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FORBIDDEN TRIPLES IMPLYING HAMILTONICITY: FOR ALL GRAPHS

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Abstract

In [2], Brousek characterizes all triples of graphs, G_1, G_2, G_3, with $G_i = K_{1,3}$ for some $i = 1, 2$, or 3, such that all $G_1G_2G_3$-free graphs contain a hamiltonian cycle. In [6], Faudree, Gould, Jacobson and Lesniak consider the problem of finding triples of graphs G_1, G_2, G_3, none of which is a $K_{1,s}$, $s \geq 3$ such that G_1, G_2, G_3-free graphs of sufficiently large order contain a hamiltonian cycle.

In this paper, a characterization will be given of all triples G_1, G_2, G_3 with none being $K_{1,3}$, such that all $G_1G_2G_3$-free graphs are hamiltonian. This result, together with the triples given by Brousek, completely characterize the forbidden triples G_1, G_2, G_3 such that all $G_1G_2G_3$-free graphs are hamiltonian.

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1. Introduction

The problem of recognizing graph properties based on forbidden subgraphs has received considerable attention. A wide variety of properties and forbidden families have been studied. In particular, the property of being hamiltonian has been considered. A series of results culminated in the characterization, by Bedrossian [1], of the pairs of forbidden subgraphs which imply all graphs are hamiltonian. In his proof, Bedrossian used some small order counterexamples to eliminate some cases. Faudree and Gould [4] extended the collection to characterize the forbidden pairs which imply all graphs of order $n \geq 10$ are hamiltonian.

Since the only single forbidden subgraph that implies a graph is hamiltonian is P_3 (the path on 3 vertices) and it forces the graph to be complete, the problem of all single or pairs of forbidden subgraphs implying hamiltonicity has been completely characterized, both for all graphs and for all sufficiently large graphs.

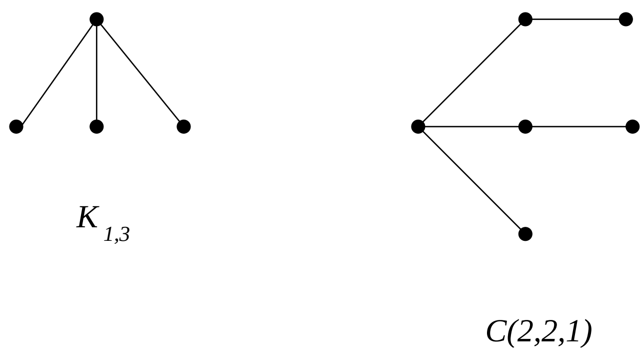

Figure 1. Common forbidden graphs

An interesting feature of both of these characterizations for pairs is that the claw $K_{1,3}$ (see Figure 1) must be one of the graphs in each pair. This led to the question: If we consider triples of forbidden subgraphs implying hamiltonicity, must the claw always be one of the graphs in the triple? This question was answered negatively in [6]. There, all triples containing no $K_{1,t}$, $t \geq 3$ which imply all sufficiently large graphs are hamiltonian were given. Brousek [2] gave the collection of all triples which include the claw that imply all 2-connected graphs are hamiltonian.

The purpose of this paper is to complete the triples question for all 2-connected graphs by providing those triples not including $K_{1,t}$, $t \geq 3$ which imply all 2-connected graphs are hamiltonian, as well as triples containing $K_{1,t}$, $t \geq 4$.

We follow the notation of [3]. In addition, we say a graph H is $G_1G_2G_3$-free if H does not contain G_i, $i = 1, 2, 3$ as an induced subgraph. We also define the graph $C(i, j, k)$ (see Figure 1 for $C(2, 2, 1)$) to be the graph obtained by identifying the end vertex of paths of lengths i, j and k, respectively. This graph may be thought of as a kind of generalized claw as $K_{1,3} = C(1, 1, 1)$.

Given a cycle with an implied orientation, we write x^+ and x^- for the successor and predecessor of x on the cycle, repsectively. Further, by $[x, y]$ we mean the subpath of C beginning at x and ending at y and following the orientation of C. We define the graphs J_1 and J_2 to be the complete graph K_m on m vertices ($m \geq 3$) with one or two edges joined to a single vertex, respectively (see Figure 2). The book B_n is obtained by identifying an edge from each of n triangles (see Figure 2 for B_2). The graph Z_i is obtained by adjoining a path of length i at one vertex of a triangle (see Figure 2 for Z_1).

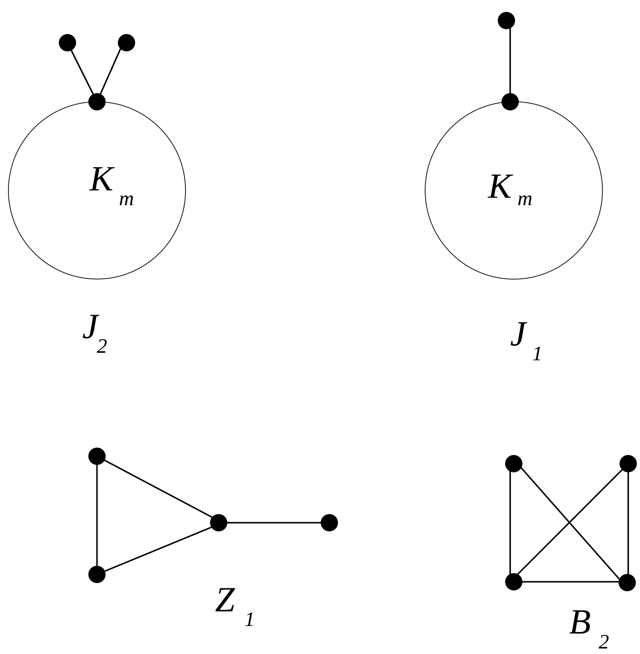

Figure 2. More common forbidden graphs

2. Main Results

Since Brousek gave all triples where one graph is a $K_{1,3}$, to complete the characterization of triples G_1, G_2, G_3 such that all 2-connected $G_1G_2G_3$-free graphs are hamiltonian, we will consider two cases. First, we consider the case where one of the $G_i \cong K_{1,t}$ for $t \geq 4$ and second is the case where no $G_i \cong K_{1,t}$ for $t \geq 3$.

In [5], Faudree, Gould and Jacobson give the following result:

Theorem 1. *If G is a 2-connected graph of sufficiently large order which is $G_1G_2G_3$-free where G_1, G_2, G_3 are one of the following triples:*

(i) $P_4,\ K_{1,t},\ J_2,\ t \geq 4$,
(ii) $P_4,\ K_{1,t},\ B_2,\ t \geq 4$,
(iii) $P_r,\ K_{1,t},\ J_1,\ r \geq 5, t \geq 4$,
(iv) $C(l,1,1),\ K_{1,t},\ Z_1,\ l \geq 2, t \geq 4$,

or G_1, G_2, G_3 is a triple of induced subgraphs of one of these triples, then G is hamiltonian. Furthermore, these are the only possible triples that contain $K_{1,t}, t \geq 4$.

But, by considering $K_{2,3}$, it is easy to see that none of these triples would imply hamiltonicity for all 2-connected graphs. So, to complete this characterization, we need to consider the triples where none of the graphs are a $K_{1,m}, (m \geq 3)$.

In [6], Faudree, Gould, Jacobson and Lesniak, characterized the triples G_1, G_2, G_3, none of which are $K_{1,t}, t \geq 3$ such that sufficiently large $G_1G_2G_3$-free graphs are hamiltonian.

The following was shown:

Theorem 2. *Let G be a 2-connected graph of sufficiently large order n, and let G_1, G_2, G_3 be connected graphs with at least three edges, none of which is a $K_{1,t}, t \geq 3$. Then G being $G_1G_2G_3$-free implies that G is hamiltonian if, and only if, G_1, G_2, G_3 is one of the following triples:*

(i) $P_4B_2K_{2,\lceil\frac{n+1}{2}\rceil}$,
(ii) $P_4B_3K_{2,3}$,
(iii) $P_5B_1K_{2,\lfloor\frac{n}{3}\rfloor}$,
(iv) $C(2,1,1), B_1, K_{2,\lfloor\frac{n}{2}\rfloor-2}$,
(v) $C(3,1,1)B_1K_{2,2}$,
(vi) $C(2,2,1)B_1K_{2,2}$,
(vii) $P_6B_1K_{2,2}$,
(viii) $P_5B_2K_{2,3}$,

or G_1, G_2, G_3 is a triple of induced subgraphs of one of these eight triples.

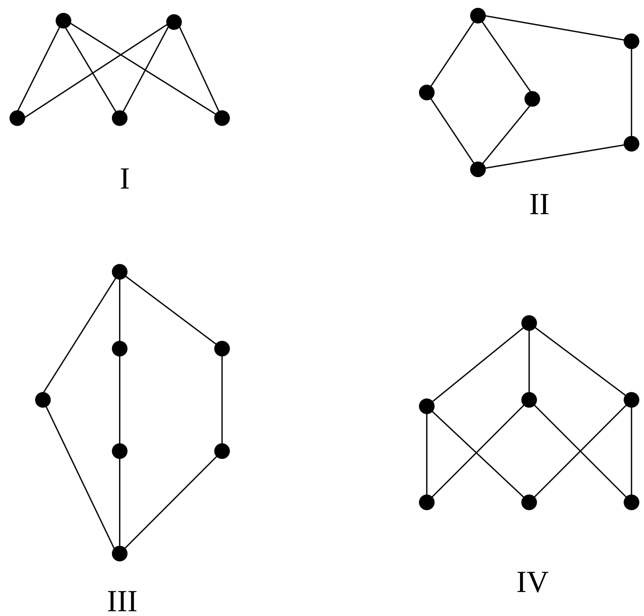

Figure 3. Forbidden graphs I–IV

By considering the graphs in Figure 3, the triples in Theorem 2 can be trimmed down to only a few possibilities.

Theorem 3. *If G_1, G_2, G_3 is a triple of graphs, none of which is a $K_{1,t}$, $t \geq 3$, and for all 2-connected G_1, G_2, G_3-free graphs G, G is hamiltonian, then it is the case that G_1, G_2, G_3 is one of the following triples:*

(a) $P_4 B_3 K_{2,3}$,

(b) $P_5 B_2 K_{2,2}$,

(c) $C(2,2,1) B_1 K_{2,2}$,

or is a triple of induced subgraphs of one of these triples.

Proof. Clearly, it follows since these graphs are hamiltonian regardless of order, they must be hamiltonian for sufficiently large order. So the triples must be triples G_1, G_2, G_3 which are induced triples from the eight possibilities in Theorem 2. Graph I in Figure 3 reduces class (i) in Theorem 2 to $P_4 B_2 K_{2,3}$, which is a triple of induced subgraphs of (a). Graph II in Figure 3 reduces class (iii) to $P_5 B_1 K_{2,2}$, which is a triple of induced subgraphs of (b), and class (viii) to $P_5 B_2 K_{2,2}$, which is triple (b). Graph III eliminates class (v) and (vii) totally, while graph IV from Figure 3 reduces class (iv) to $C(2,1,1)B_1K_{2,2}$, which is a triple of induced subgraphs of (c). Classes (ii) and (vi) remain uneffected by the graphs in Figure 3. Thus, the result follows. ■

To complete the characterization, it only remains to show that in fact forbidding any one of these triples actually does imply the graph is hamiltonian for all such 2-connected graphs. In [6], the following was shown.

Theorem C. *Let G be a 2-connected P_4-free non-hamiltonian graph. Then either B_3 or $K_{2,3}$ is an induced subgraph of G.*

From this it follows that:

Corollary 4. *If G is a 2-connected $P_4B_3K_{2,3}$-free graph (or $P_4B_2K_{2,3}$-free), then G is hamiltonian.*

We now complete the positive results with the following Theorem.

Theorem 5. *If G is a 2-connected $C(2,2,1)B_1K_{2,2}$-free or $P_5B_2K_{2,2}$-free graph, then G is hamiltonian.*

Proof. Since $K_{2,3}$ is the only 2-connected non-hamiltonian graph with fewer than 6 vertices, it follows that we may assume that G contains at least 6 vertices.

Claim 1. The longest cycle in G has length at least 6.

***Proof of Claim* 1.** Consider a longest cycle C in G. If C is a hamiltonian cycle, we are done. If not, then there exists a vertex x with at least two disjoint (except for x) paths to C. Say these paths end at y and z, respectively. Clearly, these paths do not end at consecutive vertices of C, so there is at least one vertex in each segment of C determined by y and z. Say such vertices are a and b.

Suppose first that $|V(C)| = 4$, that is, each segment has exactly one vertex. Then, the two paths from x must each be an edge, or else we would find a cycle longer than C. Now $H = \langle x, a, y, z\rangle \cong K_{2,2}$ or B_2. However, $K_{2,2}$ and B_2 are forbidden and $H = K_4$ allows us to extend C. If $|V(C)| = 5$, then a similar argument applies to $\langle z, b, y, x\rangle$ or $\langle x, a, y, z\rangle$. Thus, each segment of C between y and z must contain two or more vertices and hence, $|V(C)| \geq 6$. ■

By Claim 1, we may assume that $|V(G)| \geq 7$, for otherwise it follows that G is hamiltonian.

Case 1. Suppose that G is a $P_5B_2K_{2,2}$-free graph.
Consider a longest cycle C with a vertex $z \notin V(C)$. Again, z has at least two disjoint (except for z) paths P_1 and P_2 to C. Consider them as a shortest path P outside C except for its end vertices. If the path P contains more than two vertices not on C, say $P : z_t, z_{t-1}, \ldots, z_1, z, w_1, w_2, \ldots w_s$ where $z_1 \neq z$, $z_1 \notin V(C)$ and $z_t \in V(C)$ and where $w_1 \neq z$, $w_1 \notin V(C)$ and $w_s \in V(C)$, then since P is a shortest such path, we have that $z_1w_1 \notin E(G)$.

Now z_1, z and w_1 are not adjacent to z_t^+ by our choice of shortest path. For the same reason, z and w_1 are not adjacent to z_t, and z_1 is not adjacent to z_t unless $t = 2$. But now, $z_t^+, z_t, \ldots, z_1, z, w_1$ contains an induced P_5, contradicting our assumption. Thus, we may assume the path P off C has exactly one or two vertices not on C.

Now suppose $P : u, u_1, u_2, v$ is this path with $u, v \in V(C)$ and $u_1, u_2 \notin V(C)$. Note that u_1v and u_2u are not edges since that would contradict the minimality of P. Clearly, $|[u^+, v^-]| \geq 2$ or we could extend C. Now, $uv \notin E(G)$ since $K_{2,2}$ is forbidden. But now $\langle u^+, u, u_1, u_2, v\rangle \cong P_5$ unless $u^+v \in E(G)$. Similarly, $\langle u, u_1, u_2, v, v^+\rangle \cong P_5$ unless $uv^+ \in E(G)$. But now $\langle u, u^+, v, v^+\rangle \cong K_{2,2}$ or B_2, a contradiction completing this case.

Finally suppose the path P off C contains exactly one vertex not on C. Say the path is x, z, y where $x, y \in V(C)$. Without loss of generality we may assume z has no edges to C in $[x, y]$. Note that x^+ and y^- must be distinct or $\langle x, y, z, x^+\rangle \cong K_{2,2}$ or B_2, a contradiction.

Now, if $xy \notin E(G)$, then x^+y, xy^-, xy^+, $x^-y \notin E(G)$ or we would find $K_{2,2}$ or B_2 induced in G. Further, $x^+y^+ \notin E(G)$ or we could extend C. Similarly, $x^-y^- \notin E(G)$. But now, $\langle y^+, y, z, x, x^+\rangle \cong P_5$ a contradiction. Hence, we conclude that $xy \in E(G)$.

Now note that there exists a vertex $w \in [x^+, y^-]$ such that $xw \in E(G)$, but $xw^+ \notin E(G)$. This follows since $xx^+ \in E(G)$ and $xy^- \notin E(G)$. Observe then that $yw \notin E(G)$ or else $\langle x, y, w, z\rangle \cong B_2$. As before, if $xy^+ \in E(G)$ then $\langle x, y^+, y, z\rangle \cong B_2$, a contradiction. While if $y^+w \in E(G)$, then $\langle x, y^+, w, y\rangle \cong K_{2,2}$, again a contradiction.

But, this implies that $yw^+ \notin E(G)$ or else $\langle w, w^+, y, x\rangle \cong K_{2,2}$ or B_2. Now $y^+w^+ \in E(G)$ or else $\langle y^+, y, x, w, w^+\rangle \cong P_5$. Then $\langle y^+, w^+, w, x, z\rangle \cong P_5$ a contradiction. This completes Case 1. ∎

Case 2. Suppose G is $C(2,2,1)B_1K_{2,2}$-free.
Let C be a longest cycle in G. Recall that $|V(C)| \geq 6$. Suppose $x \notin V(C)$ and that $xy \in E(G)$, where $y \in V(C)$. Let $y^{--} = z_1$ and $y^{++} = z_2$. Consider $\langle z_1, y^-, y, y^+, z_2, x\rangle$. This graph contains $C(2,2,1)$. However, only the edge z_1z_2 can be added without producing either B_1 or $K_{2,2}$.

Now, $[z_2^+, z_1^-]$ must contain at least 3 vertices or a $K_{2,2}$ or B_1 would result. Hence, $|V(C)| \geq 8$.

Consider next $\langle z_1, z_1^+, z_1^{--}, z_1^-, z_2, z_2^-\rangle$. This graph contains $C(2,2,1)$, hence, more edges must be present. However, any edge from z_1^+ produces a B_1 or $K_{2,2}$. Similarly, there are no additional edges from z_2 or z_1^-. Hence, the only possible edge is $z_1^{--}z_2^-$.

Similarly, $\langle z_1, y^-, y, z_1^-, z_1^{--}, z_2\rangle$ implies $z_1^{--}y \in E(G)$. But then, $\langle z_1^{--}, y, z_2^-\rangle \cong B_1$, a contradiction, completing this case and the proof of the result. ∎

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