Discussiones Mathematicae
Graph Theory 24 (2004) 137–145

A SIMPLE LINEAR ALGORITHM FOR THE CONNECTED DOMINATION PROBLEM IN CIRCULAR-ARC GRAPHS*

Ruo-Wei Hung and Maw-Shang Chang

Department of Computer Science and Information Engineering
National Chung Cheng University
Ming-Hsiung, Chiayi 621, Taiwan, R.O.C.

e-mail: {rwhung,mschang}@cs.ccu.edu.tw

Abstract

A connected dominating set of a graph $G = (V, E)$ is a subset of vertices $CD \subseteq V$ such that every vertex not in CD is adjacent to at least one vertex in CD, and the subgraph induced by CD is connected. We show that, given an arc family F with endpoints sorted, a minimum-cardinality connected dominating set of the circular-arc graph constructed from F can be computed in $O(|F|)$ time.

Keywords: graph algorithms, circular-arc graphs, connected dominating set, shortest path.

2000 Mathematics Subject Classification: 05C85, 05C69.

1. Introduction

All graphs considered in this paper are finite, undirected, without loops or multiple edges. Throughout the paper, n and m denote the numbers of vertices and edges of a graph $G = (V, E)$, respectively. The *open neighborhood* of a vertex v, denoted by $N(v)$, consists of all vertices adjacent to v in G. The *closed neighborhood* of v, denoted by $N[v]$, is the set $N(v) \cup \{v\}$. A set of vertices $D \subseteq V$ is called a dominating set of G if every vertex in V is either in D or adjacent to a vertex in D. A dominating set CD of G is called

*The research was supported in part by the National Science Council of Republic of China under grant no. NSC90-2213-E194-034.

a *connected dominating set* if the subgraph induced by CD is connected. In addition, if the cardinality $|CD|$ is minimum among all possible connected dominating sets, then CD is called a *minimum connected dominating set.* Dominating sets and connected dominating sets have applications in a variety of fields, including communication theory and political science. For more background on dominating sets and connected dominating sets, we refer readers to [9], [10].

The problem of finding a minimum connected dominating set is NP-hard in general graphs [6]. The same holds true for bipartite graphs [6, 24], split graphs [16], chordal bipartite graphs [23], circle graphs [13] and weighted cocomparability graphs [3]. However, there exist polynomial time algorithms for interval graphs [2], circular-arc graphs [2], doubly chordal graphs [22], trapezoid graphs [15, 18] and distance-hereditary graphs [26].

A circular-arc family F is a collection of arcs in a circle. A graph G is a *circular-arc graph* if there exists a circular-arc family F, and a one-to-one mapping of vertices of G and the arcs in F such that two vertices in G are adjacent if, and only if, their corresponding arcs in F intersect. For a circular-arc family F, $G(F)$ denotes the graph constructed from F. Circular-arc graphs were introduced as a generalization of interval graphs (similarly defined, except that intervals on a real line are used instead of arcs on a circle) [7]. Both classes of graphs have a variety of applications involving traffic light sequencing, genetics, VLSI design and scheduling.

Tucker [25] presented an $O(n^3)$ time algorithm for testing whether a graph is a circular-arc graph and, if it is, constructing an arc family. Hsu [11] proposed an $O(mn)$ time algorithm to recognize circular-arc graphs. Eschen and Spinrad [5] proposed an $O(n^2)$ time recognition algorithm for circular-arc graphs. Recently, McConnell [21] presented an $O(n+m)$ time recognition algorithm for circular-arc graphs.

Given an arc family F with endpoints sorted, some researchers have proved that the following problems can be solved in $O(n)$ time for $G(F)$: the maximum independent set problem [8, 12, 19, 20], the single-source shortest paths problem [1], the circle-cover minimization problem [17], the dominating cycle problem [14], the domination problem [12] and the minimum clique cover problem [12].

Chang [2] proposed an $O(n+m)$ time algorithm for solving the connected domination problem on circular-arc graphs with weights on vertices (arcs). In this paper, we present a simple $O(n)$ time algorithm for finding a minimum connected dominating set of $G(F)$ given an arc family F with endpoints sorted.

2. The Algorithm

Let F be a circular-arc family with endpoints sorted where $|F| = n$. An arc v in F beginning from point c and ending at point d in clockwise direction is denoted by (c, d). We call both points c and d *endpoints* of arc v. We call point c the *head* of arc v, denoted by $h(v)$, and point d the *tail* of arc v, denoted by $t(v)$, respectively. Without loss of generality, assume that all endpoints of arcs in F are distinct and that no arc covers the entire circle. A contiguous part of the circle beginning from point c and ending at point d in the clockwise direction, is referred to as *segment* (c, d), denoted by $seg(c, d)$. We refer to an element of F as an arc and a part of the circle as a segment, respectively. We assume both arcs and segments are open, namely, they do not contain their endpoints. Note that an arc is also a segment of the circle. We say that a point p is contained *in* a segment $seg(c, d)$ if it falls within the interior of $seg(c, d)$. Denote this by $p \in seg(c, d)$. An arc u of F is said to be *contained in* another arc v if every point of arc u is contained in arc v. An arc in F is *maximal* if it is not contained in any other arc of F. Let F' denote the collection of all maximal arcs in F. Figure 1 shows a circular-arc family, where dark arcs are maximal arcs.

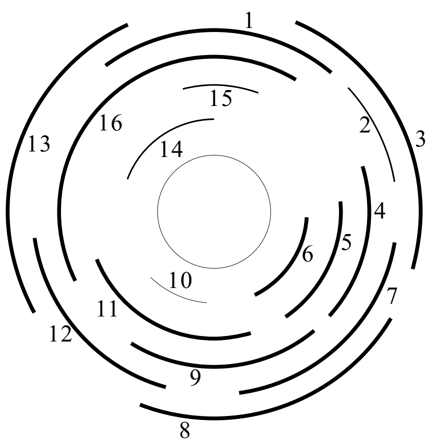


Figure 1. A circular-arc family F, where dark arcs are maximal arcs.

Remark 1. If a minimum connected dominating set D contains arc v which is not a maximal arc and is contained in a maximal arc u, then $(D\backslash\{v\})\cup\{u\}$ is still a minimum connected dominating set since $N[v] \subseteq N[u]$. Hence, there exists a minimum connected dominating set of $G(F)$ contained in F'. However, a connected dominating set of $G(F')$ is not necessarily a connected dominating set of $G(F)$.

We call a subset C of a circular-arc family F a *circle cover* in F, if the union of arcs in C covers the entire circle. A minimum circle cover in F is a circle cover of minimum cardinality among all circle covers in F. Clearly a circle cover in F is a connected dominating set of $G(F)$. If there exists an arc u in F which intersects every other arc in F, then $\{u\}$ is a minimum connected dominating set. Such an arc can be found in $O(n)$ time if it exists [12]. If F does not cover the entire circle, then $G(F)$ is an interval graph. This case can be detected easily. If $G(F)$ is an interval graph, the connected domination problem can be solved in $O(n)$ time [2]. In the following, we assume $G(F)$ is not an interval graph, and F covers the entire circle.

Definition 1. Let u and v be two maximal arcs such that $h(u)$ is not in v. Define a *clockwise path* from u to v of length $k-1$ to be a sequence $\{a_1, a_2, \cdots, a_k\}$ of distinct arcs in F such that $a_1 = u$, $a_k = v$, a_j and a_{j+1} intersect for $j \in \{1, 2, \cdots, k-1\}$, and $h(a_j)$ is contained in $seg(h(u), h(v))$ for $j \in \{2, 3, \cdots, k-1\}$. A clockwise path from u to v is called a *clockwise shortest path*, denoted by $SP_c(u, v)$, if it has the smallest length among all possible u-to-v clockwise paths. Notice that if $t(u)$ is in arc v, then $SP_c(u, v)$ visits u and v only and hence, $|SP_c(u, v)| = 2$.

For the set of arcs shown in Figure 1, we have $SP_c(4, 1) = \{4, 8, 12, 16, 1\}$, $SP_c(12, 3) = \{12, 16, 3\}$, $SP_c(1, 9) = \{1, 3, 7, 9\}$, and $SP_c(1, 7) = \{1, 3, 7\}$.

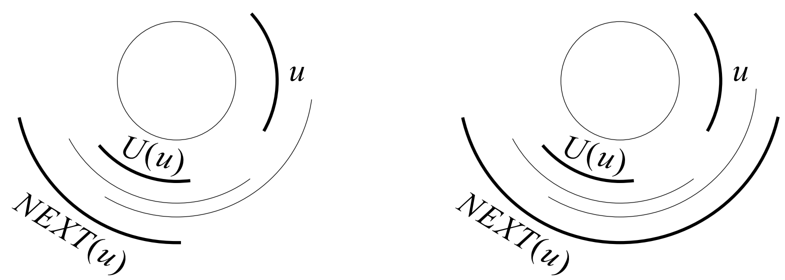
Remark 2. Let P be a clockwise path from u to v, where u and v are two maximal arcs that do not intersect each other. If $seg(t(v), h(u))$ does not contain any arc in F, then the set of arcs visited by P is a connected dominating set of $G(F)$. On the other hand, the union of arcs in a connected dominating set of $G(F)$ either covers the entire circle, or is a segment of the circle. In the former case, it is a circle cover in F. If it is a segment $seg(c, d)$, then $seg(d, c)$ contains no arc in F. Let c and d be the head and tail of arc u and v, respectively. Then the set of arcs visited by the shortest clockwise path from u to v is also a connected dominating set of $G(F)$.

Based upon the above remarks, we first find a minimum circle cover for F. Then we find two maximal arcs u and v such that $|SP_c(u,v)|$ is minimum and $seg(t(v),h(u))$ does not contain any arc in F. Then the smaller one of the minimum circle cover, and the set of arcs visited by $SP_c(u,v)$ is a minimum connected dominating set of $G(F)$. In the following, we show how to find two such arcs u and v.

Definition 2. [12] For a maximal arc u, the *first clockwise undominated* arc $U(u)$ is the arc in $F \backslash N[u]$ whose tail is first encountered in a clockwise traversal from $t(u)$. Define $NEXT(u)$ to be the arc in $N[U(u)] \cap F'$ whose tail is last encountered in a clockwise traversal from $t(U(u))$.

For example, in the arc family F shown in Figure 1, $U(1) = 2$, $NEXT(1) = 4$, $U(13) = 15$, and $NEXT(13) = 1$.

Remark 3. For a maximal arc u, $h(NEXT(u))$ is either in arc u or not, as shown in Figure 2. In case $h(NEXT(u))$ is not in arc u, we observe that $seg(t(u), h(NEXT(u)))$ does not contain any arc in F. This implies that $SP_c(NEXT(u), u)$ is a connected dominating set of $G(F)$.



(a)$h(NEXT(u))$ is not in arc u (b)$h(NEXT(u))$ is in arc u

Figure 2. The relative positions of arc u and $NEXT(u)$.

Our main theorem is given in the following.

Theorem 1. *Assume that no arcs in F intersect all other arcs in F, and that F covers the entire circle. Then, either a minimum circle cover in F is a minimum connected dominating set of $G(F)$ or there exists a maximal arc u such that $h(NEXT(u))$ is not in u and $SP_c(NEXT(u), u)$ is a minimum connected dominating set of $G(F)$.*

Proof. Suppose D, where $|D| = k$, is a minimum connected dominating set of $G(F)$ such that all arcs in D are maximal arcs. By Remark 1, such a minimum connected dominating set exists. By the assumption of the theorem, $k \geq 2$. Since the union of arcs in D either covers the entire circle or is a segment of the circle, the arcs in D can be sorted into a sequence $\{v_1, v_2, \cdots, v_k\}$ in clockwise ordering of tails such that v_i intersects v_{i+1} for $1 \leq i \leq k-1$. Let $u = v_k$. We claim that both v_1 and $NEXT(u)$ intersect $U(u)$ and hence, $NEXT(u)$ intersects v_2. Let $D' = (D \backslash \{v_1\}) \cup \{NEXT(u)\}$. If $h(NEXT(u))$ is in arc u, then the union of arcs in D' covers the entire circle. Hence D' is a minimum connected dominating set of $G(F)$. Since D' is a circle cover in F, $|D'|$ is no less than the cardinality of a minimum circle cover in F. Hence a minimum circle cover in F is also a minimum connected dominating set of $G(F)$. On the other hand, assume that $h(NEXT(u))$ is not in arc u. The union of arcs in D' is a segment $seg(h(NEXT(u)), t(u))$. Since segment $seg(t(u), h(NEXT(u)))$ contains no arc in F, D' is a minimum connected dominating set of $G(F)$. Because $|SP_c(NEXT(u), u)| \leq |D'|$, $SP_c(NEXT(u), u)$ is a minimum connected dominating set of $G(F)$. ■

Our algorithm is formally presented in the following.

Algorithm MCDS. Find a minimum connected dominating set of $G(F)$.
Input: A set F of n sorted arcs, where each arc i is represented as $(h(i), t(i))$, and all arcs in F are labelled from 1 to n.
Output: A minimum connected dominating set of $G(F)$.
Method:

1. **if** F does not cover the entire circle, **then return** the minimum connected dominating set found by Chang's algorithm [2] and **stop**;
2. **if** there exists an arc $v \in F$ such that $N[v] = F$, **then return** $\{v\}$ and **stop**;
3. compute a minimum circle cover C in F;
4. compute all maximal arcs of F;
5. compute $NEXT(v)$ for all maximal arcs v of F;
6. let H be the set of all maximal arcs v with that $h(NEXT(v))$ not in arc v;
7. **if** $H = \varnothing$, **then return** C and **stop**;
8. compute $|SP_c(NEXT(v), v)|$ for all maximal arcs v in H;

9. find a maximal arc u in H such that $|SP_c(NEXT(u), u)| \leq |SP_c(NEXT(v), v)|$ for any other maximal arc v in H;
10. find a clockwise shortest path $SP_c(NEXT(u), u)$, and let $D = SP_c(NEXT(u), u)$;
11. **output** the smaller one of C and D.

The correctness of the above algorithm can be seen from Theorem 1. In the following, we show how this algorithm can be implemented in $O(n)$ time. A minimum circle cover in F can be computed in $O(n)$ time [17]. Given an arc family F of n sorted arcs, we can compute $NEXT(v)$ for all maximal arcs v in $O(n)$ time [12]. After $O(n)$ time preprocessing, given two maximal arcs u and v with that $h(u)$ is not in arc v, $|SP_c(u, v)|$ can be computed in $O(1)$ time and a clockwise shortest path $SP_c(u, v)$ can be reported in $O(n)$ time [4]. Thus we have the following theorem.

Theorem 2. *Given a set of n sorted arcs, Algorithm MCDS solves the connected domination problem on circular-arc graphs in $O(n)$ time.*

Acknowledgements

The authors thank anonymous referees for their helpful suggestions in improving the presentation of this paper and an anonymous referee who referred us to [4].

References

[1] M.J. Atallah, D.Z. Chen, and D.T. Lee, *An optimal algorithm for shortest paths on weighted interval and circular-arc graphs, with applications*, Algorithmica **14** (1995) 429-441.

[2] M.S. Chang, *Efficient algorithms for the domination problems on interval and circular-arc graphs*, SIAM J. Comput. **27** (1998) 1671–1694.

[3] M.S. Chang, *Weighted domination of cocomparability graphs*, Discrete Appl. Math. **80** (1997) 135–147.

[4] D.Z. Chen, D.T. Lee, R. Sridhar, and C.N. Sekharan, *Solving the all-pair shortest path query problem on interval and circular-arc graphs*, Networks **31** (1998) 249–258.

[5] E.M. Eschen and J. Spinrad, *An $O(n^2)$ algorithm for circular-arc graph recognition*, in: Proceedings of the 4th Annual ACM-SIAM Symposium on Discrete Algorithm SODA'93 (1993) 128–137.

[6] M.R. Garey and D.S. Johnson, Computers and Intractability: A Guide to the Theory of NP-Completeness (W.H. Freeman, San Francisco, CA, 1979).

[7] M.C. Golumbic, Algorithmic Graph Theory and Perfect Graphs (Academic Press, New York, 1980).

[8] M.C. Golumbic and P.L. Hammer, *Stability in circular arc graphs*, J. Algorithms **9** (1988) 314–320.

[9] T.W. Haynes, S.T. Hedetniemi, and P.J. Slater, Fundamentals of Domination in Graphs (Marcel Dekker, New York, 1998).

[10] T.W. Haynes, S.T. Hedetniemi, and P.J. Slater, Domination in Graphs – Advanced Topics (Marcel Dekker, New York, 1998).

[11] W.L. Hsu, $O(M \cdot N)$ *algorithms for the recognization and isomorphism problems on circular-arc graphs*, SIAM J. Comput. **24** (1995) 411–439.

[12] W.L. Hsu and K.H. Tsai, *Linear time algorithms on circular-arc graphs*, Inform. Process. Lett. **40** (1991) 123–129.

[13] J.M. Keil, *The complexity of domination problems in circle graphs*, Discrete Appl. Math. **42** (1993) 51–63.

[14] J.M. Keil and D. Schaefer, *An optimal algorithm for finding dominating cycles in circular-arc graphs*, Discrete Appl. Math. **36** (1992) 25–34.

[15] E. Köhler, *Connected domination and dominating clique in trapezoid graphs*, Discrete Appl. Math. **99** (2000) 91–110.

[16] R. Laskar and J. Pfaff, Domination and irredundance in split graphs, Technical Report 430, Dept. Mathematical Sciences (Clemson University, 1983).

[17] C.C. Lee and D.T. Lee, *On a circle-cover minimization problem*, Inform. Process. Lett. **18** (1984) 109–115.

[18] Y.L. Lin, F.R. Hsu, and Y.T. Tsai, *Efficient algorithms for the minimum connected domination on trapezoid graphs*, Lecture Notes in Comput. Sci. **1858** (Springer Verlag, 2000) 126–136.

[19] G.K. Manacher and T.A. Mankus, *A simple linear time algorithm for finding a maximum independent set of circular arcs using intervals alone*, Networks **39** (2002) 68–72.

[20] S. Masuda and K. Nakajima, *An optimal algorithm for finding a maximum independent set of a circular-arc graph*, SIAM J. Comput. **17** (1988) 41–52.

[21] R.M. McConnell, *Linear-time recognition of circular-arc graphs*, in: Proceedings of the 42nd IEEE Symposium on Foundations of Computer Science FOCS'01 (2001) 386–394.

[22] M. Moscarini, *Doubly chordal graphs, Steiner trees, and connected domination*, Networks **23** (1993) 59–69.

[23] H. Müller and A. Brandstädt, *The NP-completeness of Steiner tree and dominating set for chordal bipartite graphs*, Theoret. Comput. Sci. **53** (1987) 257–265.

[24] J. Pfaff, R. Laskar, and S.T. Hedetniemi, NP-completeness of total and connected domination, and irredundance for bipartite graphs, Technical Report 428, Dept. Mathematical Sciences (Clemson University, 1983).

[25] A. Tucker, *An efficient test for circular-arc graphs*, SIAM J. Comput. **9** (1980) 1–24.

[26] H.G. Yeh and G.J. Chang, *Weighted connected domination and Steiner trees in distance-hereditary graphs*, Discrete Appl. Math. **87** (1998) 245–253.

Received 18 March 2002
Revised 18 January 2003