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A NOTE ON TOTAL COLORINGS OF PLANAR GRAPHS WITHOUT 4-CYCLES

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Abstract

Let G be a 2-connected planar graph with maximum degree Δ such that G has no cycle of length from 4 to k, where $k \geq 4$. Then the total chromatic number of G is $\Delta+1$ if $(\Delta, k) \in \{(7,4), (6,5), (5,7), (4,14)\}$.

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We consider finite simple graphs. Any undefined notation follows that of Bondy and Murty [1]. We use $V(G)$, $E(G)$, $\delta(G)$ and $\Delta(G)$ to denote the vertex set, the edge set, the minimum degree and the maximum degree of a graph G respectively. Let $d(v)$ denote the degree of vertex v. A *k-vertex* is a vertex of degree k.

A *total k-coloring* of a graph G is a coloring of $V(G) \cup E(G)$ using k colors such that no two adjacent or incident elements receive the same color. The *total chromatic number* $\chi_T(G)$ is the smallest integer k such that G has a total k-coloring. Behzad and Vizing (see page 86 in [8]) conjectured independently that any graph G is totally $(\Delta(G)+2)$-colorable in 1965.

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Various coloring techniques have been introduced in effort to prove this conjecture for some special graph classes (see survey papers [7] and [11]). In 1989, Sanchez-Arroyo [10] proved that for any graph G it is NP-complete to decide if $\chi_T(G) = \Delta(G)+1$. In 1997, Borodin et al [3] proved that a planar graph G with maximum degree $\Delta \geq 11$ has $\chi_T(G) = \Delta(G)+1$, and they also obtained several related results by adding girth restrictions [4]. Note that the added girth requirement in [4] prohibits the appearance of triangles. The forbidden cycle or the girth restriction plays an important role in considering list-coloring planar graphs. For example, Kratochvíl and Tuza showed that every triangle-free planar graph is 4-choosable and Thomassen observed that a planar graph is 3-choosable if the girth of the graph is at least 5 (both results can be found in Section 2.13 of [8]). Recently, Lam, Xu and Liu [9] proved that every C_4-free planar graph is 4-choosable. We shall adopt a similar approach and prove the following theorem. Note that triangles are allowed in the graph G in our theorem.

Let a planar graph G be charged by an initial charge $w(v) = d(v) - 4$ if $v \in V(G)$ and $w(f) = r(f) - 4$ if $f \in F$, where $r(f)$ is the degree of the face f. Euler's formula implies that $\sum_{x \in V \cup F} w(x) < 0$. The discharging method distributes the positive charge to neighbors so as to leave as little positive charge remaining as possible. This leads to $\sum_{x \in V \cup F} w(x) > 0$. A contradiction follows and this shows the unavoidability of a set of special elements in G (see Claims 2, 3 and 4).

Theorem. *Let G be a connected planar graph with maximum degree Δ such that G has no cycle of length from 4 to k, where $k \geq 4$. If*

(1) $\Delta \geq 7$ *and* $k \geq 4$ *or*

(2) $\Delta \geq 6$ *and* $k \geq 5$, *or*

(3) $\Delta \geq 5$ *and* $k \geq 7$, *or*

(4) $\Delta \geq 4$ *and* $k \geq 14$,

then $\chi_T(G) = \Delta(G) + 1$.

Lemma 1 [6]**.** *Every region of a planar imbedding of a graph has a simple cycle for its boundary if and only if G is 2-connected.*

This lemma is equivalent to the assertion that no three edges incident with any vertex v lie on the same face. It implies that each vertex v is incident with $d(v)$ faces. We shall use this fact often in the proof of the Theorem.

An *edge coloring* of a graph G is a coloring of $E(G)$ such that no two adjacent edges receive the same color. A graph G is said to be *edge-f-choosable* if, whenever we give lists A_e of $f(e)$ colors to each edge $e \in E(G)$, there exists an edge coloring of G where each edge is colored with a color from its own list.

Lemma 2 [5]**.** *A bipartite graph G is edge-f-choosable where $f(e) = \max\{d(u), d(v)\}$ for $e = uv \in E(G)$.*

Proof of Theorem. Let $G = (V, E, F)$ be a minimal counterexample to any of (1) – (4) in the Theorem. Then

(a) G is 2-connected and

(b) any vertex is incident with at most $\lfloor \frac{d(v)}{2} \rfloor$ 3-faces, and

(c) G contains no even cycle $v_1 v_2 \cdots v_{2t} v_1$ such that $d(v_1) = d(v_3) = \cdots = d(v_{2t-1}) = 2$, and

(d) G contains no edge uv with $\min\{d(u), d(v)\} \le \lfloor \frac{\Delta(G)}{2} \rfloor$ and $d_G(u) + d_G(v) \le \Delta(G) + 1$.

(a) and (b) are obvious. The proofs of (c) and (d) can be found in [2] and [5], respectively.

Let G_2 be the subgraph induced by the edges incident with the 2-vertices of G. Since $\Delta(G) \ge 4$ in all four cases in the Theorem, (d) implies that G does not contain two adjacent 2-vertices. Hence, G_2 does not contain any odd cycle. It follows from (c) that G_2 does not contain any even cycle. Therefore, any component of G_2 is a tree. For any component in G_2 that is a path of even length, one can easily find a set of edges saturating all 2-vertices. For any component that is not a path of even length, we can select a vertex t with $d_{G_2}(t) \ge 3$ as the root of the tree. We denote edges of distance i from the root to be at level $i + 1$ where $i = 0, 1, ..., d$ and d is the depth of the tree. Since G does not contain two adjacent 2-vertices, the distance from any leaf to the root is even. We can select all the edges at even level to form a matching saturating all 2-vertices in this component. Thus, there exists a matching M such that all 2-vertices in G_2 are saturated. If $uv \in M$ and $d(u) = 2$, v is called the 2-*master* of u and u is called the *dependent* of v. Each 2-vertex has a 2-master and each vertex of degree Δ can be the 2-master of at most one 2-vertex.

Since G is a planar graph, by Euler's formula, we have

$$(E) \qquad \sum_{v\in V}(d(v)-4)+\sum_{f\in F}(r(f)-4)=-4(|V|-|E|+|F|)=-8<0,$$

where $r(f)$ is the *degree* of the face f, that is, the number of edges around f. A k-*face* is a face of degree k. Now we define the initial charge function $w(x)$ for each $x \in V \cup F$. Let $w(x) = d(x) - 4$ if $x \in V$ and $w(x) = r(x) - 4$ if $x \in F$. It follows from (E) that $\sum_{x\in V\cup F} w(x) < 0$.

We begin the proof of (1) in the Theorem. First we prove a claim establishing a relation between the set of vertices of degree 3 or less and the set of vertices of degree at least $\Delta - 1$. We adopt the classic technique used in proving Hall's Matching Theorem (see page 72 in [1]). Let X be the set of vertices of degree at most 3 and $Y = \cup_{x\in X} N(x)$. By (d), X is an independent set of G. Let K be the induced bipartite subgraph of G with partite sets X and Y.

Claim 1. If $X \neq \emptyset$, then G contains a bipartite subgraph $B = (X, Y)$ such that $d_B(x) = 1$ and $d_B(y) \leq 2$ whenever $x \in X$ and $y \in Y$.

***Proof of Claim* 1.** Let $H = (X', Y)$, where $X' \subseteq X$, be a maximum bipartite subgraph such that $d_H(x) = 1$ and $d_H(y) \leq 2$ whenever $x \in X'$ and $y \in Y$. Note that there may be some isolated vertices in Y. Clearly, H is not empty since there is at least one edge from X to Y. Suppose that $X\backslash X' \neq \emptyset$. Let $v \in X\backslash X'$. An *alternating path*, P_v, in G is a path whose origin is v and edges are alternating between $E(K) \setminus E(H)$ and $E(H)$. By the maximality of H, there exists no alternating path that will terminate at a vertex $v' \in Y$ with $d_H(v') \leq 1$. Let Z denote the set of of all vertices connected to v by alternating paths. Set $X'' = Z \cap X'$ and $Y'' = Z \cap Y$ (see Figure 1).

Clearly, $Y'' \subseteq \cup_{x\in X''} N(x)$. Suppose $\cup_{x\in X''} N(x) \nsubseteq Y''$. It follows that there exists a vertex $x \in X''$ such that $xy \in E(G)$ and $y \notin Y''$. This implies that an alternating path P_v terminates at a vertex $y \in Y$, a contradiction. Hence, $Y'' = \cup_{x\in X''} N(x)$.

Now we show that $d_H(y) \geq 2$ for any $y \in Y''$. Suppose, on the contrary, there exists a vertex $y_i \in Y''$ where $vy_1x_1...x_{i-1}y_i$ is an alternating path such that $d_H(y_i) = 1$. Let $H' = H - \{y_1x_1, ..., y_{i-1}x_{i-1}\} + \{vy_1, x_1y_2, ..., y_ix_{i-1}\}$ if $i \geq 2$ and let $H' = H + \{vy_1\}$ if $i = 1$. It follows that $|E(H')| > |E(H)|$, a contradiction to H being maximum.

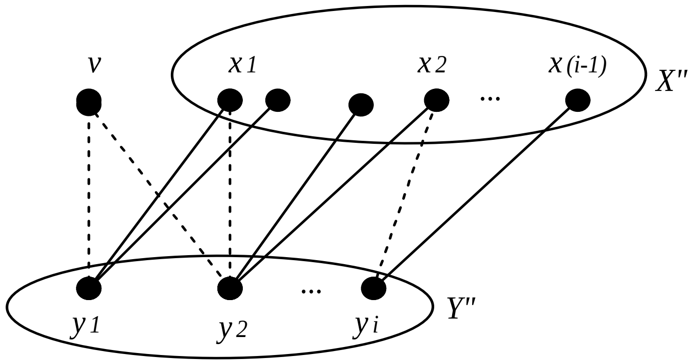

Figure 1. Subgraph F

Let $F = (X'', Y'')$. It follows that $d_F(y) \geq d_H(y) + 1 \geq 3$ for any $y \in Y''$. Note that $d_G(x) = d_F(x) \leq 3$ for any $x \in X''$.

Now $G - X''$ has a total $(\Delta + 1)$-coloring by the minimality of G. By Lemma 2, we can color all edges in F using the same set of colors by choosing the colors unused on $y \in Y''$. Since the maximum degree in X'' is 3, all vertices in X'' can be easily colored by $(\Delta{+}1)$ colors. Therefore, G has a total $(\Delta + 1)$-coloring, a contradiction with the fact that G is a counterexample. This implies $X = X'$, and which in turn, proves Claim 1. ■

We call y the 3-*master* of x if $xy \in B$ and $x \in X$. It follows from this claim that each vertex of degree at most 3 has a 3-master. Each vertex of degree at least $\Delta - 1$ can be a 3-master of at most two vertices.

Claim 2. If $\Delta \geq 7$, then G does not contain a 3-face uvw such that $d(u) = d(v) = 4$.

***Proof of Claim* 2.** Suppose it does contain such a 3-face. Let $G' = G - uv$. By the minimality of G, G' has a total $(\Delta + 1)$-coloring φ. Since $d_{G'}(u) = d_{G'}(v) = 3$ and $\Delta \geq 7$, we may assume that $\varphi(u) \neq \varphi(v)$. Let C be the set of colors used to color edges adjacent to uv. If $\varphi(w) \notin C$, then color uv with $\varphi(w)$. Otherwise, without loss of generality, we may assume that an edge e incident with u is colored with $\varphi(w)$. Then we erase the color on u. It follows that at least one color is available for uv, and then we re-color u. This is possible because $d(u) = 4$ and both e and w share the same color. Now, G has a total coloring with $(\Delta + 1)$ colors, a contradiction with the fact that G is a counterexample. ■

Claim 2 and (d) imply that every 3-face is incident with at least two vertices of degree at least 5. To prove (1), we are ready to construct a new charge $w^*(x)$ on G as follows:

R11: Each $r(\geq 5)$-face gives $1 - \frac{4}{r}$ to its incident vertices.

R12: Each 2-vertex receives $\frac{3}{5}$ from its 3-master, and receives $\frac{16}{15}$ from its 2-master if it is incident with a 3-face and receives 1 from its 2-master otherwise.

R13: Each 3-vertex receives $\frac{8}{15}$ from its 3-master. In addition, if v is incident with a 3-face f, then each 3-vertex v receives $\frac{1}{15}$ from u where u is a neighbor of v but not incident with f.

R14: Each 3-face receives $\frac{1}{2}$ from its incident vertices of degree at least 5.

By (d), $d(v) = \Delta \geq 7$ if a vertex v is the 2-master of some vertex, $d(v) \geq \Delta - 1 \geq 6$ if v is the 3-master of some vertices, and $d(u) \geq 6$ if a vertex u gives $\frac{1}{15}$ via R13. Note that a vertex can be the 3-master of most two vertices and, in turn, it may give at most $2 \times \max\{\frac{3}{5}, \frac{8}{15}\} = \frac{6}{5}$. Let f be a face of G. Clearly, $w^*(f) = 0$ if $r(f) \geq 5$. By Claim 2, each 3-face f is incident with at least two vertices of degree at least 5. Hence, $w^*(f) \geq w(f) + 1 = 0$. Let v be an arbitrary vertex of G. First, we consider the case of $d(v) = 2$. It will receive $\frac{3}{5}$ from its 3-master. By Lemma 1, v is incident with two faces. If v is incident with a 3-face, then the other incident face of v must have degree at least 6 since G is a C_4-free graph. This implies that v receives at least $\frac{1}{3}$ from the face of degree ≥ 6. If v is not incident with a 3-face, then v receives at least $2 \times \frac{1}{5}$ from its incident faces. So $w^*(v) \geq w(v) + \min\{\frac{3}{5} + \frac{16}{15} + \frac{1}{3}, \frac{3}{5} + 1 + \frac{2}{5}\} = 0$. Consider $d(v) = 3$. If it is incident with a 3-face, then the other two vertices on the same face must be of degree at least 5 and this implies that v receives at least $\frac{2}{5}$ from its incident faces. If v is not incident with a 3-face, then it must be incident with three r-faces where $r \geq 5$. It follows that it receives at least $\frac{3}{5}$ from its incident faces. Hence, $w^*(v) \geq w(v) + \min\{\frac{8}{15} + \frac{1}{15} + \frac{2}{5}, \frac{8}{15} + \frac{3}{5}\} = 0$. If $d(v) = 4$, then it is incident with at most two 3-faces and its other two incident faces must be of degree ≥ 5. Hence, $w^*(v) \geq w(v) + \frac{2}{5} > 0$. If $d(v) = 5$, then v is incident with at least three r-faces where $r \geq 5$ and at most two 3-faces. Hence, $w^*(v) \geq w(v) + \frac{3}{5} - 2 \times \frac{1}{2} > 0$. If $d(v) = 6$, it can be 3-master of at most two vertices. Consider any two neighbors of v, say u_1 and u_2. If they form a 3-face, then v gives $\frac{1}{2}$ to the 3-face. If each of them is a 3-vertex on some 3-face, then v gives $2 \times \frac{1}{15}$. However, these two cases can not happen simultaneously; that is, vu_1u_2 is a 3-face and u_1,

u_2 have another common neighbor $w \neq v$, such that either $d(u_1) = 3$ or $d(u_2) = 3$ since G is C_4-free graph. In the evaluation of the lower bound of $w^*(v)$, it suffices to consider the case when v gives $3 \times \frac{1}{15}$ to its incident 3-faces. It follows that $w^*(v) \geq w(v) + \frac{3}{5} - 2 \times \frac{8}{15} - 3 \times \frac{1}{2} > 0$. Now consider $d(v) = 7$. Suppose v is a 2-master of a vertex u. If u and v are incident with the same 3-face, then v receives at least $3 \times \frac{1}{5} + (1 - \frac{4}{6})$ from its incident faces and gives $\frac{16}{15}$ to u. Otherwise v receives at least $4 \times \frac{1}{5}$ from its incident faces and gives 1 to u. Vertex v may be incident with at most three 3-faces and the remaining neighbor of v not incident with any three 3-faces may be a 3-vertex and in another 3-face, in turn, v may give $\frac{1}{15}$ to the 3-vertex. Vertex v may also be the 3-master of two other vertices. Hence, $w^*(v) \geq w(v) + \min\{3 \times \frac{1}{5} + \frac{1}{3} - \frac{16}{15}, \frac{4}{5} - 1\} - (3 \times \frac{1}{2} + \frac{1}{15} + 2 \times \frac{3}{5}) > 0$. In general, if $d(v) \geq 8$, then $w^*(v) \geq w(v) + \min\{\lfloor \frac{d(v)}{2} \rfloor \times \frac{1}{5} + \frac{1}{3} - \frac{16}{15}, \lceil \frac{d(v)}{2} \rceil \times \frac{1}{5} - 1\} - (\lfloor \frac{d(v)}{2} \rfloor \times \frac{1}{2} + \frac{1}{15} + 2 \times \frac{3}{5}) > 0$. It follows that $\sum_{x \in V \cup F} w(x) = \sum_{x \in V \cup F} w^*(x) \geq 0$, a contradiction with (E). This completes the proof of (1).

Note that (1) implies that (2) is true if $\Delta \geq 7$. Hence, it is sufficient to prove (2) by assuming $\Delta = 6$. Similarly, we may assume that $\Delta = 5$ in the proof of (3) and $\Delta = 4$ in the proof of (4).

Claim 3. If $\Delta \geq 5$, then G does not contain a 3-face uvw such that $d(u) = d(v) = 3$.

The proof of Claim 3, which we omit, is the same as Claim 2. Claim 3 implies that each 3-face is incident with at least two vertices of degree at least 4. To prove (2), we construct the new charge $w^*(x)$ on G as follows:

R21: Each $r(\geq 6)$-face gives $1 - \frac{4}{r}$ to its incident vertices.

R22: Each 2-vertex receives $\frac{11}{7}$ from its 2-master if it is incident with a 3-face and receives $\frac{4}{3}$ from its 2-master otherwise.

R23: Each 3-vertex v receives $1/3$ from u if v is incident with a 3-face f and u is a neighbor of v but not incident with f.

R24: Each 3-face receives $\frac{1}{2}$ from its incident vertex v if $d(v) \geq 5$ and receives $\frac{1}{3}$ if $d(v) = 4$.

Clearly, we have $w^*(f) \geq 0$ for any face f. Let v be an arbitrary vertex of G. Consider the case of $d(v) = 2$. If it is incident with a 3-face, then its other incident face must have degree at least 7 since G is a C_4-free and C_5-free graph. It follows that v receives at least $1 - \frac{4}{7} = \frac{3}{7}$ from the incident face and $\frac{11}{7}$ from its 2-master; that is, $w^*(v) \geq w(v) + \frac{3}{7} + \frac{11}{7} = 0$. Otherwise if

v is not incident with any 3-face, then it receives at least $2 \times (1 - \frac{4}{6}) = \frac{2}{3}$ from its two incident faces of degree at least 6 and $\frac{4}{3}$ from its 2-master. Hence, $w^*(v) \geq w(v) + \frac{2}{3} + \frac{4}{3} = 0$. Suppose $d(v) = 3$. If v is incident with a 3-face, then v receives at least $\frac{2}{3}$ from its two incident faces and $\frac{1}{3}$ from its 3-master not lying on the same 3-face. Otherwise if v is not incident with any 3-face, then v receives at least 1 from its three incident faces. Hence, $w^*(v) \geq w(v) + 1 = 0$. Note that v gives either $\frac{1}{3}$ if $d(v) = 4$ or $\frac{1}{2}$ if $d(v) \geq 5$ to an incident 3-face, say vuw where $u, w \in N(v)$, or gives $\frac{1}{3}$ to u and $\frac{1}{3}$ to w by R23 but v will then receive at least $1 - \frac{4}{6} = \frac{1}{3}$ from the face whose partial boundary contains u, v, w sequentially if $uw \notin E(G)$. In the evaluation of the lower bound of $w^*(v)$, it suffices to consider the case when v gives $\frac{1}{3}$ or $\frac{1}{2}$ to its incident 3-faces. If $d(v) = 4$, then it receives at least $\frac{2}{3}$ from its two incident faces of degree ≥ 6 and gives at most $\frac{2}{3}$ to its incident 3-faces since any 4-vertex is incident with at most two 3-faces. It follows that $w^*(v) \geq w(v) + \frac{2}{3} - \frac{2}{3} = 0$. If $d(v) = 5$, then there are five faces incident with v by Lemma 1. It follows that v is incident with at most two 3-faces and at least three r-faces ($r \geq 6$). If four neighbors of v form two 3-faces and a 3-face is pending on the remaining neighbor of v, then v discharges at most $2 \times \frac{1}{2} + \frac{1}{3}$ via R23. This implies that $w^*(v) \geq w(v) + 3 \times \frac{1}{3} - (2 \times \frac{1}{2} + \frac{1}{3}) > 0$. Suppose $d(v) = 6$. It follows that v can be the 2-master of some vertex u. In this case, either u is on 3-face vuu' (and it follows that v gives $\frac{11}{7} + \frac{1}{2}$), or v is the 2-master of u and 3-master of u' where v and u are not on the same 3-face, and it follows that v gives $\frac{4}{3} + \frac{1}{3}$. To find a low bound for $w^*(v)$, it suffices to consider the first case when v is the 2-master of u and vuu' forms a 3-face. If v is the 3-master of some 3-vertex u_1, then v gives at most $2 \times \frac{1}{3}$ to its dependents and $\frac{1}{2}$ to another 3-face. In this case, v receives $\frac{3}{7} + 3 \times \frac{1}{3}$ from its incident faces. If v is not a 3-master of any 3-vertex, then v gives at most $2 \times \frac{1}{2}$ to its two incident 3-faces. In this case, v receives $\frac{3}{7} + 2 \times \frac{1}{3}$ from its incident faces. Hence, $w^*(v) \geq w(v) + \min\{\frac{3}{7} + 1 - (\frac{11}{7} + \frac{1}{2} + \frac{2}{3} + \frac{1}{2}), \frac{3}{7} + \frac{2}{3} - \frac{11}{7} - \frac{1}{2} - 1\} = 2 - \frac{83}{42} = \frac{1}{42} > 0$. It follows that $\sum_{x \in V \cup F} w(x) = \sum_{x \in V \cup F} w^*(x) > 0$, a contradiction. This completes the proof of (2).

To prove (3), we construct a new charge $w^*(x)$ on G as follows:

R31: Each $r(\geq 8)$-face gives $1 - \frac{4}{r}$ to its incident vertices.

R32: Each 2-vertex receives $\frac{13}{9}$ from its 2-master if it is incident with a 3-face and receives 1 from its 2-master otherwise.

R33: Each 3-face receives $\frac{1}{2}$ from its incident vertices of degree at least 4.

We also have $w^*(f) \geq 0$ for any face f. Let v be an arbitrary vertex of G. If $d(v) = 2$, then $w^*(v) \geq w(v) + \min\{\frac{13}{9} + \frac{5}{9}, 2 \times \frac{1}{2} + 1\} = 0$. If $d(v) = 3$, then v is incident with at most one 3-face and at least two faces of degree ≥ 8. It follows that v receives at least $2 \times \frac{1}{2} = 1$ from its incident faces, and in turn, $w^*(v) \geq w(v) + 1 = 0$. If $d(v) = 4$, then it receives at least $2 \times \frac{1}{2} = 1$ from its incident faces and gives at most $2 \times \frac{1}{2} = 1$ to its incident 3-faces, that is, $w^*(v) \geq w(v) + 1 - 1 = 0$. Suppose $d(v) = 5$. If v is the 2-master of a 2-vertex u, and u is incident with a 3-face, then v receives at least $3 \times \frac{1}{2}$ from its incident faces and gives at most $\frac{13}{9} + 2 \times \frac{1}{2}$ to its dependent and 3-faces. Otherwise v receives at least $3 \times \frac{1}{2}$ from its incident faces and gives at most $1 + 2 \times \frac{1}{2}$ to its dependent and 3-faces. It follows that $w^*(v) \geq w(v) + \min\{\frac{3}{2} - \frac{13}{9} - 1, \frac{3}{2} - 2\} = \frac{1}{18} > 0$. This implies that $\sum_{x \in V \cup F} w(x) = \sum_{x \in V \cup F} w^*(x) > 0$, a contradiction. This completes the proof of (3).

We will prove the following claim before we prove (4).

Claim 4. If $\Delta = 4$, then G contains no 4-vertex z where z is incident with two 3-faces zux, zvy and $d(x) = d(y) = 2$.

***Proof of Claim* 4.** Suppose, on the contrary, such vertex z does exist. We can totally color the edges and vertices of $G - \{xz, yz\}$ with a set of five colors, say C, by the minimality of G. First, we erase the colors assigned on x and y. Let c_1, c_2, c_3, c_4, c_5 be colors used on xu, zu, zv, yv, z, respectively.

We will show that $c_1 \neq c_4$. Otherwise if $c_1 = c_4$, we claim that $c_1 \neq c_5$. If $c_1 = c_5$, then we can color xz by $\alpha \in C \backslash \{c_1, c_2, c_3\}$ and yz by a color in $C \backslash \{c_1, c_2, c_3, \alpha\}$. It easy to see that x and y can be colored because they are only adjacent to two vertices and incident with two edges. This implies that G can be totally colored by five colors, a contradiction. Now we show it is impossible that $c_1 = c_4$ and $c_1 \neq c_5$. If $c_1 = c_4$, then we can interchange colors c_3 and c_1 at v and color zx by c_3. It follows that we can also color zy by a color in $C \backslash \{c_1, c_2, c_3, c_5\}$. Similarly we can color vertices x and y since they are both vertices of degree 2. This implies that G can be totally colored by five colors, a contradiction.

Similarly, we can show that $c_1 \neq c_3$ and $c_1 \neq c_5$. Since $c_1 \notin \{c_2, c_3, c_4, c_5\}$, c_1 can be assigned to zy and there is a color available for zx, x and y. This implies that G can be totally colored by five colors, a contradiction. ■

To prove (4), construct a new charge $w^*(x)$ on G as follows:

R41: Each $r(\geq 15)$-face gives $1 - \frac{4}{r}$ to its incident vertices.

R42: Each 2-vertex receives $\frac{19}{24}$ from its neighbors if it is incident with a 3-face and receives $\frac{8}{15}$ from its 2-master otherwise.

R43: Each 3-face receives $\frac{1}{3}$ from its incident vertices.

It is obvious that $w^*(f) = 0$ for any face f. Let v be an arbitrary vertex of G. First consider the case of $d(v) = 2$. If it is incident with a 3-face, then its other incident face f must have degree at least 16. From (d), any neighbor of v should be of degree at least $(\Delta + 2) - 2 = 4$. Hence, they can not be 2-vertices. It follows that v receives at least $1 - \frac{4}{16} = \frac{3}{4}$ from f and $2 \times \frac{19}{24} = \frac{19}{12}$ from its neighbors, and gives $\frac{1}{3}$ to its incident 3-face. Otherwise v receives at least $2 \times \frac{11}{15} = \frac{22}{15}$ from its incident faces and $\frac{8}{15}$ from its 2-master. Hence, $w^*(v) \geq w(v) + \min\{\frac{3}{4} + \frac{19}{12} - \frac{1}{3}, \frac{22}{15} + \frac{8}{15}\} = 0$. Now consider the case of $d(v) = 3$. v receives at least $2 \times \frac{11}{15} = \frac{22}{15}$ from its incident faces. Hence, $w^*(v) = w(v) + \frac{22}{15} - \frac{1}{3} = \frac{2}{15} > 0$. If $d(v) = 4$ and it is incident with two 3-faces, then v is adjacent to at most one 2-vertex by Claim 4. It follows that $w^*(v) \geq w(v) + \frac{22}{15} - (\frac{2}{3} + \frac{19}{24}) = \frac{1}{120} > 0$. Otherwise it receives at least $3 \times \frac{11}{15}$ from its incident faces, and gives at most $\frac{1}{3}$ to its incident 3-face and $\frac{19}{24} + \frac{8}{15}$ to its adjacent 2-vertices. It follows that $w^*(v) \geq w(v) + \frac{33}{15} - (\frac{1}{3} + \frac{19}{24} + \frac{8}{15}) = \frac{13}{24} > 0$. This implies that $\sum_{x \in V \cup F} w(x) = \sum_{x \in V \cup F} w^*(x) > 0$, a contradiction. This completes the proof of (4). ∎

In the proof of the Theorem, we showed that $\sum_{x \in V \cup F} w(x) = \sum_{x \in V \cup F} w^*(x) > 0$. It implies the following corollary.

Corollary 1. *Let G be a graph with maximum degree Δ embedded in a surface of nonnegative characteristic, and G has no cycle of length from 4 to k, where $k \geq 4$. Then $\chi_T(G) = \Delta+1$ if $(\Delta, k) \in \{(7, 4), (6, 5), (5, 7), (4, 14)\}$.*

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