Discussiones Mathematicae
Graph Theory 24 (2004) 85–107

LIGHT CLASSES OF GENERALIZED STARS IN POLYHEDRAL MAPS ON SURFACES

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Dedicated to Professor Hanjo Walther on the occasion of his 60th birthday

Abstract

A generalized s-star, $s \geq 1$, is a tree with a root Z of degree s; all other vertices have degree ≤ 2. S_i denotes a generalized 3-star, all three maximal paths starting in Z have exactly $i+1$ vertices (including Z). Let $\mathbb{M}$ be a surface of Euler characteristic $\chi(\mathbb{M}) \leq 0$, and $m(\mathbb{M}) := \lfloor \frac{5+\sqrt{49-24\chi(\mathbb{M})}}{2} \rfloor$. We prove:

(1) Let $k \geq 1, d \geq m(\mathbb{M})$ be integers. Each polyhedral map G on $\mathbb{M}$ with a k-path (on k vertices) contains a k-path of maximum degree $\leq d$ in G or a generalized s-star $T, s \leq m(\mathbb{M})$, on $d+2-m(\mathbb{M})$ vertices with root Z, where Z has degree $\leq k \cdot m(\mathbb{M})$ and the maximum degree of $T \setminus \{Z\}$ is $\leq d$ in G. Similar results are obtained for the plane and for large polyhedral maps on $\mathbb{M}$.

(2) Let k and i be integers with $k \geq 3, 1 \leq i \leq \frac{k}{2}$. If a polyhedral map G on $\mathbb{M}$ with a large enough number of vertices contains a k-path then G contains a k-path or a 3-star S_i of maximum degree $\leq 4(k+i)$ in G. This bound is tight. Similar results hold for plane graphs.

Keywords: polyhedral maps, embeddings, light subgraphs, path, star, 2-dimensional manifolds, surface.

2000 Mathematics Subject Classification: 05C10, 05C75, 52B10.

1. Introduction

In this paper all manifolds are compact 2-dimensional manifolds. We shall consider graphs without loops and multiple edges. Multigraphs can have multiple edges and loops. If a multigraph G is embedded in a manifold $\mathbb{M}$ then the connected components of $\mathbb{M} - G$ are called the *faces* of G. If each face is an open disc then the embedding is called a 2-*cell embedding.* If each vertex of a 2-cell embedding has degree ≥ 3 and each vertex of degree h is incident with h different faces then G is called a *map* in $\mathbb{M}$. If, in addition, G is 3-connected and the embedding has representativity at least three, then G is called a *polyhedral map* in $\mathbb{M}$, see e.g. Robertson and Vitray [19] or Mohar [17]. Let us recall that the *representativity* $\text{rep}(G, \mathbb{M})$ (or the *face width*) of a (2-cell) embedded graph G into a compact 2-manifold $\mathbb{M}$ is equal to the smallest number k such that $\mathbb{M}$ contains a noncontractible closed curve that intersects the graph G in k points.

Let $\mathbb{S}_g$ ($\mathbb{N}_q$) be an orientable (a non-orientable) compact 2-dimensional manifold (called also a surface , see [18]) of genus g (q, respectively). Let us recall that the relationship between Euler characteristic and the genus of a surface is the following

$$\chi(\mathbb{S}_g) = 2 - 2g \quad \text{and} \quad \chi(\mathbb{N}_q) = 2 - q.$$

We say that H is a *subgraph* of a polyhedral map G if H is a subgraph of the underlying graph of the map G.

The boundary of a face α of an embedded graph consists of all vertices and edges incident with α. Note that the boundary of α can be disconnected. Let $D_1, D_2, \cdots, D_s$ be the components of the boundary of α. Let W_i be the shortest closed walk induced by all edges of D_i, and let $\partial(W_i)$ be its length, i.e., the number of edges met at the walk W_i (edges met twice are counted twice). The degree of a face α is

$$\deg_G(\alpha) = \sum_{i=1}^{s} \partial(W_i).$$

Hence the *degree* $\deg_G(\alpha)$ of a face α of a 2-cell embedding is the length of its facial walk. Vertices and faces of degree i are called i-vertices and i-faces, respectively. The number of i-vertices and j-faces in a map is denoted by v_i and f_j, respectively. For a map G let $V(G), E(G)$ and $F(G)$ be the vertex set, the edge set and the face set of G, respectively. The degree of a vertex A in G is denoted by $\deg_G(A)$ or $\deg(A)$ if G is known from the context. A path and a cycle on k distinct vertices is defined to be the *k-path* and the *k-cycle*, respectively. P_k will denote a k-path. The *length* of a path or a cycle is the number of its edges.

A generalized s-star, $s \geq 1$, is a tree with a root Z of degree s; all other vertices have degree ≤ 2. The maximal paths starting in Z are called beams. The symbol $S_i, i \geq 0$, denotes a generalized 3-star, all three beams of it are paths with $i+1$ vertices (including the root). Obviously, $S_0 = K_1$, and $S_1 = K_{1,3}$.

It is a consequence of Euler's formula that each planar graph contains a vertex of degree at most 5. It is well known that any graph embedded in a surface $\mathbb{M}$ with Euler characteristic $\chi(\mathbb{M})$ has minimum degree

(1) $\delta(G) \leq \left\lfloor \frac{5+\sqrt{49-24\chi(\mathbb{M})}}{2} \right\rfloor =: m(\mathbb{M})$, if $\mathbb{M} \neq \mathbb{S}_0$, and
$\delta(G) \leq 5 =: m(\mathbb{S}_0)$, where $\mathbb{S}_0$ is the sphere.

(For a proof see e.g. Sachs [20], p. 227).

A further consequence of Euler's formula is

$$\sum_{A \in V(G)} (\deg(A) - 6) + 2 \sum_{\alpha \in F(G)} (\deg(\alpha) - 3) = 6(-\chi(\mathbb{M})).$$

For any graph G embedded in a surface $\mathbb{M}$ of Euler characteristic $\chi(\mathbb{M}) \leq 0$ this implies

(2) if $\sum_{\deg(A)>6}(\deg(A) - 6) > 6|\chi(\mathbb{M})|$ then $\delta(G) \leq 5$, and

(3) if G has more than $6|\chi(\mathbb{M})|$ vertices then $\delta(G) \leq 6$.

A theorem of Kotzig [15] states that every 3-connected planar graph contains an edge with degree-sum of its endvertices being at most 13. This result was further developed in various directions and served as a starting point for discovering many structural properties of embeddings of graphs. For example Ivančo [6] has proved that every polyhedral map on $\mathbb{S}_g$ contains an edge with degree sum of their end vertices being at most $2g+13$ if $0 \le g \le 3$ and at most $4g+7$, if $g \ge 4$. The bounds are best possible. For other results in this topic see e.g. [1, 4, 14, 21].

2. The General Problem

In the past subgraphs have been investigated which are light in a family of graphs (see our survey article [14]). There we have generalized this concept to a light class $\mathcal{L}$ of subgraphs in a family $\mathcal{H}$ of graphs.

Problem. Let $\mathcal{H}$ be a family of graphs and $\mathcal{L}$ be a finite class of connected graphs having the property that every member of $\mathcal{L}$ is isomorphic to a proper subgraph of at least one member of $\mathcal{H}$. Let $\varphi(\mathcal{L},\mathcal{H})$ be the smallest integer with the property that every graph $G \in \mathcal{H}$, which has a subgraph isomorphic with a member of $\mathcal{L}$, also contains a subgraph $K, K \simeq H, H \in \mathcal{L}$, such that for every vertex $A \in V(K)$

$$\deg_G(A) \le \varphi(\mathcal{L},\mathcal{H}).$$

If such a $\varphi(\mathcal{L},\mathcal{H})$ does not exist we write $\varphi(\mathcal{L},\mathcal{H}) = +\infty$. If $\varphi(\mathcal{L},\mathcal{H}) < +\infty$ we call the class $\mathcal{L}$ *light in the family* $\mathcal{H}$. Obviously, if $\mathcal{L}' \subseteq \mathcal{L}$ then $\varphi(\mathcal{L},\mathcal{H}) \le \varphi(\mathcal{L}',\mathcal{H})$. The corresponding problem of a light subgraph H is again obtained if $\mathcal{L} = \{H\}$ is chosen. In this case let $\varphi(\{H\},\mathcal{H}) = \varphi(H,\mathcal{H})$.

3. Results

A. Polyhedral maps

Let $\mathcal{G}(\delta,\rho;\mathbb{M})$ denote the set of all polyhedral maps on the surface $\mathbb{M}$ of Euler characteristic $\chi(\mathbb{M})$ having minimum vertex degree at least δ and minimum face degree at least ρ. The following theorem has been proved for the planes $\mathbb{S}_0$ and $\mathbb{N}_1$ by Fabrici and Jendrol' [1] and for 2-dimensional manifolds $\mathbb{M}$ other than the planes by Jendrol' and Voss [8].

Theorem 1 ([1], [8]). *Let k be an integer, $k \ge 1$, and $\mathbb{M}$ a surface with Euler characteristic $\chi(\mathbb{M})$. Then*

(i) $\varphi(P_k, \mathcal{G}(3,3;\mathbb{S}_0)) = \varphi(P_k, \mathcal{G}(3,3;\mathbb{N}_1)) = 5k$,

(ii) $2\lfloor \frac{k}{2} \rfloor \cdot m(\mathbb{M}) \leq \varphi(P_k, \mathcal{G}(3,3;\mathbb{M})) \leq k \cdot m(\mathbb{M})$, *if* $\mathbb{M} \notin \{\mathbb{S}_0, \mathbb{N}_1\}$,

(iii) $\varphi(H, \mathcal{G}(3,3;\mathbb{M})) = \infty$ *for any connected graph* $H \neq P_k$.

By the same arguments used in the proof of (iii) for the sphere $\mathbb{S}_0$ and the projective plane $\mathbb{N}_1$ by Fabrici and Jendrol' [1] it can be proved that a class $\mathcal{L}$ of plane graphs is light in $\mathcal{G}(3,3;\mathbb{M})$ if and only if $\mathcal{L}$ contains a path. So, if $\mathcal{L}$ contains P_k then $\varphi(\mathcal{L}, \mathcal{G}(3,3;\mathbb{M})) \leq \varphi(P_k, \mathcal{G}(3,3;\mathbb{M}))$. We will study how small can $\varphi(\mathcal{L}, \mathcal{G}(3,3;\mathbb{M}))$ be if besides P_k the class $\mathcal{L}$ contains some trees different from P_k.

The class $\mathcal{T}_k$ of all trees of order k contains a k-path. Obviously, $\varphi(\mathcal{T}_k, \mathcal{G}(3,3;\mathbb{M})) = \varphi(P_k, \mathcal{G}(3, 3;\mathbb{M}))$ for $k \in \{1,2,3\}$. For the sphere $\mathbb{S}_0$ Fabrici and Jendrol' [2] and for each surface $\mathbb{M}, \mathbb{M} \neq \mathbb{S}_0$, Jendrol' and Voss [13] proved

Theorem 2 ([2], [13])**.** *Let* k *be an integer,* $k \geq 4$*, and* $\mathbb{M}$ *a surface with Euler characteristic* $\chi(\mathbb{M})$*. Then*

(i) $\varphi(\mathcal{T}_k, \mathcal{G}(3,3;\mathbb{S}_0)) = \varphi(\mathcal{T}_k, \mathcal{G}(3,3;\mathbb{N}_1)) = 4k+3$,

(ii) $\left\lfloor \frac{2k+2}{3} \right\rfloor \left(\left\lfloor \frac{5+\sqrt{49-24\chi(\mathbb{M})}}{2} \right\rfloor - \frac{3}{2} \right) \leq \varphi(\mathcal{T}_k, \mathcal{G}(3,3;\mathbb{M})) \leq$
$\left\lfloor (k+1)\frac{5+\sqrt{49-24\chi(\mathbb{M})}}{3} \right\rfloor$ *if* $\mathbb{M} \notin \{\mathbb{S}_0, \mathbb{N}_1\}$.

In the Theorem 1(i) not all vertices of a P_k must have the degree $5k$. Really, Madaras [16] improved Theorem 1(i) by showing

Theorem 3 ([16])**.** *Let* k *be an integer,* $k \geq 2$*. Then each map of* $\mathcal{G}(3,3;\mathbb{S}_0)$ *containing a path* P_k *has also a path* P_k *such that one vertex has a degree* $\leq 5k$ *and all other* $k-1$ *vertices have a degree* $\leq \frac{5k}{2}$*.*

Let $\mathbb{M}$ be a surface of Euler characteristic $\chi(\mathbb{M})$ and $m(\mathbb{M})$ as defined in (1). Using the arguments of Madaras [16] we can show that G contains at least one tree from a family of specified trees with given degree constraints.

Theorem 4. *Let* $\mathbb{M}$ *be a surface of Euler characteristic* $\chi(\mathbb{M})$*. Let* $k \geq 1$ *and* $d \geq m(\mathbb{M})$ *be integers. Let* $G \in \mathcal{G}(3,3;\mathbb{M})$ *contain a* k*-path. Then* G *contains at least one of the following subgraphs:*

(i) *a* k*-path of maximum degree* $\leq d$ *in* G*, or*

(ii) *a generalized s-star $T, s \leq m(\mathbb{M})$, on $d + 2 - m(\mathbb{M})$ vertices with root Z, where Z has a degree $\leq k \cdot m(\mathbb{M})$ in G and the maximum degree of $T \setminus \{Z\}$ is $\leq d$ in G.*

The generalized star T contains a path with $2\frac{d+1-m(\mathbb{M})}{m(\mathbb{M})} + 1$ vertices.

If $d = \lfloor \frac{k}{2} m(\mathbb{M}) \rfloor$ then the generalized star T contains a P_k. Hence Theorem 4 implies the validity of the following result.

Theorem 5. *Let $\mathbb{M}$ be a surface of Euler characteristic $\chi(\mathbb{M})$ and $k \geq 1$ an integer. Then each map $G \in \mathcal{G}(3,3;\mathbb{M})$ containing a k-path, has a k-path P_k with the property: besides one vertex Z all vertices have a degree $\leq \frac{k}{2} \cdot m(\mathbb{M})$ and the vertex Z has a degree $\leq k \cdot m(\mathbb{M})$ in G.*

For the sphere $m(\mathbb{S}_0) = 5$ holds and Theorem 5 implies the validity of Theorem 3. If $d = k \cdot m(\mathbb{M})$ then the generalized star T contains a P_k not meeting the root of T. Hence Theorem 4 implies the validity of the upper bound in Theorem 1. Interesting special variants of Theorem 4 are also obtained for $d = k$ and $d = k + 4$.

B. Large polyhedral maps

Let $\chi(\mathbb{M}) \leq 0$ throughout section B.

For large maps of $\mathcal{G}(3,3;\mathbb{M})$ we await a smaller bound for the maximum degree of light paths. A *large* polyhedral map is one with a large number of vertices or a large positive charge. A positive *k-charge $ch_k(G)$* is defined $ch_k(G) := \sum_{\deg_G(A)>6k}(\deg_G(A) - 6k)$. Let $\mathcal{G}(3,3;\mathbb{M}, n(a))$ and $\mathcal{G}(3,3;\mathbb{M}, c_k(b))$ denote the sets of the graphs G of $\mathcal{G}(3,3;\mathbb{M})$ with $> a$ vertices or a k-charge $ch_k(G) > b$, respectively. Let b_k denote the largest number of vertices in a connected graph with maximum degree $\leq 6k$ containing no path of k vertices. Obviously, $b_k \leq (6k)^{k/2+2}$.

Let $l_k(\mathbb{M}) := 3 \cdot 10^4(|\chi(\mathbb{M})| + 1)^3(b_k + 3(|\chi(\mathbb{M})| + 1))$. We have proved

Theorem 6 ([9], [10]). *For any surface $\mathbb{M}$ with Euler characteristic $\chi(\mathbb{M}) \leq 0$, any integer $k \geq 1$, any integer $a > l_k(\mathbb{M})$ and any integer $b > 6k|\chi(\mathbb{M})|$,*

(i) $\varphi(P_k, \mathcal{G}(3,3;\mathbb{M}, n(a))) = \begin{cases} 6k, & \text{if } k = 1 \text{ or } k \text{ is even,} \\ 6k-2, & \text{if } k \geq 3 \text{ is odd,} \end{cases}$

(ii) $\varphi(P_k, \mathcal{G}(3,3;\mathbb{M}, c_k(b))) = 5k,$

(iii) $\varphi(H, \mathcal{G}(3,3;\mathbb{M}, n(a))) = \varphi(H, \mathcal{G}(3,3;\mathbb{M}, c_k(a))) = \infty$ *for any* $H \not\cong P_k$ *and any* a.

In [9] we could show that $\varphi(P_k, \mathcal{G}(3,3;\mathbb{M}, n(a))) \leq 6k$ even for the smaller bound $a > (14(k-1)b_k + 6)|\chi(\mathbb{M})|$. For the class $\mathcal{T}_k$ of all trees of order k we could prove [11]

Theorem 7 ([11]). *For any surface* $\mathbb{M}$ *with Euler characteristic* $\chi(\mathbb{M}) \leq 0$, *any integer* $k \geq 3$, *and any integer* $a > (8k^2 + 6k - 6)|\chi(\mathbb{M})|$

(i) $\varphi(\mathcal{T}_k, \mathcal{G}(3,3;\mathbb{M}, n(a))) = 4k+4$, *and*

(ii) $\varphi(\mathcal{T}_k, \mathcal{G}(3,3;\mathbb{M}, c_k(a))) = 4k+3$.

With the arguments of Madaras [16] we will prove: for the graphs of $\mathcal{G}(3,3;\mathbb{M}, n(a))$ and $\mathcal{G}(3,3;\mathbb{M}, c_k(b))$ with large a and b the Theorem 4 is again valid, if $m(\mathbb{M})$ is replaced by 6 or 5, respectively.

Theorem 8. *Let* $\mathbb{M}$ *be a surface of Euler characteristic* $\chi(\mathbb{M}) \leq 0$. *Let* k, d *and* a, b *be integers with* $k \geq 1, d \geq 6$, $a > (14(k-1)b_k + 6)|\chi(\mathbb{M})|$, *and* $b > 6k|\chi(\mathbb{M})|$. *Let* $G_1 \in \mathcal{G}(3,3;\mathbb{M}, n(a))$ *and* $G_2 \in \mathcal{G}(3,3;\mathbb{M}, c_k(b))$ *contain a* k*-path. Let* $m_1 := 6$ *and* $m_2 := 5$.
Then for $i = 1, 2$ *the map* G_i *contains at least one of the following subgraphs:*

(i) *a* k*-path of maximum degree* $\leq d$ *in* G_i, *or*

(ii) *a generalized* s*-star* $T, s \leq m_i$ *on* $d + 2 - m_i$ *vertices with root* Z, *where* Z *has a degree* $\leq k \cdot m_i$ *in* G_i *and the maximum degree of* $T \setminus \{Z\}$ *is* $\leq d$ *in* G_i.

Finally we deal with light classes $\mathcal{H} \neq \mathcal{T}_k,\ k \geq 1$.

Since by Theorem 7 each polyhedral map G on $\mathbb{M}$ of large order contains a tree of order k such that each vertex has a degree at most $4k+4$, if $k \geq 3$, the map G also contains a P_k or a $K_{1,3}$ with the same bound. Examples in [11] show that the bound is best possible.

Theorem 9. *For any surface* $\mathbb{M}$ *of Euler characteristic* $\chi(\mathbb{M}) \leq 0$ *and any integer* $k \geq 3$ *let* $a > (8k^2 + 6k - 6)|\chi(\mathbb{M})|$. *Then*

(i) $\varphi(\{P_k, K_{1,3}\}, \mathcal{G}(3,3;\mathbb{M}, n(a))) = 4k+4$, *and*

(ii) $\varphi(\{P_k, K_{1,3}\}, \mathcal{G}(3,3;\mathbb{M}, c_k(a))) = 4k+3$.

Next the class $\{P_k, S_i\}$, $i \geq 2$, will be considered. If $\chi(\mathbb{M}) \leq 0$, $a > 6k(2b_k+1)|\chi(\mathbb{M})|$ and $i \geq \frac{k}{2}$ then $P_k \subseteq S_i$ and $\varphi(\{P_k, S_i\}, \mathcal{G}(3,3;\mathbb{M}, n(a))) = \varphi(\{P_k\}, \mathcal{G}(3,3;\mathbb{M}, n(a)))$. For different i we will prove the following theorem.

Theorem 10. *Let $\mathbb{M}$ be a surface with Euler characteristic $\chi(\mathbb{M})$, and let $k \geq 3, i \geq 1$ be integers.*

(i) *If $\mathbb{M}$ is the sphere $\mathbb{S}_0$ or the projective plane $\mathbb{N}_1$, then*

$$\varphi(\{P_k, S_i\}, \mathcal{G}(3,3;\mathbb{S}_0)) = \varphi(\{P_k, S_i\}, \mathcal{G}(3,3;\mathbb{N}_1)) \leq 4(k+i) - 1.$$

(ii) *If $\chi(\mathbb{M}) \leq 0$, then for each integer $b > 4(k+i)|\chi(\mathbb{M})|$ it holds*

$$\varphi(\{P_k, S_i\}, \mathcal{G}(3,3;\mathbb{M}, c_k(b))) \leq 4(k+i) - 1.$$

(iii) *If $\chi(\mathbb{M}) \leq 0$, then for each integer $a > (6k+1)(2b_k+1)|\chi(\mathbb{M})|$ it holds*

$$\varphi(\{P_k, S_i\}, \mathcal{G}(3,3;\mathbb{M}, n(a)) \leq 4(k+i).$$

Taking into the consideration the Theorems 1 and 6 we obtain tight bounds in some subclasses of $\mathcal{G}(3,3;\mathbb{M})$.

Theorem 11. *Let k and i be integers with $k \geq 3$ and $i \geq 1$. If $\mathbb{M}$ is the sphere $\mathbb{S}_0$ or the projective plane $\mathbb{N}_1$, then*

$$\varphi(\{P_k, S_i\}, \mathcal{G}(3,3;\mathbb{S}_0)) = \varphi(\{P_k, S_i\}, \mathcal{G}(3,3;\mathbb{N}_1)) = \min\{4(k+i) - 1; 5k\}.$$

Theorem 12. *For any surface $\mathbb{M}$ with Euler characteristic $\chi(\mathbb{M}) \leq 0$, any integers $k \geq 3, i \geq 1$, and $b > 6k|\chi(\mathbb{M})|$ it holds:*

$$\varphi(\{P_k, S_i\}, \mathcal{G}(3,3;\mathbb{M}, c_k(b))) = \min\{4(k+i) - 1; 5k\}.$$

Theorem 13. *For any surface $\mathbb{M}$ with Euler characteristic $\chi(\mathbb{M}) \leq 0$, any integers $k \geq 3, i \geq 1$, and $a > l_k(\mathbb{M})$ it holds:*

$$\varphi(\{P_k, S_i\}, \mathcal{G}(3,3;\mathbb{M}, n(a))) = \begin{cases} \min\{4(k+i); 6k\} & for\ even\ k, \\ \min\{4(k+i); 6k-2\} & for\ odd\ k. \end{cases}$$

4. Proof of Theorems 4 and 8

Assume there is a counterexample G to Theorem 4 or 8 having $v = |V(G)|$ vertices, where in Theorem 8 the number of vertices $v > (14(k-1)b_k + 6)|\chi(\mathbb{M})|$ or the positive k-charge

$$ch_k(G) = \sum_{\deg_G(A) > 6k} (\deg_G(A) - 6k) > 6k|\chi(\mathbb{M})|.$$

Let G be a counterexample with the maximum number of edges among all counterexamples having v vertices. A vertex A of the graph G is *major* (*minor*) if its degree is $\geq d+1 (\leq d$, respectively). The assertions $(1)-(3)$ can be found in the introduction.

(4) Each path P_k of k vertices contains a major vertex.

Hence G contains at least one major vertex.

(5) Each r-face $\alpha, r \geq 4$, contains at most two major vertices; if α has precisely two major vertices then they are adjacent.

***Proof of* (5).**
Suppose G has an r-face with two nonadjacent vertices of degree $\geq d+1$. Since G is a polyhedral map we can join these two vertices by an edge. The resulting embedding is again a counterexample but with one edge more, a contradiction. ■

Let H denote the subgraph of G induced by the major vertices, and let $v(H)$ be the number of vertices of H.

(6) The subgraph H contains a vertex Z of degree $s := \deg_H(Z)$ with

(i) $s \leq m(\mathbb{M})$ if $G \in \mathcal{G}(3,3;\mathbb{M})$, or

(ii) $s \leq 6$, if $G \in \mathcal{G}(3,3;\mathbb{M},n(a)), \chi(\mathbb{M}) \leq 0$, or

(iii) $s \leq 5$, if $G \in \mathcal{G}(3,3;\mathbb{M},c_k(b)), \chi(\mathbb{M}) \leq 0$.

***Proof of* (6).**

(i) This assertion follows from (1)(see the introduction).

(ii) Suppose there is a $G \in \mathcal{G}(3,3;\mathbb{M},n(a))$ with the subgraph H of major vertices of G with minimum degree $\delta(H) > 6$. In Lemma 5 of [9] we have proved that $v(H) > 6|\chi(\mathbb{M})|$. By (3) the subgraph H has $v(H) \leq 6|\chi(\mathbb{M})|$ vertices. This contradiction completes the proof of (ii).

(iii) Suppose there is a $G \in \mathcal{G}(3,3;\mathbb{M},c_k(b))$ with the subgraph H of major vertices of G with minimum degree $\delta(H) > 5$. By (2) we have $\sum(\deg_H(A) - 6) \leq 6|\chi(\mathbb{M})|$ where the sum is taken over all vertices A of H with $\deg_H(A) > 6$.

Since G is a polyhedral map the union of all faces incident with Z forms a wheel with nave Z and cycle $\mathcal{C}_Z$; it may be that some vertices of the cycle $\mathcal{C}_Z$ are not joined with Z by an edge. By (5) each vertex of $\mathcal{C}_Z$ not adjacent with Z is a minor vertex. Hence all major vertices of $\mathcal{C}_Z$ are neighbours of Z. These neighbours partition $\mathcal{C}_Z$ into $\deg_H(Z)$ paths which have at most $k-1$ minor vertices according (4). Therefore, $\deg_G(Z) \leq k\ \deg_H(Z)$. This together with $ch_k(G) > 6k|\chi(\mathbb{M})|$, $\delta(H) \geq 6$ and $\sum_{\deg_H(A)\geq 6}(\deg_H(A) - 6) \leq 6|\chi(\mathbb{M})|$ implies:

$$6k|\chi(\mathbb{M})| < ch_k(G) = \sum_{\deg_G(A)>6k} (\deg_G(A) - 6k) \leq \sum_{\deg_G(A)>6k} (k\ \deg_H(A) - 6k)$$

$$\leq k \sum_{A\in V(H)} (\deg_H(A) - 6) \leq 6k|\chi(\mathbb{M})|.$$

This contradiction completes the proof of assertion (iii). ∎

The s neighbours $Y_1, Y_2, \ldots, Y_s$ of Z in H are the only major vertices on the cycle $\mathcal{C}_Z$. An upper bound for s is known by (6). If $\mathcal{C}_Z$ has no major vertices then let $s := 1$ and Y_1 be an arbitrary neighbour of Z on $\mathcal{C}_Z$. The cycle $\mathcal{C}_Z$ contains altogether $\deg_G(Z) \geq d+1$ neighbours of Z. Next $\mathcal{C}_Z \backslash \{Y_1, \ldots, Y_s\}$ consists of s paths $p_1, p_2, \ldots, p_s$ of minor vertices. These paths and Z induce a subgraph which contains a generalized star T with root Z of degree $\deg_T(Z) \leq s$ and containing all $\geq \deg_G(Z) - s \geq d+1-s$ minor neighbours of Z. By (4) each path p_i has at most $k-1$ vertices. Hence the cycle $\mathcal{C}_Z$ has at most $s \cdot k$ vertices and $\deg_G(Z) \leq s \cdot k$. Consequently, G contains a generalized star T of order $d+2-s$ with root Z of degrees $\deg_T(Z) \leq s$ and $\deg_G(Z) \leq s \cdot k$, all other vertices of T have a degree $\leq d$ in G. This contradicts our assumption that G is a counterexample to Theorem 4 or 8. Thus the proof of the Theorems 4 and 8 is complete. ∎

5. Proof of Theorems 10–13 — Upper Bounds

Theorem 1, 6, and 10 imply the validity of the upper bounds in Theorems 11–13. Hence it suffices to prove Theorem 10.

If $i \geq \frac{k}{2}$ then $P_k \subseteq S_i$ and $\varphi(\{P_k, S_i\}, \mathcal{L}) = \varphi(\{P_k\}, \mathcal{L})$ for each class $\mathcal{L}$ of graphs. This bound is $\leq 6k$ in each class $\mathcal{L}$ we have considered. Hence $\varphi(\{P_k, S_i\}, \mathcal{L}) \leq 6k \leq 4(k+i)$, and it suffices to accomplish the proof for all $i \leq \frac{k-1}{2} \leq \frac{k}{2}$.

The proof follows the ideas of [1] and [13]. Suppose that there is a counterexample to one version of our theorem having v vertices. Let G be a counterexample with the maximum number of edges among all counterexamples having v vertices. Obviously, G contains a P_k or an S_i.

(A) If G is a counterexample to Theorem 10(i) then $\mathbb{M}$ is the sphere $\mathbb{S}_0$ or is the projective plane $\mathbb{N}_1$ and each P_k and each S_i of G contains a vertex of degree $\geq 4(k+i)$.

(B) If G is a counterexample to Theorem 10(ii) then $\chi(\mathbb{M}) \leq 0$, the map G has a positive k-charge $ch_k(G) > 4(k+i)|\chi(\mathbb{M})|$, and each P_k and each S_i of G contains a vertex of degree $\geq 4(k+i)$.

(C) If G is a counterexample to Theorem 10(iii) then $\chi(\mathbb{M}) \leq 0$, the map G has an order $v(G) > (6k+1)(2b_k+1)|\chi(\mathbb{M})|$ and each P_k and each S_i of G contains a vertex of degree $\geq 4(k+i)+1$.

In the cases (A) and (B) a vertex A is a *minor vertex* if $\deg_G(A) \leq 4(k+i)-1$ and is a *major* vertex if $\deg_G(A) \geq 4(k+i)$. In case (C) a vertex A is a *minor vertex* if $\deg_G(A) \leq 4(k+i)$ and is a *major vertex* if $\deg_G(A) \geq 4(k+i)+1$. Since G is a counterexample it holds.

(1) Each k-path and each generalized star S_i in G contains a major vertex.

(2) Every r-face $\alpha, r \geq 4$, of G is incident only with minor vertices.

***Proof of* (2).** Suppose there is a major vertex B incident with an r-face $\alpha, r \geq 4$. Let C be a diagonal vertex on α with respect to B i.e., BC is no edge of the boundary of α. Because G is a polyhedral map we can insert the edge BC into the r-face α The resulting embedding is again a counterexample but with one edge more, a contradiction. ∎

Let $H = H(G)$ and $H' = H'(G)$ be the subgraphs of G induced on all major or minor vertices of G, respectively.

(3) H is not empty.

***Proof of* (3).** Since G is a counterexample it contains a k-path P_k or a 3-star S_i. By (1) P_k or S_i contains a major vertex. ∎

(2) directly implies:

(4) All faces incident with a major vertex X induce a wheel with nave X and a cycle $\mathcal{C}$ of length $\geq 4(k+i)$ consisting of all neighbours of X. The cycle $\mathcal{C}$ contains at least 5 major vertices.

***Proof of* (4).** The first assertion is clear. If $\mathcal{C}$ would contain at most 4 major vertices then $\mathcal{C}$ would also contain at most 4 paths of $\leq k-1$ minor vertices and $\mathcal{C}$ would have a length $\leq 4 + 4(k-1) = 4k < 4(k+i)$. This contradiction proves (4). ■

(5) By (4) the minimum degree of H is at least 5.

Note that a triangle is always a 3-face. For the following we need Lemma 1. ■

Lemma 1. *The three vertices of each triangle D of H are joint with the minor vertices inside D by at most* $2(k-1+i)-1$ *edges.*

Proof. Let $D = [PQR]$ be a triangle of H. Let K denote the subgraph of G induced by the minor vertices of G lying in the interior of $[PQR]$. By (2) all faces incident with P induce a wheel W_P with nave P and a cycle containg all neighbours of P. Correspondingly, Q and R are the naves of a wheel W_Q and W_R, respectively. Let p, q, and r denote the path of $W_P \cap K, W_Q \cap K$, and $W_R \cap K$, respectively. Then p, q and q, r and r, p have a common endvertex Q', R', and P', respectively (a sketch of the situation is depicted in Figure 1).

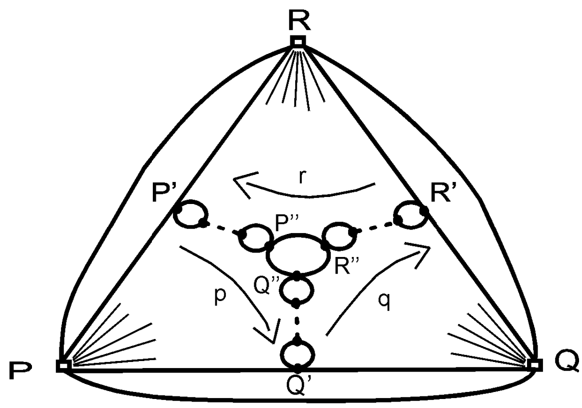

Figure 1

Case 1. Let p and q have precisely one common vertex, namely, Q'. Then $p \cup q$ is a path of K having $\leq k-1$ vertices. Hence the paths $p \cup q$ and r having at most $k-1$ vertices each, and P, Q, R are joined by $\leq 2k-1 < 2(k-1+i)$ edges with K.

Case 2. Let p, q and q, r and r, p have a second common vertex. Let Q'', and R'', and P'' denote the last common vertex of p, q, and q, r, and r, p by walking along p, or q, or r with start in Q', or R', or P', respectively. Since K does not contain a generalized star S_i, w.l.o.g. the paths $R''qR'$ and $R'rR''$ have at most i vertices. The paths $P'rR''$ and $R''qQ'$ have precisely one common vertex, namely, R''. Hence $P'rR''qQ'$ is a path of K with $\leq k-1$ vertices. The path p has also $\leq k-1$ vertices, and $R'rR'' - \{R''\}$ and $R''qR' - \{R''\}$ have at most $i-1$ vertices each. Therefore, P, Q, R are joined with K by $\leq (k-1)+(k-1)+1+(i-1)+(i-1) = 2(k-1+i)-1$ edges with K. ■

Let α be a face of H. Let $D_1, D_2, \cdots, D_s$ be the components of the boundary of α. Let W_i be the shortest closed walk induced by all edges of D_i, and let $\partial(W_i)$ be its length. Then the degree of the face α is $\deg_H(\alpha) = \sum_{i=1}^{s} \partial(W_i)$. Since G is a polyhedral map any three consecutive vertices on the boundary of α (i.e., in the walk W_i for some i) are pairwise different. Hence $\partial(W_i) \geq 3$, and

(6) $\deg_H(\alpha) \geq 3s \geq 3$.

Let X, Y, Z be three consecutive vertices on the boundary of α. We call XYZ a *corner* of α at the vertex Y. Assertion (4) implies: In G the vertices X and Z are joined by a path q completely lying in α and containing all minor neighbours of Y in this corner (Y can have some other minor neighbours at some other corners of α at the vertex Y, because Y can appear on the boundary of α more than once).

The path $q \smallsetminus \{X, Z\}$ consists of all minor neighbours of Y in this corner.

(7) In each corner XYZ of α at Y the vertex Y has at most $k-1$ minor neighbours. They form a path of $H'(G)$.

It is obvious that

(8) each face α of H has precisely $\deg_H(\alpha)$ corners.

Let $w(\alpha)$ denote the number of edges joining the minor vertices inside α with all major vertices of H (i.e., the major vertices on the boundary of α).

With (8) it follows:

(9) The minor vertices inside α are joined with all major vertices by $w(\alpha) \leq (k-1)\deg_H(\alpha)$ edges.

Thus the number w of all edges of G joining minor vertices with major vertices is

(10) $w = \sum_{\alpha \in F(H)} w(\alpha) \leq \sum_{\alpha \in F(H)} (k-1)\deg_H(\alpha)$.

By Lemma 1 we have a better bound if α is a 2-cell 3-face (triangle).

(11) If α is a triangle of H then $w(\alpha) \leq 2(k-1+i)-1$.

We proceed in three steps. First, we assign to each face α of H the charge $w(\alpha)$. Next, we triangulate each face α of H by introducing diagonals into the face α (a diagonal is an edge joining two vertices of the boundary of α such that no 1-face or 2-face is generated). By this method α is splitted into at least $t-2$ triangles, $t = \deg_H(\alpha)$. The obtained semitriangulation H^* can have loops or muliple edges (A *triangulation* (*semitriangulation*) is an embedding of a graph (multigraph) such that each face is a triangle). In the third step the charge $w(\alpha)$ is equally distributed to the triangles inside α. The charge of a triangle D of H^* is denoted by $w^*(D)$. Distributing the old charges no charge has been lost. Hence,

(12) $w = \sum_{\alpha \in F(H)} w(\alpha) = \sum_{D \in F(H^*)} w^*(D)$.

(13) Each triangle D of H^* has a charge $w^*(D) \leq 2(k-1+i)-1$.

***Proof of* (13).** Let α be a face of H. We consider two cases. ■

Case 1. Let $t := \deg_H(\alpha) \geq 4$. Then with (9) each triangle D inside α has a charge

$$\begin{aligned} w^*(D) &\leq \frac{w(\alpha)}{t-2} \leq \frac{t(k-1)}{t-2} \leq \left(1 + \frac{2}{t-2}\right)(k-1) \\ &\leq 2(k-1) < 2(k-1+i)-1. \end{aligned}$$

(Note $i \geq 1$).

Case 2. Let $t = \deg_H(\alpha) = 3$.
If α is a triangle (2-cell 3-face) of H then α is also a triangle of H^*. Hence with (11) the charge $w^*(\alpha) = w(\alpha) \leq 2(k-1+i)$.

Next let α not be a triangle (2-cell 3-face). Then at least one diagonal d can be added so that no new face is created. The diagonal is counted twice on the boundary of α, i.e., $\deg_{H+d}(\alpha) = 5$. The charge $w(\alpha) \leq 3(k-1)$ is equally distributed to at least three (new) triangles of H^*, each receiving a charge $\leq \frac{w(\alpha)}{3} \leq \frac{3(k-1)}{3} = k-1 \leq 2(k-1+i)-1$. Thus the proof of (13) is complete. ∎

Properties (12) and (13) imply:

$$(14)\quad w = \sum_{\alpha \in F(H)} w(\alpha) = \sum_{D \in F(H^*)} w^*(D) \leq (2(k-1+i)-1) f(H^*).$$

where $f(H^*) = |F(H^*)|$.

The semitriangulation H^* satisfies the equation

$$2e(H^*) = 3f(H^*),$$

and Euler's formula

$$v(H^*) - e(H^*) + f(H^*) = \chi(\mathbb{M}).$$

Hence

$$f(H^*) = 2(v(H^*) - \chi(\mathbb{M})), \tag{15}$$

and

$$e(H^*) = 3(v(H^*) - \chi(\mathbb{M})). \tag{16}$$

The number of the edges joining vertices of H and the number w of the edges joining minor vertices with major vertices in G contribute to the degree sum $\sum_{A \in V(H)} \deg_G(A)$. Consequently, with (14) it holds

$$\sum_{A \in V(H)} \deg_G(A) = \sum_{A \in V(H)} \deg_H(A) + w$$

$$\leq \sum_{A \in V(H)} \deg_H(A) + (2(k-1+i)-1) f(H^*).$$

With (15)

$$\sum_{A \in V(H)} \deg_G(A) \leq 2e(H) + (4(k-1+i)-2)(v(H^*) - \chi(\mathbb{M})). \tag{17}$$

With $e(H) \leq e(H^*)$ and (16) we have

$$\sum_{A \in V(H)} \deg_G(A) \leq (6 + 4(k-1+i)-2)(v(H^*) - \chi(\mathbb{M})), \quad \text{and}$$

$$\sum_{A \in V(H)} \deg_G(A) \leq 4(k+i)(v(H^*) - \chi(\mathbb{M})). \tag{18}$$

The inequality (18) implies with $v(H) = v(H^*)$ the existence of a major vertex B of degree

$$\deg_G(B) \leq 4(k+i)\left(1 - \frac{\chi(\mathbb{M})}{v(H^*)}\right). \tag{19}$$

If $\mathbb{M} = \mathbb{S}_0$ or $\mathbb{N}_1$ then $\chi(\mathbb{M}) \geq 1$ and by (5) H has at least 6 vertices. Moreover (18) implies the existence of a major vertex B of degree $\deg_G(B) \leq 4(k+i)-1$. But by condition (A) each major vertex has a degree $\geq 4(k+i)$. This contradiction completes the proof of Theorem 10(i). ■

Next Theorem 10(ii) can be proved in the following way. By condition (B) with $6k \geq 4(k+i)$ (i.e., with $i \leq \frac{k}{2}$) and $\chi(\mathbb{M}) \leq 0$

$$\begin{aligned}\sum_{\deg_G(A) \geq 4(k+i)} (\deg_G(A) - 4(k+i)) \;&\geq \sum_{\deg_G(A) > 6k} (\deg_G(A) - 6k) \\ &= ch_k(G) > 4(k+i)|\chi(\mathbb{M})|.\end{aligned}$$

With

$$\sum_{\deg_G(A) \geq 4(k+i)} (\deg_G(A) - 4(k+i)) = \left(\sum_{\deg_G(A) \geq 4(k+i)} \deg_G(A)\right) - 4(k+i)v(H^*)$$

this implies

$$\begin{aligned}\sum_{\deg_G(A) \geq 4(k+i)} \deg_G(A) \;&>\; 4(k+i)(v(H^*) + |\chi(\mathbb{M})|) \\ &=\; 4(k+i)(v(H^*) - \chi(\mathbb{M})).\end{aligned} \tag{20}$$

The assertion (20) contradicts (18). Thus the proof of Theorem 10(ii) is complete. ■

Finally Theorem 10(iii) will be proved. For these purposes we need an upper bound for the number $v(H')$ of vertices of H' in dependence on $f(H^*)$. Recall that H' is the subgraph of G induced by the minor vertices of G. Let l denote the number of components of H'. Since G is 3-connected each component K of H' contains the minor vertices of at least three corners of a face of H. The number of corners of H is not greater than the number of corners of H^*, and H^* has at most $3f(H^*)$ corners. Hence $3l \leq 3f(H^*)$, and $l \leq f(H^*)$. Since each component K of H' has no path with k vertices and each (minor) vertex of K has a degree $\leq 4(k+i) \leq 6k$ in G the number of vertices of K is $v(K) \leq b_k$, and the number of vertices of H' is $v(H') \leq l \cdot b_k \leq f(H^*) \cdot b_k$. Therefore,

$$v(G) = v(H) + v(H') \leq v(H^*) + f(H^*) \cdot b_k.$$

Assertion (15) implies

$$v(G) \leq v(H^*) + 2(v(H^*) + |\chi(\mathbb{M})|) \cdot b_k \leq 2(v(H^*) + |\chi(\mathbb{M})|) \left(b_k + \frac{1}{2} \right).$$

With the hypothesis $v(G) > (6k+1)(2b_k+1)|\chi(\mathbb{M})|$ the number of vertices of H^* is

$$v(H^*) \geq \frac{v(G)}{2b_k+1} - |\chi(\mathbb{M})| > \frac{12k(b_k+\frac{1}{2})|\chi(\mathbb{M})|}{2b_k+1} = 6k|\chi(\mathbb{M})|,$$

and

$$v(H^*) > 6k|\chi(\mathbb{M})| \geq 4(k+i)|\chi(\mathbb{M})|. \tag{21}$$

(19) and (21) imply: there is a vertex $B \in V(H)$ such that its degree

$$\begin{aligned} \deg_G(B) \; &\leq \; 4(k+i) + \frac{4(k+i)|\chi(\mathbb{M})|}{v(H^*)} \\ &< \; 4(k+i) + \frac{4(k+i)|\chi(\mathbb{M})|}{4(k+i)|\chi(\mathbb{M})|} = 4(k+i)+1. \end{aligned} \tag{22}$$

Therefore, the degree of the major vertex B in G is $\leq 4(k+i)$. But by the condition (C) each major vertex has a degree $\geq 4(k+i)+1$. This contradiction completes the proof of Theorem 10(iii). ■

6. Proof of Theorem 13 for Polyhedral Maps — Lower Bound

The main goal of this part is to prove that $\varphi(\{P_k, S_i\}, \mathcal{G}(3,3;\mathbb{M}, n(a))) \geq 4k+4i,\ k \geq 3,\ \chi(\mathbb{M}) \leq 0$, that is to construct a large polyhedral map G on surface $\mathbb{M}$ with Euler characteristic $\chi(\mathbb{M}) \leq 0$ so that each path P_k with k vertices and each generalized 3-star S_i contains a vertex of degree at least $4k+4i$. This construction is very similar to our construction presented in Sections 3 and 4 of [11].

Let $P_n \times P_n$ be the cartesian product of two n-paths with vertex set $\{(x,y)|x,y \in \mathbb{Z}, 1 \leq x \leq n, 1 \leq y \leq n\}$ and edge set $\{\{(x,y),(x,y+1)\}|1 \leq x \leq n, 1 \leq y \leq n-1\} \cup \{\{(x,y),(x+1,y)\}|1 \leq x \leq n-1, 1 \leq y \leq n\}$. Add the edge set $\{\{(x,y),(x+1,y-1)\}|1 \leq x \leq n-1, 2 \leq y \leq n\}$. The so obtained plane graph with $2(n-1)^2$ triangles and an outer $4(n-1)$-face is denoted by R_n.

Into each triangle D of the obtained graph we insert a generalized 3-star $S(r)$, $0 \leq r \leq k-2i$, consisting of a central vertex Z and three paths p_1, p_2 and p_3 starting in Z, the path p_1 of length $k-(i+r)$, the path p_2 of length $i+r$, and the path p_3 of length i. Let the paths p_1, p_2, and p_3 be in this anticlockwise cyclic order in D. If $D_{x,y} = ((x,y),(x+1,y),(x,y+1))$ then (x,y) is joined to all vertices of p_1 and $p_2, (x+1,y)$ is joined to all vertices of p_2 and p_3, and $(x,y+1)$ is joined to all vertices of p_3 and p_1 (see Figure 2). We do the same in $D'_{x,y} = ((x,y),(x-1,y),(x,y-1))$. The resulting plane graph is denoted by R_n^*.

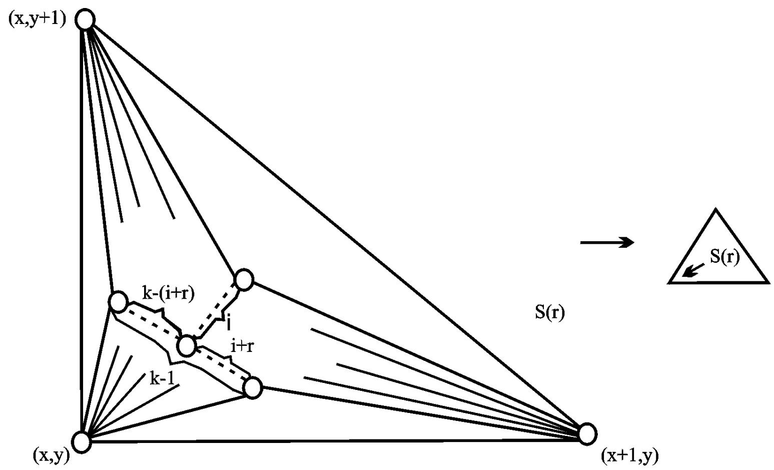

Figure 2

The situation is presented in Figure 3, where in each triangle D an arrow indicates which vertex of D is joined with all vertices of p_1 and p_2. In this part of the proof the labels 0 and $k-2i, \cdots$ have no meaning. For the further proof of the lower bound $4(k+i)$ choose a fixed r, $0 \le r \le k-2i$.

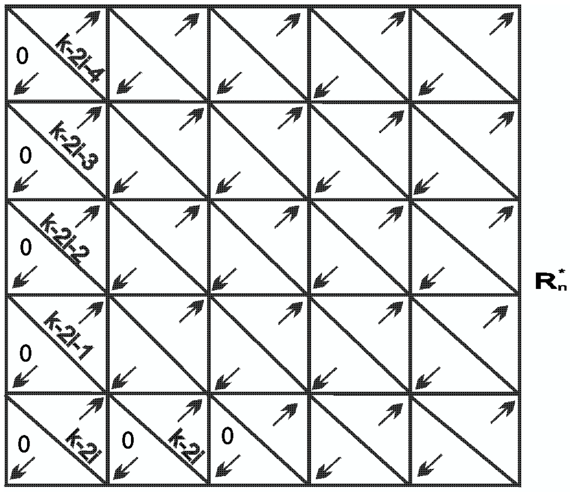

Figure 3

The inserted trees have $k-(i+r)+(i+r)+i-2 = k+i-2$ vertices, and the degree of each inner vertex (x,y), $2 \le x, y \le n-1$ is

$$\begin{aligned} \deg(x,y) = 6 \; &+ \; 2((k-(i+r)) + (i+r) - 1) \\ &+ \; 2((i+r) + i - 1) \\ &+ \; 2(i + (k-(i+r)-1) = 4k+4i. \end{aligned}$$

Deleting the outer face of R_n^* and identifying opposite sides of the "quadrangle" results in a toroidal map T_n, and reversing one side of this "quadrangle" and then identifying opposite sides of this "quadrangle" results in a map Q_n on the Klein bottle, respectively, both satisfying the degree requirements.

The required polyhedral map on an orientable surface $\mathbb{S}_g$ of genus $g \ge 2$ will be constructed from the toroidal triangulation T_n^* with the triangulation T_n. We choose $2g-2$ triangles of T_n so that any two of them have a distance ≥ 2 in T_n (this is possible if n is large enough). In T_n^* from each of these triangles D we delete the interior part so that the bounding 3-cycle of D

bounds now a hole of the torus. We join repeatedly two holes of T_n^* by a handle, and $g-1$ handles are added to the torus in this way.

The handles are triangulated in the following way: if $[X_1X_2X_3]$ and $[Y_1Y_2Y_3]$ are the bounding cycles of some handle which are around the handle in the same cyclic order then add the cycle $[X_1Y_1X_2Y_2X_3Y_3]$. In each of the new triangles a generalized 3-star $S(r)$ can be placed in such a manner that the obtained polyhedral triangulation of $\mathbb{S}_g$ fulfils also the degree requirements.

The required polyhedral map on an unorientable surface $\mathbb{N}_q$ of genus $q \geq 3$ will be constructed from the triangulation Q_n^* of the Klein bottle with triangulation Q_n. We choose $q-2$ triangles of Q_n so that any two of them have a distance ≥ 4 in Q_n.

Let D be one of these triangles with bounding cycle $[X_1X_2X_3]$ and D_1, D_2, D_3 the three neighbouring triangles in Q_n with bounding cyles $[Y_1X_3X_2]$, $[Y_2X_1X_3]$, and $[Y_3X_2X_1]$ (see Figure 4). In Q_n^* we delete the inserted trees of D, D_1, D_2, D_3 and the separating edges X_1X_2, X_2X_3 and X_3X_1. A greater face F with bounding 6-cycle $\mathcal{C} = [X_1Y_3X_2Y_1X_3Y_2]$ is obtained (for the notation see Figure 5).

In F a crosscap is placed and the edges X_1X_2, X_2X_3, and X_3X_1 are again added so that the "interior" of $\mathcal{C}$ is subdivided into three quadrangles (see Figure 5). These quadrangles are subdivided by the edges X_iY_i, $i = 1, 2, 3$ (see Figure 6). Finally in each of the new triangles a generalized 3-star $S(r)$ can be placed in such a manner that the obtained polyhedral triangulation of $\mathbb{N}_q$ fulfils the degree requirements.

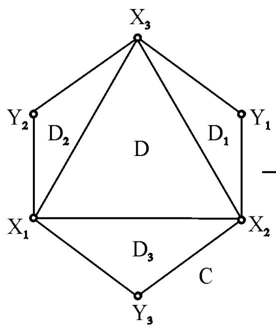

Figure 4

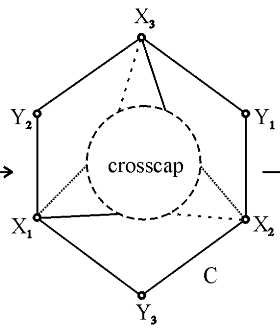

Figure 5

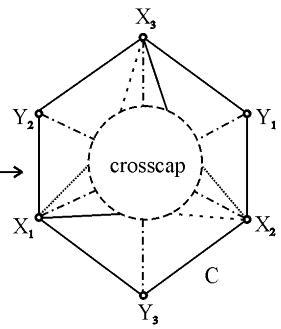

Figure 6

7. Proof of Theorems 11 and 12 for Polyhedral Maps — Lower Bounds

Let $\mathbb{M}$ be a surface with Euler characteristic $\chi(\mathbb{M})$. Firstly we construct a polyhedral graph of the plane with degree sum $\sum_{j>4k+4}(j-6k)v_j > (4k+4i)|\chi(\mathbb{M})|$, $k \geq 3$, such that each subgraph P_k and each subgraph S_i contains a vertex of degree at least $4(k+i)-1$, $k \geq 3$. Our method used here is very similar to that one used in [12]. We start with a plane graph R_{n+1} with $n > (k+1)|\chi(\mathbb{M})| + 3k$ as described in Section 6. Next the outer $4n$-face is deleted and the opposite "vertical sides" are identified, i.e., the two paths $(1,1),(1,2),\ldots,(1,n+1)$ and $(n+1,1),(n+1,2),\ldots,(n+1,n+1)$ are identified in the given order.

The result is a triangulated cylinder Z_n^*. A plane polyhedral graph Z_n is obtained by adding a bottom n-face F_1 and a top n-face F_2 which are the only n-faces of a Z_n, all other faces of Z_n are triangles. We use the same notation as in R_n. If in all triangles of Z_n a generalized 3-star $S(r)$ with a fixed r is inserted then all inner vertices of Z_n have the degree $4(k+i)$. For instance choose $r = 0$. We want to increase the degrees of the vertices of the boundaries of Z_n, i.e., for the vertices $(1,1)$,$(1,2)$,…,$(1,n)$ and $(n+1,1)$,$(n+1,2)$,…,$(n+1,n)$. In order to do this we vary r so that from each inner vertex near to these boundaries a degree unit is transferred to one of these boundaries. We achieve this in the following way: According to Figure 3 insert the 3-star $S(k-2i)$ into $D'_{x,1}$, the 3-star $S(k-2i-1)$ into $D'_{x,2},\ldots,$ the 3-star $S(0)$ into $D'_{x,k-2i+1}$ for all $x, 1 \leq x \leq n$. According to Figure 3 insert the 3-star $S(k-2i)$ into $D_{x,n+1}$, the 3-star $S(k-2i-1)$ into $D_{x,n},\ldots,$ the 3-star $S(0)$ into $D_{x1,n-(k-2i)M}$, for all $2 \leq x \leq n+1$. Into all other triangles insert the 3-star $S(0)$ according to Figure 3.

By the construction the vertices (x,y), $1 \leq x \leq n$, $2+(k-2i) \leq y \leq n-(k-2i)$ have degree $4(k+i)$. The vertices $(x,y), 1 \leq x \leq n$, $2 \leq y \leq k-2i+1$ or $n-(k-2i)+1 \leq y \leq n$ have the degree $4(k+i)-1$. The vertices on the boundaries, i.e., the vertices $(x,1)$ and $(x,n+1)$, $1 \leq x \leq n$ have the degree $3k+1$. In order to complete our construction we put into F_i a new vertex X_i and join X_i with all bounding vertices of F_i, $i = 1,2$. In each new triangle Δ a k-path p of F_1 and F_2 is inserted. One endvertex of p is joined with all three vertices of Δ, and all other vertices of p are joined with each of the two remaining vertices of Δ. In the obtained triangulation Z_n^{**} the vertices bounding F_i have degree $3k+1+3+2(k-1)-2 = 5k$, and X_i has degree $\deg X_i \geq 2n > 2(k+i)|\chi(\mathbb{M})| + 6k$. Thus $ch_k(Z_n^{**}) \geq (\deg X_1 - 6k) + (\deg X_2 - 6k) > 4(k+i)|\chi(\mathbb{M})|$.

Next the wanted polyhedral maps of $\mathbb{M}$ will be constructed. If $\mathbb{M}$ is an orientable 2-manifold $\mathbb{S}_g$ of genus g then g handles have to be added. If $\mathbb{M}$ is a nonorientable 2-manifold $\mathbb{N}_q$ of genus q then q crosscaps have to be added. In both cases this is accomplished in the same way as in Section 6. The addition of g handles or of q crosscaps causes no problems according Section 6.

Acknowledgement

1. Support of Slovak VEGA Grant 1/0424/03 is acknowledged.
2. The authors are gratitude to anonymous referees who read original versions of the paper very carefully. Their suggestions helped us to improve the presentation of the paper.

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Received 10 January 2002
Revised 20 May 2003