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# ON TRACEABILITY AND 2-FACTORS IN CLAW-FREE GRAPHS

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**Abstract**

If $G$ is a claw-free graph of sufficiently large order $n$, satisfying a degree condition $\sigma_k > n+k^2-4k+7$ (where $k$ is an arbitrary constant), then $G$ has a 2-factor with at most $k-1$ components. As a second main result, we present classes of graphs $\mathcal{C}_1, \ldots, \mathcal{C}_8$ such that every sufficiently large connected claw-free graph satisfying degree condition $\sigma_6(k) > n + 19$ (or, as a corollary, $\delta(G) > \frac{n+19}{6}$) either belongs to $\cup_{i=1}^{8} \mathcal{C}_i$ or is traceable.

**Keywords:** traceability, 2-factor, claw, degree condition, closure

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## 1. Introduction

We consider finite undirected graphs $G = (V(G), E(G))$ without loops and multiple edges. We follow the most common terminology and notation and for concepts not defined here we refer e.g. to [1]. For any set $A \subset V(G)$ we denote by $\langle A \rangle_G$ the subgraph of $G$ induced on $A$ and $G - A$ stands for $\langle V(G) \setminus A \rangle$. A graph $G$ is *$H$-free* (where $H$ is a graph), if $G$ does not contain an induced subgraph isomorphic to $H$. In the special case $H = K_{1,3}$ we say that $G$ is *claw-free*. The *independence number* of $G$ is denoted by $\alpha(G)$ and the *clique covering number* of $G$ (i.e., the minimum number of cliques necessary for covering $V(G)$) by $\theta(G)$. We denote by $\delta(G)$ the *minimum degree* of $G$ and by $\sigma_k(G)$ $(k \geq 1)$ the *minimum degree sum* over all independent sets of $k$ vertices in $G$ (for $k > \alpha(G)$ we set $\sigma_k(G) = \infty$). The *circumference* of $G$, i.e., the length of a longest cycle in $G$, is denoted by $c(G)$, and the length of a longest path in $G$ is denoted by $p(G)$. A graph $G$ of order $n$ is *hamiltonian* or *traceable* if $c(G) = n$ or $p(G) = n$, respectively.

The *line graph* of a graph $H$ is denoted by $L(H)$. If $G = L(H)$, then we also denote $H = L^{-1}(G)$ and say that $H$ is the *line graph preimage* of $G$ (recall that for any line graph $G$ nonisomorphic to $K_3$, its line graph preimage is uniquely determined).

A vertex $x \in V(G)$ is said to be *locally connected* if its neighborhood $N(x)$ induces a connected graph. The *closure* of a claw-free graph $G$ (introduced in [12] by the first author) is defined as follows: the closure $\mathrm{cl}(G)$ of $G$ is the (unique) graph obtained by recursively completing the neighborhood of any locally connected vertex of $G$, as long as this is possible. The closure $\mathrm{cl}(G)$ remains a claw-free graph and its connectivity is at least equal to the connectivity of $G$. The following basic properties of the closure $\mathrm{cl}(G)$ were proved in [12], [3] and [13].

**Theorem A.** *Let $G$ be a claw-free graph and $\mathrm{cl}(G)$ its closure. Then*

(i) [12] *there is a triangle-free graph $H_G$ such that $\mathrm{cl}(G) = L(H_G)$,*

(ii) [12] $c(G) = c(\mathrm{cl}(G))$*,*

(iii) [3] $p(G) = p(\mathrm{cl}(G))$*,*

(iv) [13] *$G$ has a 2-factor with at most $k$ components if and only if $\mathrm{cl}(G)$ has a 2-factor with at most $k$ components.*

Consequently, $G$ is hamiltonian (traceable) if and only if $\mathrm{cl}(G)$ is hamiltonian (traceable). If $G$ is a claw-free graph such that $G = \mathrm{cl}(G)$, then we say that $G$ is *closed*. It is apparent that a claw-free graph $G$ is closed if and only if every vertex $x \in V(G)$ is either *simplicial* (i.e., $\langle N(x)\rangle_G$ is a clique), or is *locally disconnected* (i.e., $\langle N(x)\rangle_G$ consists of two vertex disjoint cliques).

In [12], the closure concept was used to answer an old question by showing that every 7-connected claw-free graph is hamiltonian. H. Li [10] extended this result as follows.

**Theorem B** [10]**.** *Every 6-connected claw-free graph with at most 34 vertices of degree 6 is hamiltonian.*

In [5], the following result on 2-factors with limited number of components was proved.

**Theorem C** [5]**.** *If $G$ is a claw-free graph of order $n$ and minimum degree $\delta \geq 4$, then $G$ contains a 2-factor with at most $\frac{6n}{\delta+2} - 1$ components.*

This result was improved by Gould and Jacobson [8].

**Theorem D** [8]**.** *Let $k \geq 2$ be an integer and let $G$ be a claw-free graph of order $n \geq 16k^3$ and minimum degree $\delta \geq \frac{n}{k}$. Then $G$ contains a 2-factor with at most $k$ components.*

In the first main result of this paper, Theorem 3, we give a strengthening of this result.

A trail $T$ (closed or not) in a graph $H$ is said to be *dominating* if every edge of $H$ has at least one vertex on $T$. Harary and Nash-Williams [11] proved the following result, showing that hamiltonicity of a line graph is equivalent to the existence of a dominating closed trail in its preimage.

**Theorem E** [11]**.** *Let $H$ be a graph without isolated vertices. Then $L(H)$ is hamiltonian if and only if either $H$ is isomorphic to $K_{1,r}$ (for some $r \geq 3$) or $H$ contains a dominating closed trail.*

It is straightforward to verify the following analogue of Theorem E for traceability.

**Theorem F.** *Let $H$ be a graph without isolated vertices. Then $L(H)$ is traceable if and only if either $H$ is isomorphic to $K_{1,r}$ (for some $r \geq 3$) or $H$ contains a dominating trail.*

Using the closure concept in claw-free graphs [12], Favaron, Flandrin, Li and Ryjáček [6] observed that there is a close relation between the minimum degree sum $\sigma_k(G)$ (or the minimum degree $\delta(G)$, respectively) of a closed claw-free graph $G$ and its clique covering number. These connections are established in the following results [6].

**Theorem G** [6]**.** *Let $k \geq 2$ be an integer and let $G$ be a claw-free graph of order $n$ such that $\delta(G) > 3k-5$ and $\sigma_k(G) > n+k^2-2k$. Then $\theta(\mathrm{cl}(G)) \leq k-1$.*

**Corollary H** [6]**.** *Let $k \geq 2$ be an integer and let $G$ be a claw-free graph of order $n \geq 2k^2-3k$ and minimum degree $\delta(G) > \frac{n}{k}+k-2$. Then $\theta(\mathrm{cl}(G)) \leq k-1$.*
The bounds on $\sigma_k(G)$ ($\delta(G)$) in the previous results are sharp (this can be easily seen considering the cartesian product of cliques).

It was shown in [6] and [9] that these results can be slightly strengthened under an additional assumption that $G$ is not hamiltonian, and this result was used to obtain degree conditions for hamiltonicity (by characterizing the classes of all 2-connected nonhamiltonian closed claw-free graphs with small clique covering number). In the second main result of this paper, Theorem 6, we follow up with this study by considering analogous questions for traceability.

## 2. Main Results

We begin with a structural result that can be considered, in a sense, as a strengthening of Theorem G.

**Theorem 1.** *Let $k \geq 2$ be an integer, let $G$ be a claw-free graph of order $n$ and let $\kappa = \kappa(\mathrm{cl}(G))$. Suppose that $G$ is such that $n \geq 3k^2 + k - (k+1)\kappa - 2$, $\delta(G) \geq 3k - 4$ and*

$$\sigma_k(G) > n + k^2 - 4k + 2 + \kappa.$$

*Then either $\theta(\mathrm{cl}(G)) \leq k - 1$, or $\alpha(\mathrm{cl}(G)) \leq \kappa$.*

Before proving Theorem 1, we first recall the following auxiliary results that were proved in [6].

**Lemma I** [6]**.** *Let $G$ be a closed claw-free graph of order $n$ and $\{a_1, a_2, \ldots, a_t\} \subset V(G)$ an independent set. Then*

(i) $|N(a_i) \cap N(a_j)| \leq 2,\ \ 1 \leq i < j \leq t,$

(ii) $\sum_{i=1}^{t} d(a_i) \leq n + t^2 - 2t.$

**Lemma J** [6]**.**

(i) *Any triangle-free graph $H$ whose matching number $\nu(H)$ and vertex covering number $\tau(H)$ satisfy $\nu(H) < \tau(H)$, contains an edge $xy$ such that $d(x) + d(y) \leq \nu(H) + \tau(H)$.*

(ii) *Let $G$ be a closed claw-free graph. If $\alpha(G) < \theta(G)$, then $\delta(G) \leq \alpha(G) + \theta(G) - 2$.*

**Lemma K** [6]**.** *Let $G$ be a closed claw-free graph. Then $\theta(G) \leq 2\alpha(G)$.*

**Lemma L** [6]**.** *Let $G$ be a closed claw-free graph of order $n$ and connectivity $\kappa(G)$ such that $1 \leq \kappa(G) < \alpha(G)$ and let $A = \{a_1, \ldots, a_\alpha\}$ be a maximum independent set in $G$. Then*

$$\sum_{i=1}^{\alpha} d(a_i) \leq n + \alpha^2 - 4\alpha + 2 + \kappa(G).$$

***Proof of Theorem* 1.** If $G$ is a counterexample to Theorem 1 such that $G$ satisfies the assumptions but $\kappa < \alpha(\mathrm{cl}(G))$ and $\theta(\mathrm{cl}(G)) \geq k$, then so is the closure $\mathrm{cl}(G)$. Hence we can suppose that $G$ is closed.

If $\alpha(G) \geq k+1$, then by Lemma I we have $\sigma_{k+1}(G) \leq n + (k+1)^2 - 2(k+1) = n + k^2 - 1$, implying $\sigma_k(G) \leq \frac{k}{k+1}(n+k^2-1) \leq n+k^2-4k+2+\kappa$ for $n \geq 3k^2 + k - (k+1)\kappa - 2$, a contradiction. Hence $\alpha(G) \leq k$.

If $\alpha(G) \leq k-1$, then $\alpha(G) < \theta(G)$ and, by Lemma J and Lemma K, $\delta(G) \leq \alpha(G) + \theta(G) - 2 \leq (k-1) + 2(k-1) - 2 = 3k-5$, a contradiction.

Hence we have $\alpha(G) = k$. Since $\kappa(G) < \alpha(G)$, Lemma L gives $\sigma_k(G) \leq n + k^2 - 4k + \kappa + 2$, a contradiction. ■

From Theorem 1 we obtain the following minimum degree result.

**Corollary 2.** *Let $k \geq 2$ be an integer, let $G$ be a claw-free graph of order $n$ and let $\kappa = \kappa(\mathrm{cl}(G))$. Suppose that $G$ is such that $n \geq 3k^2 - k - \kappa - 2$ and*

$$\delta(G) > \frac{n + k^2 - 4k + 2 + \kappa}{k}.$$

*Then either $\theta(\mathrm{cl}(G)) \leq k-1$, or $\alpha(\mathrm{cl}(G)) \leq \kappa$.*

***Proof.*** We can again suppose that $G$ is closed. If $n \geq 3k^2 - k - \kappa - 2$, then obviously $\delta(G) > \frac{n+k^2-4k+2+\kappa}{k} \geq 3k-5$ and hence $\delta(G) \geq 3k-4$. The rest of the proof follows immediately from Theorem 1. ■

Now we can prove our first main result that gives a degree condition for the existence of a 2-factor with limited number of components.

**Theorem 3.** *Let $k \geq 2$ be an integer, let $G$ be a claw-free graph of order $n$ and let $\kappa = \kappa(\mathrm{cl}(G))$. Suppose that $G$ is such that $n \geq 3k^2 + k - (k+1)\kappa - 2$, $\delta(G) \geq 3k-4$ and*

$$\sigma_k(G) > n + k^2 - 4k + 2 + \kappa.$$

*Then $G$ has a 2-factor with at most $k - \kappa$ components.*

***Proof.*** If $G$ satisfies the assumptions of the theorem but has no 2-factor with at most $k-\kappa$ components, then, since $\delta(G) \leq \delta(\mathrm{cl}(G))$ and by Theorem A(iv), so does its closure $\mathrm{cl}(G)$. Since $\mathrm{cl}(G)$ is nonhamiltonian, by the well-known theorem of Chvátal and Erdős (see [2]), $\alpha(\mathrm{cl}(G)) > \kappa(\mathrm{cl}(G))$. By Theorem 1, we have $\theta(\mathrm{cl}(G)) \leq k-1$.

Let $\mathcal{P} = \{K_1, \ldots, K_\theta\}$ be a minimum clique covering of $\mathrm{cl}(G)$ such that each of the cliques $K_1, \ldots, K_\theta$ is maximal. We show that each clique of $\mathcal{P}$ has at least $2k-1$ vertices. This follows immediately from $\delta(\mathrm{cl}(G)) \geq 3k-4$ for those $K_i$'s that contain at least one simplicial vertex. Thus, suppose that (say) $K_1$ contains no simplicial vertex. By the minimality of $\mathcal{P}$, there is a clique $K'$ with $K_1 \cap K' \neq \emptyset$ and $K' \notin \mathcal{P}$ (otherwise $\mathcal{P} \setminus \{K_1\}$ is also a clique covering of $\mathrm{cl}(G)$). Since clearly every clique in $\mathrm{cl}(G)$ that contains a simplicial vertex must be in $\mathcal{P}$, $K'$ has no simplicial vertex. By the structure of the closure, $|K_1 \cap K'| = 1$.

Denote $K_1 \cap K' = \{x\}$, $|K_1| = t$ and $|K'| = r$. Then we have $d(x) = t-1+r-1 \geq \delta(\mathrm{cl}(G)) \geq 3k-4$, implying $t+r \geq 3k-2$. Since $K' \notin \mathcal{P}$, there are $r-1$ further cliques $K_{i_1}, \ldots, K_{i_{r-1}} \in \mathcal{P}$ having a common vertex with $K'$. By the structure of the closure, $K_{i_j} \neq K_{i_\ell}$, $j \neq \ell$, $j, \ell = 1, \ldots, r-1$, implying $\theta \geq r$. Since $\theta \leq k-1$, we have $3k-2 \leq t+r \leq t+k-1$, from which $t \geq 2k-1$.

Now, since $\theta \leq k-1$, each clique of $\mathcal{P}$ contains at least $2k-1-(k-2) = k+1$ vertices that are in no other clique of $\mathcal{P}$. Since $k \geq 2$, every $K_i \in \mathcal{P}$ contains a cycle $C_i$ that is vertex-disjoint from all other cliques of $\mathcal{P}$. Let $x_i \in K_i$, $i = 1, \ldots, \theta$ be such that each $x_i$ is in no other clique of $\mathcal{P}$. Since $\kappa = \kappa(\mathrm{cl}(G))$, by a well-known theorem by Dirac [4], there is a cycle $C$ in $\mathrm{cl}(G)$ containing all the vertices $x_1, \ldots, x_\kappa$. Let $\mathcal{C}$ be the collection of those of the cycles $C_{\kappa+1}, \ldots, C_\theta$, which are vertex-disjoint with $C$. Then the collection of cycles $\{C\} \cup \mathcal{C}$ can be easily extended to a 2-factor of $\mathrm{cl}(G)$ with at most $k-\kappa$ components. The result then follows by Theorem A(iv). ■

**Corollary 4.** *Let $k \geq 4$ be an integer and $G$ be a connected claw-free graph of order $n \geq 3k^2-3$, $\delta(G) \geq 3k-4$ and*

$$\sigma_k(G) > n + k^2 - 4k + 7.$$

*Then $G$ has a 2-factor with at most $k-1$ components.*

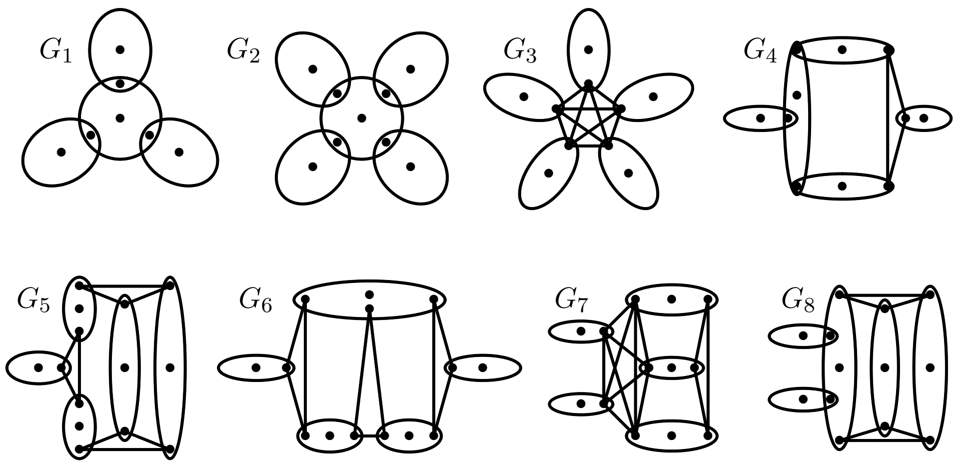
***Proof.*** We can again suppose that $G$ is closed. If $\kappa(\mathrm{cl}(G)) \geq 6$, then $G$ has a required 2-factor since $G$ is hamiltonian by Theorem B (note that $\delta(G) \geq 3k-4 \geq 8$). Hence suppose $1 \leq \kappa(\mathrm{cl}(G)) \leq 5$. Then we have $n \geq 3k^2-3 \geq 3k^2-(k+1)\kappa+2$ since $\kappa \geq 1$ and $\sigma_k(G) > n+k^2-4k+7 \geq n+k^2-4k+\kappa+2$ since $\kappa \leq 5$. Then $G$ has a 2-factor with at most $k-1$ components by Theorem 3. ■

**Remark.** It would be possible to formulate minimum degree results corresponding to Theorem 3 and Corollary 4. Details are left to the reader.

Next we turn our attention to traceability. Let $\mathcal{C}_i$, $i = 1, \ldots, 8$, be the class of all spanning subgraphs of the graphs $G_i$, $i = 1, \ldots, 8$, shown in Figure 1 (where the circular and elliptical parts represent cliques of arbitrary order). Using a technique similar to that of [6], we can prove the following result.

**Theorem 5.** *Let $G$ be a connected closed claw-free graph with clique covering number $\theta \leq 5$. Then either $G \in \cup_{i=1}^{8} \mathcal{C}_i$, or $G$ is traceable.*

Proof of Theorem 5 is lengthy and it is therefore postponed to Section 3.




$\mathcal{C}_i$ is the class of all spanning subgraphs of $G_i$, $i = 1, \ldots, 8$.

Figure 1

Combining Theorem 5 and Theorem 1, we can now obtain the following theorem, which is the second main result of this paper.

**Theorem 6.** *Let $G$ be a connected claw-free graph of order $n \geq 112 - 7\kappa(\mathrm{cl}(G))$ such that $\delta(G) \geq 14$ and*

$$\sigma_6(G) > n + 14 + \kappa(\mathrm{cl}(G)).$$

*Then either $\mathrm{cl}(G) \in \cup_{i=1}^{8} \mathcal{C}_i$, or $G$ is traceable.*

***Proof.*** If $G$ is a nontraceable graph satisfying the asumptions of the theorem, then clearly so is $\mathrm{cl}(G)$. Thus, suppose that $G$ is closed. By a well-known consequence of a theorem by Chvátal and Erdős [2] (see e.g. [7], Part I, Corollary 4.17), nontraceability of $G$ implies $\alpha(G) > \kappa + 1$. From Theorem 1 (for $k = 6$) we then obtain $\theta(G) \leq 5$. The rest of the proof follows from Theorem 5. ■

**Corollary 7.** *Let $G$ be a connected claw-free graph of order $n \geq 105$ such that $\delta(G) \geq 14$ and*

$$\sigma_6(G) > n + 19.$$

*Then either $\mathrm{cl}(G) \in \cup_{i=1}^{8} \mathcal{C}_i$, or $G$ is traceable.*

***Proof.*** We can again suppose that $G$ is closed. If $G$ is nontraceable, then Theorem B implies $1 \leq \kappa(G) \leq 5$. Rest of the proof follows immediately from Theorem 6. ■

**Corollary 8.** *Let $G$ be a connected claw-free graph of order $n \geq 105$ with minimum degree*

$$\delta(G) > \frac{n+19}{6}.$$

*Then either $\mathrm{cl}(G) \in \cup_{i=1}^{8} \mathcal{C}_i$, or $G$ is traceable.* ■

## 3. Proof of Theorem 5

We basically follow the terminology and notation introduced in [6] and [9]. Let $\mathcal{G}_\theta$ be the class of all connected nontraceable closed claw-free graphs with clique covering number $\theta$. By Theorem A, every $G \in \mathcal{G}_\theta$ is the line graph of some (unique) triangle-free graph $H$. Let $D_1(H)$ be the set of all degree 1 vertices of $H$ and put $H' = H - D_1(H)$. Set $\mathcal{H}_\theta = \{L^{-1}(G) |\ G \in \mathcal{G}_\theta\}$ and $\mathcal{H}'_\theta = \{H - D_1(H) |\ H \in \mathcal{H}_\theta\}$.

In every $G \in \mathcal{G}_\theta$ choose a fixed minimum clique covering $\mathcal{P}_G = \{B_1, \ldots, B_\theta\}$ of $G$ such that each clique $B_i$ is maximal. Since $\mathcal{P}_G$ is minimum, every $B_i$ contains at least one *proper vertex*, i.e., a vertex belonging to no other clique of $\mathcal{P}_G$. The centers $B_1, \ldots, B_\theta$ of the stars of $H = L^{-1}(G)$ that correspond to the cliques of $G$ will be called the *black vertices* of $H$. The other vertices of $H$ are called *white*. The set of black (white) vertices of $H$ is denoted by $B(H)$ ($W(H)$), respectively. Since $B(H)$ is a vertex covering of $H$ (i.e., every edge of $H$ has at least one vertex in $B(H)$), the set $W(H)$ is independent.

It is easy to see that for any $G \in \mathcal{G}_\theta$, any graph obtained from $G$ by adding/removing simplicial vertices to/from cliques of $\mathcal{P}_G$ also belongs to $\mathcal{G}_\theta$ as long as (in the case of removal) at least one simplicial vertex in the clique remains (while the removal of the last simplicial vertex of a clique can turn $G$ into a traceable graph). Hence we can without loss of generality denote for any $H' \in \mathcal{H}'_\theta$ by $L(H)$ the graph obtained from the line graph of $H'$ by adding one simplicial vertex to every clique corresponding to a black vertex of $H'$.

Let $G_1, G_2 \in \mathcal{G}_\theta$. We say that $G_1$ is an *ss-subgraph* of $G_2$, if $G_1$ is isomorphic to a spanning subgraph of a graph, which is obtained from $G_2$ by adding an appropriate number of simplicial vertices to some cliques of $\mathcal{P}_{G_2}$, and that $G_1$ is a *proper ss-subgraph* of $G_2$ if $G_1$ is an $ss$-subgraph of $G_2$ and $G_1$, $G_2$ are nonisomorphic. In the following we present a method for finding a subset $\mathcal{F}_\theta \subset \mathcal{H}'_\theta$ such that

(i) every $G \in \mathcal{G}_\theta$ is an $ss$-subgraph of $L(F)$ for some $F \in \mathcal{F}_\theta$,

(ii) for any $F_1, F_2 \in \mathcal{F}_\theta$, $L(F_1)$ is not an $ss$-subgraph of $L(F_2)$.

By the previous observations, the class $\mathcal{G}_\theta$ is fully characterized by $\mathcal{F}_\theta$.

If, for some $H \in \mathcal{H}_\theta$, the corresponding $H' \in \mathcal{H}'_\theta$ has a *black trail* (abbreviated BT), i.e., a trail containing all black vertices of $H'$, then clearly $H$ has a dominating trail. Since, by Theorem F, no $H \in \mathcal{H}_\theta$ has a dominating trail, no $H' \in \mathcal{H}'_\theta$ has a BT.

For a trail $T$ in $H' \in \mathcal{H}'_\theta$ we denote by $\mathrm{bla}(T)$ the black length of $T$, i.e., the number of black vertices of $H'$ that are on $T$, and by $\mathrm{cro}(T)$ the number of "crossings" of $T$, i.e., the number of vertices of $H'$ that are visited by $T$ at least twice.

Two vertices of $H'$ are said to be *related* if they are adjacent or if they are both black and have a common white neighbor. If $T$ is a (fixed) trail in $H'$ and $x, y$ are vertices of $H'$, then we say that $x, y$ are $\bar{T}$*-related* (denoted $x \sim y$) if $xy \in E(H') \setminus E(T)$ or $x$ and $y$ have a white common neighbor outside $T$.

Let now $H' \in \mathcal{H}'_\theta$, and let $T$ be a trail in $H'$ such that

(i) $\mathrm{bla}(T)$ is maximum,

(ii) subject to $(i)$, $\mathrm{cro}(T)$ is minimum,

(iii) subject to $(i)$ and $(ii)$, the length of $T$ is minimum.

Then $T$ has two black vertices of degree 1. We will always denote by $b_1, \ldots, b_k$ the black vertices of $T$ labelled along $T$, and by $w_i$ the white successor of $b_i$ on $T$, if it exists. Note that, since $T$ is a trail, possibly $b_i = b_j$ or $w_i = w_j$ for some $i \neq j$. If $b_i \sim b_j$, then the (possible) white common neighbor of $b_i$, $b_j$ outside $T$ will be denoted by $w_{ij}$.

*Case* $\theta = 3$.
Let $B = \{b_1, b_2, b_3\}$. Then, clearly, $\mathrm{bla}(T) = 2$, $\mathrm{cro}(T) = 0$ and $T = b_1(w_1)b_2$. Since $H'$ is connected and the set $\{b_1, b_2, b_3\}$ is dominating, $b_3$ is $\bar{T}$-related to some vertex of $T$. Clearly both $b_3 \sim b_1$ and $b_3 \sim b_2$ imply traceability of $L(H')$, hence $b_3 \sim w_1$, implying $b_3w_1 \in E(H')$. The existence of any further relation implies traceability of $L(H')$, hence $V(H') = \{b_1, b_2, b_3, w_1\}$ and $E(H') = \{w_1b_1, w_1b_2, w_1b_3\}$, implying $L(H') \in \mathcal{C}_1$.

*Case* $\theta = 4$.
Let $B = \{b_1, b_2, b_3, b_4\}$. Then obviously $2 \leq \mathrm{bla}(T) \leq 3$ and $\mathrm{cro}(T) = 0$, i.e., $T$ is a path. We have two subcases.

*Subcase* $\mathrm{bla}(T) = 3$.
Then $T = b_1(w_1)b_2(w_2)b_3$. Suppose first that $b_4$ is $\bar{T}$-related to some black vertex. Then necessarily $b_4 \sim b_2$. If $b_4 \sim x$ for some $x \in \{b_1, (w_1), b_3, (w_2)\}$, then we immediately have a trail $T'$ with $\mathrm{bla}(T') = 4$. Similarly, $b_1 \sim b_2$ yields $T' = b_3(w_2)b_2(w_1)b_1(w_{12})b_2(w_{24})b_4$, $b_1 \sim w_2$ gives $T' = b_3w_2b_1(w_1)$ $b_2(w_{24})b_4$, and for $b_1 \sim b_3$ we have $T' = b_3(w_{13})b_1(w_1)b_2(w_{24})b_4$. By symmetry and since $H'$ is triangle-free, these are all possibilities. Hence there are no further $\bar{T}$-relations, implying $L(H') \in \mathcal{C}_1$.

Hence $b_4$ is related to white vertices only. If both $b_4 \sim w_1$ and $b_4 \sim w_2$, then $H'$ contains no more relations, implying $L(H') \in \mathcal{C}_2$. If (by symmetry) $b_4 \sim w_1$ and $b_4 \not\sim w_2$, then the only possible additional relation that does not create a trail $T'$ with $\mathrm{bla}(T) = 4$ is $b_1 \sim w_2$. Then for $b_1 \not\sim w_2$ we have $L(H') \in \mathcal{C}_1$ and for $b_1 \sim w_2$ we have $L(H') \in \mathcal{C}_4$.

*Subcase* $\mathrm{bla}(T) = 2$.
Then $T = b_1(w_1)b_2$. Then immediately $b_3 \sim w_1$ and $b_4 \sim w_1$. If $b_3 \sim b_4$, then $L(H')$ is traceable; hence $b_3 \not\sim b_4$, implying $L(H') \in \mathcal{C}_2$.

*Case* $\theta = 5$.
Let $B = \{b_1, b_2, b_3, b_4, b_5\}$. We have obviously $2 \leq \mathrm{bla}(T) \leq 4$. If $2 \leq \mathrm{bla}(T) \leq 3$, then $\mathrm{cro}(T) = 0$ (since $H'$ is triangle-free), for $\mathrm{bla}(T) = 4$

we have $0 \leq \mathrm{cro}(T) \leq 1$. We will denote these subcases by $k/\ell$, where $k = \mathrm{bla}(T)$ and $\ell = \mathrm{cro}(T)$. Thus, we have subcases 4/0, 4/1, 3/0 and 2/0. The subcase 4/1 splits into two subcases 4/1w and 4/1b according to whether the vertex visited twice by $T$ is white or black, respectively. We consider these subcases separately.

*Subcase* 4/0.
Then $T = b_1(w_1)b_2(w_2)b_3(w_3)b_4$ is a path. It is straightforward to check that $b_5$ can be $\bar{T}$-related to at most one black vetex of $T$ (for otherwise $L(H')$ is traceable). Thus, we have two possibilities.

*Subcase* 4/0-1. *$b_5$ is $\bar{T}$-related to exactly one black vertex of $T$.*
By symmetry, we can suppose $b_5 \sim b_2$. We distinguish two subcases.

*Subcase* 4/0-1-1. *$b_5$ is $\bar{T}$-related to some white vertex on $T$.*
Then the only possibility that does not imply $L(H')$ is traceable is $b_5 \sim w_3$. Then it is straightforward to check that any further $\bar{T}$-relation between vertices of $T$ implies $L(H')$ is traceable, but then $L(H') \in \mathcal{C}_4$.

*Subcase* 4/0-1-2. *$b_2$ is the only $\bar{T}$-relation of $b_5$ on $T$.*
We consider possible $\bar{T}$-relations between vertices of $T$.

If $b_1 \sim w_3$, then we are in a situation symmetric to the subcase 4/0-1-1 and hence $L(H') \in \mathcal{C}_4$. All the other relations of $b_1$ on $T$ imply $L(H')$ is traceable. Hence we can assume $b_1$ has no $\bar{T}$-relation on $T$. Now, if also $b_4$ has no $\bar{T}$-relation on $T$, then we have $L(H') \in \mathcal{C}_1$. Hence we can suppose $b_4 \sim x$ for some $x \in V(T)$. If $x \in \{b_1, w_1, b_2\}$, then $L(H')$ is traceable. Hence $x \in \{w_2, b_3\}$. Now, if there is no $\bar{T}$-relation $y \sim z$ for any $y \in \{w_1, b_2\}$, $z \in \{w_2, b_3, w_3, b_4\}$, then we have $L(H') \in \mathcal{C}_1$. It is straightforward to check that all such relations $y \sim z$ imply $L(H')$ is traceable.

*Subcase* 4/0-2. *$b_5$ is $\bar{T}$-related only to white vertices on $T$.*
In this subcase, we have three further possibilities.

*Subcase* 4/0-2-1. *$b_5$ is $\bar{T}$-related to $w_1$, $w_2$ and $w_3$.*
Then there is no further $\bar{T}$-relation on $T$ and $L(H') \in \mathcal{C}_6$.

*Subcase* 4/0-2-2. $b_5$ *is* $\bar{T}$*-related to two white vertices on* $T$.

By symmetry, we can suppose that $b_5 \sim w_1$ and either $b_5 \sim w_2$ or $b_5 \sim w_3$.

Let first $b_5 \sim w_2$. If no vertices on $T$ are $\bar{T}$-related, then $L(H') \in \mathcal{C}_4$ (with $b_3, (w_3), b_4$ in one clique of $L(H')$). Hence suppose there is a $\bar{T}$-relation between some vertices of $T$. Clearly $b_1 \not\sim b_2$, $b_1 \not\sim w_2$, $b_1 \not\sim b_3$, $b_1 \not\sim b_4$, $w_1 \not\sim b_3$, $w_1 \not\sim b_4$, $b_2 \not\sim b_3$, $b_2 \not\sim b_4$ and $w_2 \not\sim b_4$, since any of these relations implies $L(H')$ is traceable. It remains to consider the possibilities $b_1 \sim w_3$, $b_2 \sim w_3$ and $b_3 \sim b_4$.

If $b_1 \sim w_3$, then both $b_2 \not\sim w_3$ and $b_3 \not\sim b_4$ (otherwise $L(H')$ is traceable), and then $L(H') \in \mathcal{C}_5$; if $b_2 \sim w_3$, then $b_1 \not\sim w_3$ and $b_3 \not\sim b_4$, implying $L(H') \in \mathcal{C}_6$; and if $b_3 \sim b_4$, then similarly $b_1 \not\sim w_3$, $b_2 \not\sim w_3$ and $L(H') \in \mathcal{C}_4$ (in which $b_3, b_4$ and their common neighbors are in one clique).

Hence suppose $b_5 \sim w_3$. Similarly as before, no $\bar{T}$-relation between vertices of $T$ implies $L(H') \in \mathcal{C}_4$ with $b_2, (w_2), b_3$ in one clique. Thus, suppose some vertices of $T$ are $\bar{T}$-related. Immediately $b_1 \not\sim b_2$, $b_1 \not\sim b_3$, $b_1 \not\sim w_3$, $b_1 \not\sim b_4$ and $w_1 \not\sim b_3$, since any of these relations implies $L(H')$ is traceable. By symmetry, it remains to consider the possibilities $b_1 \sim w_2$ and $b_2 \sim b_3$. If $b_1 \sim w_2$, then $b_2 \not\sim b_3$ (otherwise $L(H')$ is traceable), implying $L(H') \in \mathcal{C}_5$; if $b_2 \sim b_3$, then similarly $b_1 \not\sim w_2$ and $L(H') \in \mathcal{C}_4$ (with $b_2, b_3$ and their common neighbors in one clique).

*Subcase* 4/0-2-3. $b_5$ *is* $\bar{T}$*-related to exactly one white vertex on* $T$.

By symmetry, either $b_5 \sim w_1$ or $b_5 \sim w_2$.

Let first $b_5 \sim w_1$. If $b_1$ is not $\bar{T}$-related to any of $b_2, w_2, b_3, w_3, b_4$, then $L(H') \in \mathcal{C}_1$ (with $b_2, b_3$ and $b_4$ in one clique). The relations $b_1 \sim b_2$ and $b_1 \sim b_4$ immediately imply traceability. Hence $b_1$ is $\bar{T}$-related to $w_2$, $b_3$ or $w_3$.

If $b_1 \sim b_3$ and, at the same time, $b_1 \sim w_2$ or $b_1 \sim w_3$, then $L(H')$ is traceable, and if $b_1 \sim w_2$ and $b_1 \sim w_3$, then we are in Subcase 4/0-2-1 (where $b_1$ plays the role of $b_5$). Hence $b_1$ is $\bar{T}$-related to exactly one of $b_3$, $w_2$, $w_3$.

If $b_1 \sim b_3$, then any additional relation implies $L(H')$ is traceable, and hence we have $L(H') \in \mathcal{C}_4$.

If $b_1 \sim w_2$, then for $b_2 \sim w_3$ we are in Subcase 4/0-2-1 (where $b_2$ plays the role of $b_5$) and $L(H') \in \mathcal{C}_6$. Any other additional relation except $b_3 \sim b_4$

implies $L(H')$ is traceable. If $b_3 \sim b_4$, or if there is no additional relation, we have $L(H') \in \mathcal{C}_4$ (with $b_3$, $b_4$ in one clique).

If $b_1 \sim w_3$, then any additional relation except for $b_2 \sim b_3$ or $w_2 \sim b_4$ implies $L(H')$ is traceable. For $w_2 \sim b_4$ we have $L(H') \in \mathcal{C}_5$, and if $b_2 \sim b_3$ or if there is no additional relation, then $L(H') \in \mathcal{C}_4$ (with $b_2$ and $b_3$ in one clique).

Let now $b_5 \sim w_2$. First observe that there is no $\bar{T}$-relation containing $w_2$ since $H'$ is triangle-free and both $w_2 \sim b_1$ and $w_2 \sim b_4$ imply traceability. Secondly, if there is no $\bar{T}$-relation $x \sim y$ with $x \in \{b_1, w_1, b_2\}$ and $y \in \{b_3, w_3, b_4\}$, then $L(H') \in \mathcal{C}_1$. Since $b_1 \sim b_3$ and $b_1 \sim b_4$ imply traceability, by symmetry, we have $b_2 \sim b_3$, $w_1 \sim b_3$ or $b_1 \sim w_3$. We consider these possibilities separately.

If $b_2 \sim b_3$, then there is no additional $\bar{T}$-relation containing $b_1$ (or symmetrically $b_4$), for otherwise $L(H')$ is traceable. This implies $L(H') \in \mathcal{C}_1$.

If $w_1 \sim b_3$, then similarly $L(H') \in \mathcal{C}_1$, unless there is an additional $\bar{T}$-relation containing $b_1$ or $b_4$. The only such relations that do not imply traceability are $b_1 \sim w_3$ or $b_3 \sim b_4$, but then $L(H') \in \mathcal{C}_6$ or $L(H') \in \mathcal{C}_4$, respectively.

Finally, if $b_1 \sim w_3$, then we already know there is no further relation, and we have $L(H') \in \mathcal{C}_4$.

*Subcase* 4/1w.
Recall that in this subcase $T$ visits twice one white vertex. Choose the notation such that $T = b_1 w_1 b_2 w_2 b_3 w_1 b_4$. Clearly, $b_5 \not\sim b_1$, $b_5 \not\sim b_4$ and $b_5$ cannot be $\bar{T}$-related to both $b_2, b_3$ (since in each of these cases $L(H')$ is traceable). Thus, $b_5$ is $\bar{T}$-related to at most one black vertex on $T$. There are two subcases.

*Subcase* 4/1w-1. *$b_5$ is $\bar{T}$-related to one black vertex on $T$.*
By symmetry, let $b_5 \sim b_2$. Then $L(H') \in \mathcal{C}_1$ (with $w_1, b_2, w_2, b_3$ in one clique), since any additional $\bar{T}$-relation involving any of $b_1$, $b_4$, $b_5$ implies $L(H')$ is traceable.

*Subcase* 4/1w-2. *$b_5$ is $\bar{T}$-related only to white vertices.*
In this subcase, we distinguish three further possibilities.

*Subcase* 4/1w-2-1. $b_5 \sim w_1$, $b_5 \sim w_2$.
Then there is no other relation and $L(H') \in \mathcal{C}_7$.

*Subcase* 4/1w-2-2. $b_5 \sim w_1$, $b_5 \not\sim w_2$.
If there is no other relation involving any of $b_1$, $b_2$, then we have $L(H') \in \mathcal{C}_1$ (with $b_2, b_3$ and their common neighbors in one clique). It is straightforward to check that any further $\bar{T}$-relation involving $b_1$ or $b_4$ gives $L(H') \in \mathcal{C}_8$ (if some of $b_1, b_2$ is $\bar{T}$-related to $w_2$), or traceability of $L(H')$.

*Subcase* 4/1w-2-3. $b_5 \sim w_2$, $b_5 \not\sim w_1$.
Then $L(H') \in \mathcal{C}_1$ (with $w_1, b_2, w_2, b_3$ in one clique) and any $\bar{T}$-relation between any of $b_2, w_2, b_3$ and the rest implies traceability of $L(H')$.

*Subcase* 4/1b.
Choose the notation such that the vertex $b_2$ is visited twice by $T$, i.e., $T = b_1(w_1)b_2w_2b_3w_3b_2(w_4)b_4$. Similarly as before, $b_5$ is $\bar{T}$-related to at most one black vertex on $T$, and neither to $b_1$ nor to $b_4$. We distinguish three subcases.

*Subcase* 4/1b-1. $b_5 \sim b_2$, $b_5 \not\sim b_3$.
In this case $b_2$ is the only $\bar{T}$-relation of $b_5$ on $T$ (since any other relation implies traceability). Now $L(H') \in \mathcal{C}_2$ and any other $\bar{T}$-relation between vertices of $T$ gives $L(H') \in \mathcal{C}_1$ or traceability. We distinguish three possibilities.

*Subcase* 4/1b-2. $b_5 \sim b_3$, $b_5 \not\sim b_2$.
In this subcase immediately $L(H') \in \mathcal{C}_1$ with $\{b_2, w_2, b_3, w_3\}$ in one clique and any relation joining a vertex from this set to the rest gives traceability.

*Subcase* 4/1b-3. $b_5$ *is $\bar{T}$-related only to white vertices.*
Then $b_5$ can have $\bar{T}$-relations in at most one of the sets $\{w_1, w_4\}$, $\{w_2, w_3\}$ (otherwise $L(H')$ is traceable). We have two further possibilities.

*Subcase* 4/1b-3-1. $b_5 \sim w_1$.
For $b_5 \not\sim w_4$ we have $L(H') \in \mathcal{C}_1$, and for $b_5 \sim w_4$ we have $L(H') \in \mathcal{C}_4$ (with $b_2, w_2, b_3, w_3$ in one clique).

*Subcase* 4/1b-3-2. $b_5 \sim w_2$.
If $b_5 \not\sim w_3$, then $L(H') \in \mathcal{C}_1$ (with $b_2, w_2, b_3, w_3$ in one clique), and if $b_5 \sim w_3$, then $L(H') \in \mathcal{C}_8$.

*Subcase* 3/0.
Let $T = b_1(w_1)b_2(w_2)b_3$.

*Subcase* 3/0-1. $b_4 \sim b_5$.
If $b_4$ or $b_5$ is $\bar{T}$-related to any vertex on $T$, then we have a path $T'$ with $\mathrm{bla}(T') \geq 4$, except for the case if $w_{45} = w_{24} = w_{25}$. In this case, $L(H') \in \mathcal{C}_2$.

*Subcase* 3/0-2. $b_4 \not\sim b_5$.
If $b_4$ has two relations on $T$, then the only possibility that does not create a trail $T'$ with $\mathrm{bla}(T') \geq 4$ is $b_4 \sim w_1$, $b_4 \sim w_2$, but then, for any $\bar{T}$-relation of $b_5$ on $T$ we again have a trail $T'$ with $\mathrm{bla}(T') \geq 4$. Hence both $b_4$ and $b_5$ have one $\bar{T}$-relation on $T$. Then it is straightforward to check that in all nontraceable cases we have $L(H') \in \mathcal{C}_2$ or $L(H') \in \mathcal{C}_1$.

*Subcase* 2/0.
Let $T = b_1(w_1)b_2$. If any two of $b_3, b_4, b_5$ are $\bar{T}$-related, we have a trail $T'$ with $\mathrm{bla}(T') \geq 3$. Hence $b_3 \sim w_1$, $b_4 \sim w_1$, $b_5 \sim w_1$, implying $L(H') \in \mathcal{C}_3$. ■

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