

**A NOTE ON MINIMALLY 3-CONNECTED GRAPHS \***

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**Abstract**

If  $G$  is a minimally 3-connected graph and  $C$  is a double cover of the set of edges of  $G$  by irreducible walks, then  $|E(G)| \geq 2|C| - 2$ .

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## 1. INTRODUCTION

A *walk*  $\alpha$  in a simple graph  $G$  is a sequence  $w_0, w_1, \dots, w_s$  of vertices of  $G$ , not necessarily different, such that  $w_{i-1}w_i$  is an edge of  $G$  for  $i = 1, 2, \dots, s$ . An edge  $e$  of  $G$  is said to be *traversed* in a walk  $\alpha$  if its vertices are consecutive in  $\alpha$ ; an edge may be traversed more than once in a given walk.

A walk  $\alpha$  in a graph  $G$  is *irreducible* if  $a \neq b$  for every pair  $a, b$  of edges which are traversed consecutively in  $\alpha$ . A set  $C$  of irreducible closed walks in a graph  $G$  is a *walk double cover* of  $G$  if each edge of  $G$  is traversed exactly two times, either once in two different walks in  $C$  or twice in the same walk in  $C$ .

For any simple graph  $G$  and any edge  $e = uv$  of  $G$  we denote by  $G - e$  the graph obtained from  $G$  by deleting the edge  $e$ , and by  $G \cdot e$  the simple graph obtained from  $G$  by identifying the vertices  $u$  and  $v$  and deleting loops and multiple edges. A *minimally 3-connected* graph is a 3-connected graph  $G$  such that, for every edge  $e$  of  $G$ , the graph  $G - e$  is no longer 3-connected.

Whenever possible we follow the terms and notation given in [1]. A *wheel*  $W_t$  is a graph with  $t + 1$  vertices, obtained from a cycle  $C_t$  with  $t$  vertices by adding a new vertex  $w$  adjacent to each vertex in  $C_t$ . The cycle  $C_t$  and the vertex  $w$  are called the rim and the hub of  $W_t$ , respectively. In this note we prove the following result.

**Theorem 1.1.** *Let  $G$  be a minimally 3-connected graph with  $m$  edges. If  $C$  is a walk double cover of  $G$  with  $k$  walks, then  $m \geq 2k - 2$ . Moreover if  $m \leq 2k - 1$ , then  $G$  is a planar graph and  $C$  is the set of planar faces of  $G$ ; in particular if  $m = 2k - 2$ , then  $G$  is a wheel.*

## 2. PROOF OF THEOREM 1.1

The following result due to R. Halin [2] will be used in the proof of Theorem 1.1.

**Theorem 2.1.** *If  $e = uv$  is an edge of a minimally 3-connected graph  $G$  with  $\min\{d(u), d(v)\} \geq 4$ , then  $e$  lies in no cycle of  $G$  of length 3 and  $G \cdot e$  is also minimally 3-connected.*

For any graph  $G$  and any walk double cover  $C$  of  $G$ , we denote by  $m(G)$  and by  $k(C)$  the number of edges of  $G$  and the number of walks in  $C$ , respectively.

**Remark 1.** Let  $G$  be a 3-connected graph and  $C$  be a walk double cover of  $G$ . If two edges  $uw$  and  $wv$  are consecutive edges in two walks in  $C$ , then the degree of  $w$  is at least 4.

**Proof of Theorem 1.1.** The smallest 3-connected graph is the wheel  $W_3$  which is planar and has 6 edges. Since each irreducible walk has at least 3 edges, no walk double cover of  $W_3$  has more than 4 walks. Moreover, the only walk double cover of  $W_3$  with 4 walks consists of the planar faces of  $W_3$ .

We proceed by induction assuming  $m \geq 7$  and that the result holds for every minimally 3-connected graph with less than  $m$  edges.

If  $G$  has an edge  $e = uv$  with  $\min\{d(u), d(v)\} \geq 4$ , then by Halin's theorem,  $G \cdot e$  is also minimally 3-connected. Let  $C \cdot e$  denote the set of walks of  $G \cdot e$  obtained from the walks in  $C$  by contracting the edge  $e$ .

Also by Halin's theorem, the edge  $e$  lies in no cycle of  $G$  of length 3; this implies that all walks in  $C \cdot e$  are irreducible. Because  $C$  is a walk double cover of  $G$  and  $e$  is not an edge of  $G \cdot e$ ,  $C \cdot e$  is a walk double cover of  $G \cdot e$ . By induction,  $m(G \cdot e) \geq 2k(C \cdot e) - 2$ ; therefore  $m \geq 2k - 1$ , since  $m(G \cdot e) = m - 1$  and  $k(C \cdot e) = k$ .

If  $m = 2k - 1$ , then  $m(G \cdot e) = 2k(C \cdot e) - 2$ ; by induction  $G \cdot e$  is a wheel  $W_t$  and  $C \cdot e$  is the set of planar faces of  $W_t$ . Let  $x$  be the vertex of  $W_t$  obtained by identifying  $u$  and  $v$ . Since  $u$  and  $v$  have degree at least 4 in  $G$ , the vertex  $x$  must be the hub of  $W_t$ ; let  $w_0, w_1, \dots, w_{t-1}$  be the rim of  $W_t$ .

Since  $e$  is in no cycle of  $G$  of length 3,  $G$  is a graph consisting of the cycle  $w_0, w_1, \dots, w_{t-1}$ , the two adjacent vertices  $u$  and  $v$ , and one edge joining each vertex  $w_i$  to either  $u$  or  $v$ .

Suppose there are distinct integers  $a, b$  and  $c$  such that  $w_a, w_{b+1}$  and  $w_c$  are adjacent to  $u$  in  $G$  and  $w_{a+1}, w_b$  and  $w_{c+1}$  are adjacent to  $v$  in  $G$ . The walks  $w_a, x, w_{a+1}$ ,  $w_b, x, w_{b+1}$  and  $w_c, x, w_{c+1}$  lie in  $C$ , since they are faces of  $G \cdot e$ . This implies that  $w_a, u, v, w_{a+1}$ ,  $w_b, v, u, w_{b+1}$  and  $w_c, u, v, w_{c+1}$  are walks in  $C$  which is not possible, since the edge  $e = uv$  cannot lie in three walks in  $C$ .

Therefore there are integers  $i$  and  $j$  such that  $w_i, w_{i+1}, \dots, w_{j-1}$  are adjacent to  $u$  in  $G$  and  $w_j, w_{j+1}, \dots, w_{i-1}$  are adjacent to  $v$  in  $G$ . This shows that  $G$  is a planar graph.

Since  $C \cdot e$  is the set of faces of  $G \cdot e = W_t$  and each walk in  $C \cdot e$  is either a walk in  $C$  or is obtained from a walk in  $C$  by contracting the edge  $e$ , the set  $C$  must be the set of faces of  $G$ .

We can now assume that each edge of  $G$  has at least one end with degree 3. If  $C$  contains no cycle of length 3, then  $2m \geq 4k$  and  $m \geq 2k$ . Therefore we can also assume that  $C$  contains at least one cycle of length 3. Let  $C_3$  be the set of cycles in  $C$  of length 3; two cases are considered.

*Case 1.* There is a cycle  $\alpha$  in  $C_3$  such that no pair of edges of  $\alpha$  are traversed consecutively in any other walk in  $C$ .

Let  $u, v$  and  $w$  be the vertices of  $\alpha$ . Since each edge of  $G$  has an end with degree 3, without loss of generality, we can assume  $d_G(u) = d_G(v) = 3$ . Let  $u_1$  and  $v_1$  denote the third vertex of  $G$  adjacent to  $u$  and the third vertex of  $G$  adjacent to  $v$ , respectively; notice that  $u_1 \neq v_1$ , since  $G$  is 3-connected and has at least 5 vertices.

*Subcase 1.1.* If  $d_G(w) = 3$ , let  $w_1$  denote the third vertex of  $G$  adjacent to  $w$ ; as above  $u_1 \neq w_1 \neq v_1$ . Let  $G'$  be the graph obtained from  $G$  by contracting the cycle  $\alpha$  to a single point  $x$ . We claim that  $G'$  can also be obtained from  $G$  by a *delta to wye* transformation (see Figure 1), and therefore it is also a 3-connected graph.

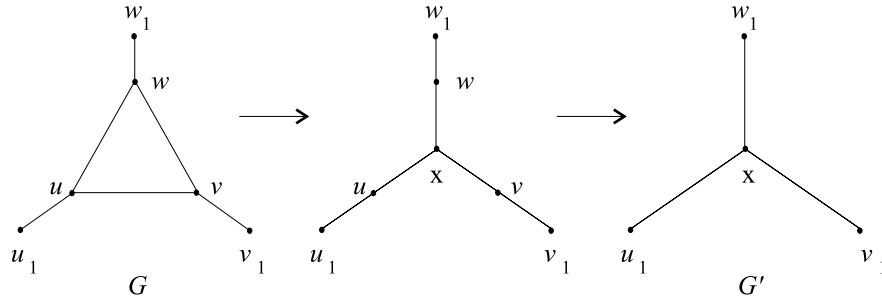


Figure 1

Since  $d_{G'}(x) = 3$  and  $d_{G'}(z) = d_G(z)$  for each vertex  $z \neq x$  of  $G'$ , every edge of  $G'$  has an end with degree 3; therefore  $G'$  is minimally 3-connected.

Let  $C'$  be the set of  $k - 1$  walks of  $G'$  obtained from the walks in  $C \setminus \{\alpha\}$  by contracting the edges  $uv, vw$  and  $wu$ . Since no pair of edges of  $\alpha$  are consecutive edges in any walk in  $C \setminus \{\alpha\}$ , all walks in  $C'$  are irreducible. Moreover,  $C'$  is a walk double cover of  $G'$ , since  $C$  is a walk double cover of  $G$  and  $uv, vw$  and  $wu$  are not edges of  $G'$ .

By induction  $m(G') \geq 2k(C') - 2$ ; hence  $m \geq 2k - 1$ , since  $m(G') = m - 3$  and  $k(C') = k - 1$ . If  $m = 2k - 1$ , then  $m(G') = 2k(C') - 2$ . Again by

induction  $G \cdot e$  is a wheel  $W_t$  and  $C'$  is the set of planar faces of  $W_t$ . Since  $x$  has degree 3 in  $G'$ , we can assume without loss of generality that  $x$  lies in the rim of  $G' = W_t$  and that  $w_1$  is the hub; this implies that  $G$  is a graph as in Figure 2 and therefore it is a planar graph in which  $\alpha$  is a face.

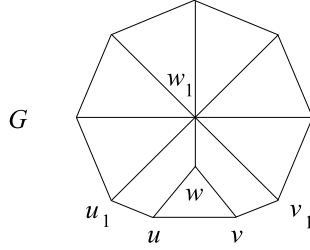


Figure 2

Since  $C'$  is the set of faces of  $G'$  and every walk in  $C'$  is either a walk in  $C \setminus \{\alpha\}$  or is obtained from a walk in  $C \setminus \{\alpha\}$  by contracting some of the edges  $uv$ ,  $vw$  and  $wu$ , the set  $C$  must be the set of planar faces of  $G$ .

*Subcase 1.2.* If  $d_G(w) \geq 4$ , we consider the graph  $G \cdot uv$ . We claim that  $u$  and  $v$  cannot be contained in a 3-vertex cut of  $G$  and, therefore,  $G \cdot uv$  is 3-connected.

Since  $d_{G \cdot uv}(x) = 3$  and  $d_{G \cdot uv}(z) \leq d_G(z)$  for each vertex  $z \neq x$  of  $G \cdot uv$ , every edge of  $G \cdot uv$  has an end with degree 3; therefore  $G \cdot uv$  is minimally 3-connected.

Let  $C \cdot uv$  be the set of  $k - 1$  walks of  $G \cdot uv$  obtained from the walks in  $C \setminus \{\alpha\}$  by contracting the edge  $uv$  to a vertex  $x$  and substituting each of the edges  $uw$  and  $vw$  by the edge  $xw$ . Each walk in  $C \cdot uv$  is irreducible, because no pair of edges of  $\alpha$  are traversed consecutively in any other walk in  $C$ . Since  $C$  is a walk double cover of  $G$  and  $uv$  is not an edge of  $G \cdot uv$ , the set  $C \cdot uv$  is a walk double cover of  $G \cdot uv$ .

By induction  $m(G \cdot uv) \geq 2k(C \cdot uv) - 2$ ; hence  $m \geq 2k - 2$ , since  $m(G \cdot uv) = m - 2$  and  $k(C \cdot uv) = k - 1$ . If  $m \leq 2k - 1$ , then  $m(G \cdot uv) \leq 2k(C \cdot uv) - 1$ ; again by induction,  $G \cdot uv$  is a planar graph and  $C \cdot uv$  is the set of planar faces of  $G \cdot uv$ .

Since  $G \cdot uv$  is 3-connected, there is a planar drawing  $\overline{G \cdot uv}$  of  $G \cdot uv$  in which  $x$  is an interior vertex. Let  $R$  be the region formed by the three faces of  $\overline{G \cdot uv}$  in which  $x$  is a vertex. Since  $w$ ,  $u_1$  and  $v_1$  lie in the boundary of  $R$

and  $x$  is in the interior of  $R$ , a planar drawing  $\overline{G}$  of  $G$  can be obtained from  $\overline{G \cdot uv}$  by replacing (within the interior of  $R$ ) the vertex  $x$  with two adjacent vertices  $u$  and  $v$ , and the edges  $wx$ ,  $u_1x$  and  $v_1x$  with the edges  $wu$ ,  $wv$ ,  $u_1u$  and  $v_1v$  as in Figure 3.

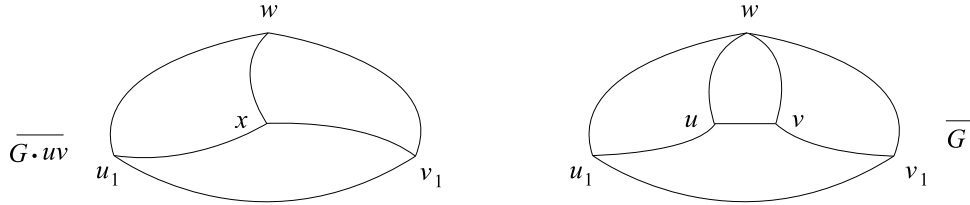


Figure 3

Therefore  $G$  is a planar graph and  $\alpha$  is a face of  $G$ . Furthermore,  $C$  is the set of faces of  $G$ , since  $C \cdot uv$  is the set of planar faces of  $G \cdot uv$  and each walk in  $C \cdot uv$  is either a walk in  $C \setminus \{\alpha\}$  or is obtained from a walk in  $C \setminus \{\alpha\}$  by contracting the edge  $uv$  to the vertex  $x$  and substituting each of the edges  $uw$  and  $vw$  by the edge  $xw$ .

If  $m = 2k - 2$ , then  $m(G \cdot uv) = 2k(C \cdot uv) - 2$ ; again by induction,  $G \cdot uv$  is a wheel  $W_t$ . Since  $d_{G \cdot uv}(x) = 3$ , we can assume that  $x$  lies in the rim of  $G \cdot uv$ .

If  $w$  is the hub of  $G \cdot uv$ , then  $G$  is the wheel  $W_{t+1}$ , also with hub  $w$ . If  $u_1$  is the hub of  $G \cdot uv$ , then  $G$  is a graph as in Figure 4. Notice that if  $t > 3$ , then  $G - u_1w$  is 3-connected which is not possible since  $G$  is minimally 3-connected. Therefore  $t = 3$  and  $G$  is the wheel  $W_4$  with hub  $w$ . Analogously, if  $v_1$  is the hub of  $G \cdot uv$ , then  $G$  is the wheel  $W_4$ .

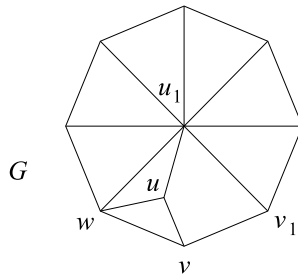


Figure 4

*Case 2.* For every cycle  $\alpha \in C_3$  there is walk  $\sigma_\alpha \neq \alpha$  in  $C$  such that two edges of  $\alpha$  are traversed consecutively in  $\sigma_\alpha$ .

For this case, we shall prove that the average length of the walks in  $C$  is at least 4 and therefore  $2m \geq 4k$  and  $m \geq 2k$ .

For each  $\alpha \in C_3$  let  $u_\alpha, w_\alpha$  and  $v_\alpha$  denote the vertices of  $\alpha$ . Without loss of generality we assume that  $u_\alpha w_\alpha$  and  $w_\alpha v_\alpha$  are traversed consecutively in  $\sigma_\alpha$ . Notice that the walk  $\sigma_\alpha$  is uniquely determined since  $C$  is a walk double cover of  $G$ .

By Remark 1,  $d_G(w_\alpha) \geq 4$ ; therefore  $d_G(u_\alpha) = d_G(v_\alpha) = 3$ , since every edge of  $G$  has an end with degree 3. Let  $u'_\alpha$  and  $v'_\alpha$  denote the third vertex of  $G$  adjacent to  $u_\alpha$  and the third vertex of  $G$  adjacent to  $v_\alpha$ , respectively.

Again by Remark 1, the edges  $w_\alpha u_\alpha$  and  $u_\alpha v_\alpha$  are not traversed consecutively in  $\sigma_\alpha$ ; therefore  $\sigma_\alpha$  must traverse the edge  $u_\alpha u'_\alpha$ ; analogously  $\sigma_\alpha$  traverses the edge  $v_\alpha v'_\alpha$ . If  $u'_\alpha = v'_\alpha$ , then  $u_\alpha$  and  $v_\alpha$  are adjacent only to  $u'_\alpha = v'_\alpha$ , to  $w_\alpha$  and to each other which is not possible since  $G$  is a 3-connected graph with at least 5 vertices; therefore  $\sigma_\alpha$  has length at least 5 for each  $\alpha \in C_3$ . For each  $\tau \in C$  let  $l(\tau)$  denote the length of  $\tau$ .

Consider the equivalence relation in  $C_3$  given by  $\beta \sim \gamma$  if and only if  $\sigma_\beta = \sigma_\gamma$ . For  $\alpha \in C_3$  let  $[\alpha]$  denote the equivalence class of  $\alpha$ .

Let  $\beta$  and  $\gamma$  be two distinct cycles in  $[\alpha]$  and assume, without loss of generality, that the edges  $u_\beta w_\beta, w_\beta v_\beta, u_\gamma w_\gamma$  and  $w_\gamma v_\gamma$  are traversed in  $\sigma_\alpha = \sigma_\beta = \sigma_\gamma$  in that relative order. The edges  $u_\beta w_\beta$  and  $w_\beta v_\beta$  are not edges of  $\gamma$  since they are traversed in  $\beta$  and in  $\sigma_\beta \neq \beta$ ; analogously  $u_\gamma w_\gamma$  and  $w_\gamma v_\gamma$  are not edges of  $\beta$ .

Suppose that  $w_\beta v_\beta$  and  $u_\gamma w_\gamma$  are traversed consecutively in  $\sigma_\alpha$ . Then  $v_\beta = u_\gamma$  and  $w_\beta \neq w_\gamma$ , since  $\sigma_\alpha$  is an irreducible walk. Moreover,  $u_\beta = v_\gamma$  since  $d_G(v_\beta = u_\gamma) = 3$  and  $w_\beta, w_\gamma, u_\beta$  and  $v_\gamma$  are all adjacent to  $v_\beta = u_\gamma$ . This implies that the vertices  $v_\beta = u_\gamma$  and  $u_\beta = v_\gamma$  are adjacent in  $G$  only to  $w_\beta$ , to  $w_\gamma$  and to each other which is not possible since  $G$  is 3-connected and has at least 5 vertices.

Therefore, no edges of two distinct cycles in  $[\alpha]$  are traversed consecutively in  $\sigma_\alpha$ . This implies that  $\sigma_\alpha$  has at least  $3|[\alpha]|$  edges.

By the above arguments

$$\frac{l(\sigma_\alpha) + l(\alpha)}{2} \geq \frac{5 + 3}{2} = 4$$

for each  $\alpha \in C_3$  with  $|\alpha| = 1$ , and

$$\frac{l(\sigma_\alpha) + \sum_{\beta \in [\alpha]} l(\beta)}{|\alpha| + 1} \geq \frac{3|\alpha| + 3|\alpha|}{|\alpha| + 1} = \frac{6|\alpha|}{|\alpha| + 1} \geq 4$$

for each  $\alpha \in C_3$  with  $|\alpha| \geq 2$ .

Since all walks in  $C$  which are not in  $C_3$  have length at least 4, the average length in  $C$  must also be at least 4.

**Corollary 2.2.** *Let  $G$  be a minimally 3-connected graph with  $n$  vertices. If  $C$  is a walk double cover of  $G$  with  $k$  walks, then  $k \leq \frac{3n-4}{2}$ .*

**Proof.** Let  $m$  denote the number of edges in  $G$ . W. Mader proved in [3] that  $m \leq 3n - 6$ ; by Theorem 1.1,  $k \leq \frac{m+2}{2} \leq \frac{(3n-6)+2}{2} = \frac{3n-4}{2}$ . ■

**Corollary 2.3.** *If  $G$  is a minimally 3-connected planar graph with  $n$  vertices, then  $G$  has at most  $n$  faces. Moreover if  $G$  has exactly  $n$  faces, then  $G$  is a wheel.*

**Proof.** Since  $G$  is 3-connected, its set of faces is a walk double cover. By Theorem 1.1,  $m \geq 2r - 2$ , where  $m$  and  $r$  are the number of edges and faces of  $G$ , respectively. Since  $n - m + r = 2$ , it follows  $r \leq n$ .

Also by Theorem 1.1, if  $G$  is not a wheel, then  $m \geq 2r - 1$ , in which case  $r \leq n - 1$ . ■

**Corollary 2.4.** *If  $G$  is a minimally 3-connected graph with  $n$  vertices embedded in a closed surface  $S$  with Euler characteristic  $\chi \neq 2$ , then  $G$  has at most  $n - \chi$  faces.*

**Proof.** As in Corollary 2.3, the set of faces of  $G$  is a walk double cover of  $G$ . Since  $S$  is not the sphere,  $C$  is not the set of planar faces of  $G$ . By Theorem 1.1,  $m \geq 2r$ , where  $m$  and  $r$  are the number of edges and faces of  $G$ , respectively. Since  $\chi = n - m + r$ , it follows  $r \leq n - \chi$ . ■

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