y = x + |Y \cap X| + |Y \cap Z|$$

$$\leq x + (x_1 + x_2) + t + \left(\frac{n - 2x - x_2 - t}{2}\right) \tag{2}$$

$$= x_1 + \frac{x_2}{2} + \frac{t}{2} + \frac{n}{2}.$$

The blue epns in X_3 are distinct and so $x_3 \geq t + x_1$, i.e.,

$$\frac{t}{2} \leq \frac{x_3}{2} - \frac{x_1}{2}. \tag{3}$$

From (2) and (3) we obtain

$$x + y \leq \left(\frac{x_1 + x_2 + x_3}{2}\right) + \frac{n}{2} = \frac{x}{2} + \frac{n}{2}.$$

Therefore $y \leq \frac{n}{2} - \frac{x}{2}$ and $xy \leq \frac{nx}{2} - \frac{x^2}{2}$. By elementary calculus, xy attains its maximum $\frac{n^2}{8}$ when $x = \frac{n}{2}$.

Case 2. $|Y \cap X'| \geq 2$.
In this case $x_1 = 0$, each $w \in Y \cap Z$ has $w_b \in Z$ and for each $w \in Y \cap X'$, $w_b \notin X'$ i.e., $w_b \in X_3 \cup Z$.

Subcase 2(a). $w \in Y \cap X'$ has $w_b \in X_3$.
This implies $|Y \cap X'| = 2$. Let $Y \cap X' = \{w, v\}$. Now

$$x + y = x + |Y \cap X| + |Y \cap X'| + |Y \cap Z|$$

$$\leq x + x_2 + 2 + \frac{(n - 2x - x_2 - \lambda)}{2}$$

where $\lambda = 1$ (resp. 0) if $v_b \in Z(X_3)$. Hence

$$x + y \leq \frac{n}{2} + \frac{x_2}{2} - \frac{\lambda}{2} + 2. \tag{4}$$

By counting blue epns in X_3, we obtain $x_3 \geq 2 - \lambda$ and since $|Z| \geq x_2$, we deduce $x_2 \leq n - 2x$. Use of these gives

$$x_2 \leq n - 2\,(x_1 + x_2 + x_3) = n - 2\,(x_2 + x_3)\,.$$

Therefore

$$x_2 \leq \frac{n-2x_3}{3} \leq \frac{n-4-2\lambda}{3}. \tag{5}$$

From (4) and (5)

$$x+y \leq \frac{2n+4}{3} - \frac{5\lambda}{6} \leq \frac{2n+4}{3},$$

so that $xy \leq x(\frac{2n+4}{3} - x)$. Calculus shows that $xy \leq \lfloor (\frac{n+2}{3})^2 \rfloor \leq \frac{n^2}{8}$ $(n \geq 32)$.

Subcase 2(b). Each $w \in Y \cap X'$ has $w_b \in Z$.
In this situation every $v \in Y$ has $v_b \in Z$. Therefore $y \leq |Z| = n - 2x$ and $xy \leq nx - 2x^2$. The maximum of this for $x \in [\frac{n}{4}, \frac{n}{2}]$ is $\frac{n^2}{8}$.

Case 3. $|Y \cap X'| = \{v\}$.
Define λ as in subcase 2(a) and let $\mu(= 0$ or $1)$ be the number of vertices in $Y \cap Z$ with blue epns in X_3.

The set Z contains $\lambda + x_2$ blue epns of vertices in $Y \cap (X \cup X')$ and μ vertices of $Y \cap Z$ have blue epns in X_3. Hence using Lemma 2 we obtain

$$\begin{aligned} x+y &= x + |Y \cap X| + |Y \cap X'| + |Y \cap Z| \\ &\leq x + (x_1 + x_2) + 1 + \mu + \left(\frac{n - 2x - \mu - x_2 - \lambda}{2} \right) \\ &= \frac{n}{2} + x_1 + \frac{x_2}{2} + \frac{(\mu - \lambda)}{2} + 1\,. \end{aligned} \tag{6}$$

By counting blue epns in X_3 we obtain $x_3 \geq (1-\lambda) + x_1 + \mu$ and since $|Z| \geq x_2$ we have $x_2 \leq n - 2x$. Use of these gives

$$x_2 \leq n - 2\,(x_1 + x_2 + x_3)\,.$$

Hence

$$\begin{aligned} x_2 &\leq \frac{n - 2(x_1 + x_3)}{3} \\ &\leq \frac{n - 2x_1 - 2[(1-\lambda) + x_1 + \mu]}{3} \\ &= \frac{n - 4x_1 - 2 - 2(\mu - \lambda)}{3}\,. \end{aligned} \tag{7}$$

Combining (6) and (7) we obtain

$$x + y \leq \frac{2n+2}{3} + \frac{x_1}{3} + \frac{\mu - \lambda}{6}. \tag{8}$$

However hypothesis and the private neighbour property imply that $x_1 + \mu \leq 1$. Hence from (8) we deduce

$$x + y \leq \frac{2n+3}{3} - \left(\frac{\lambda + \mu}{6}\right) \leq \frac{2n+3}{3}.$$

Calculus shows that $xy \leq (\frac{2n+3}{6})^2 \leq \frac{n^2}{8}$ $(n \geq 32)$. This completes the proof of Proposition 7. ■

Proposition 8. *If $n \geq 32$ and $|Y \cap X| \leq 2$, then $xy \leq n^2/8$.*

Proof. Define $Y' = \{v_f | v \in Y\}$. If $|Y \cap X'|$ ($|Y' \cap X|$ or $|Y' \cap X'|$) > 2, then we may apply Proposition 7 to the open irredundant sets Y, X' (Y', X or Y', X') of $\overline{G}$, G and infer the result. Thus we assume that $|Y \cap X'|$ $|Y' \cap X|$ and $|Y' \cap X'|$ are at most two. Then

$$\begin{aligned} n &\geq |X| + |X'| + |Y| + |Y'| - |Y \cap X| - |Y' \cap X| - |Y \cap X'| - |Y' \cap X'| \\ &\geq 2x + 2y - 2 - 2 - 2 - 2. \end{aligned}$$

Hence $x + y \leq \frac{n+8}{2}$ and therefore by elementary calculus $xy \leq (\frac{n+8}{4})^2 \leq \frac{n^2}{8}$ $(n \geq 32)$. ■

The preceding propositions have established a bound for $Q_{21}\overline{Q}_{21}$.

Theorem 9. *If $n \geq 32$, then $Q_{21}\overline{Q}_{21} \leq n^2/8$.*

Proof. Immediate from Propositions 7 and 8. ■

We now use Theorems 6 and 9 to determine exact Nordhaus-Gaddum bounds for the remaining values of i.

Theorem 10. *If $n \geq 32$ and $i \in \{1, 4, 5, 16, 17, 20, 21\}$, then $\max_G(Q_i + \overline{Q}_i) \leq 3n/4$, $\max_G(Q_i\overline{Q}_i) \leq n^2/8$ and these bounds are attained for infinitely many values of n.*

Proof. For any $i \in \{1, 4, 5, 16, 17, 20, 21\}$,

$$f_1 \implies f_i \implies f_{21},$$
$$f_4 \implies f_i \implies f_{21}$$

or

$$f_{16} \implies f_i \implies f_{21}.$$

Hence by Lemma 1, Theorems 6 and 9, for any G

$$Q_j + \overline{Q}_j \leq Q_i + \overline{Q}_i \leq Q_{21} + \overline{Q}_{21} \leq \frac{3n}{4}$$

and

$$Q_j\overline{Q}_j \leq Q_i\overline{Q}_i \leq Q_{21}\overline{Q}_{21} \leq \frac{n^2}{8},$$

where $j \in \{1, 4, 16\}$. Thus the bounds of the theorem are established. To show that they are attained it is sufficient to exhibit for each $j \in \{1, 4, 16\}$ graphs satisfying

$$Q_j + \overline{Q}_j \geq \frac{3n}{4} \qquad \text{and} \qquad Q_j\overline{Q}_j \geq \frac{n^2}{8}. \qquad \blacksquare$$

In order to describe the three examples we need the following definition. Let A, B be disjoint m-vertex subsets of a graph L. We say there is an *induced matching from A to B in L* if the bipartite subgraph of L defined by A, B is isomorphic to mK_2.

We form the graph H as follows. Let $V(H) = X \cup Y \cup Y'$ (disjoint union) where $|X| = \frac{n}{2}$ where $n \equiv 0$ (mod4), $n \geq 32$, $|Y| = |Y'| = \frac{n}{4}$ and $X' = Y \cup Y'$. Add edges so that there are induced matchings from X to X' in H and from Y to Y' in $\overline{H}$.

Each of the three examples will be formed by adding edges to H. For each of the three values of j it is easily checked that X and Y are f_j-sets of the constructed graph H^* and $\overline{H^*}$ respectively, so that H^* satisfies (9). In each case we remind the reader of the f_j-set definition.

$j = 1$: Subset S is an f_1-set if each $s \in S$ is a S-*spn* and has an S-*epn*. Form H^* from H by adding edges so that $H^*[Y]$ is complete.

$j = 4$: Subset S is an f_4-set if each $s \in S$ has both an S-*ipn* and an S-*epn*. In this case we require $n \equiv 0$ (mod 8). Form H^* from H by adding edges so that $H^*[X]$ and $\overline{H^*}[Y]$ are isomorphic to $\frac{n}{4}K_2$ and $\frac{n}{8}K_2$, respectively.

$j = 16$: Subset S is an f_{16}-set if each $s \in S$ has an S-*epn*, has no S-*ipn* and is not an S-*spn*. Form H^* from H by adding edges so that $H^*[X]$ and $\overline{H^*}[Y]$ are isomorphic to $C_{\frac{n}{2}}$ and $C_{\frac{n}{4}}$ respectively.

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