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CHARACTERIZATIONS OF PLANAR PLICK GRAPHS

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Abstract

In this paper we present characterizations of graphs whose plick graphs are planar, outerplanar and minimally nonouterplanar.

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1. Introduction

By a graph we mean a finite, undirected graph without loops or multiple edges. We refer the terminology of [3]. For any graph G, $L(G)$ denote the line graph of G.

The *inner vertex number* $i(G)$ of a planar graph G is the minimum number of vertices not belonging to the boundary of the exterior region in any embedding of G in the plane. A graph G is said to be *k-minimally nonouterplanar* if $i(G) = k, k \geq 1$. An 1-minimally nonouterplanar graph is called *minimally nonouterplanar* (see [4]). A graph G is outerplanar if and only if $i(G) = 0$ (see [2]).

A graph G^+ is the *endedge graph* of a graph G if G^+ is obtained from G by adjoining an endedge $u_i u_i'$ at each vertex u_i of G.

The *plick graph* $P(G)$ of a graph G is obtained from the line graph by adding a new vertex corresponding to each block of the original graph and joining this vertex to the vertices of the line graph which correspond to the edges of the block of the original graph (see [5]).

The following will be useful in the proof of our results.

Theorem A [5]**.** *A graph G is a cycle if and only if the plick graph $P(G)$ is a wheel.*

Theorem B [6]**.** *The line graph $L(G)$ of a planar graph G is planar if and only if $\Delta(G) \leq 4$ and if $\deg v = 4$ for a vertex v of G, then v is a cutvertex.*

Theorem C [1]**.** *The line graph $L(G)$ of a graph G is outerplanar if and only if $\Delta(G) \leq 3$ and if $\deg v = 3$ for a vertex v of G, then v is a cutvertex.*

Remark 1 [5]**.** For any graph, $L(G)$ is a subgraph of $P(G)$.

2. Results

We first prove the following theorem, which will be useful to prove some of our next results.

Theorem 1. *Let G be a connected graph. Then $P(G)$ and $L(G)^+$ are isomorphic if and only if G is a tree.*

Proof. Suppose G is a tree. Then every block B_i of G is K_2. Therefore, there is a one-to-one correspondence between the vertices of $L(G)$ and the edges of G such that two vertices of $L(G)$ are adjacent if the corresponding edges of G are adjacent. $L(G)^+$ is obtained from $L(G)$ by adjoining a new edge at each vertex of $L(G)$ such that this edge has exactly one vertex in common with $L(G)$. By the definition of $P(G)$, the vertices u_i, u_i' in $P(G)$ corresponding to edge e_i and block B_i of G, respectively, are adjacent. By Remark 1, $L(G)$ is a subgraph of $P(G)$. Every vertex of the subgraph isomorphic to $L(G)$ is in both $P(G)$ and $L(G)^+$ adjacent to exactly one end vertex. Therefore $P(G)$ is isomorphic to $L(G)^+$.

Conversely, suppose $P(G)$ is isomorphic to $L(G)^+$. Assume G has a cycle. Then the number of edges of G is greater than the number of blocks of G. Clearly $P(G)$ has less number of vertices than $L(G)^+$. Hence $P(G)$ is not isomorphic to $L(G)^+$, a contradiction. This completes the proof. ∎

We now characterize graphs whose plick graphs are planar.

Theorem 2. *The plick graph $P(G)$ of a graph G is planar if and only if G satisfies the following conditions:*

(1) $\Delta(G) \leq 4$, *and*

(2) *every block of G is either a cycle or a K_2.*

Proof. Suppose $P(G)$ is planar. Assume $\Delta(G) \geq 5$. Then $L(G)$ contains K_5 as a subgraph and by Remark 1, $P(G)$ is nonplanar, a contradiction. This proves (1). If G has a block B which is neither a cycle nor a K_2, then this block has a subgraph homeomorphic to $K_{2,3}$ and $K_{2,3}$ is homeomorphic to $K_4 - x$ where x is any edge of K_4. The vertex u of $P(G)$ which corresponds to the block B of the graph G is, in $P(G)$, adjacent to every vertex of $L(B)$. This produces the subgraph homeomorphic to K_5 in $P(G)$, a contradiction. This proves (2).

Conversely, suppose G satisfies (1) and (2). It is easy to see that G is planar and, by Theorem B, $L(G)$ is planar. By Remark 1, $L(G)$ is a subgraph of $P(G)$. Assume a cycle C_r is a block of G. Then $L(C_r)$ is the cycle in the planar graph $L(G)$. Since in the planar drawing of $L(G)$ every such cycle can be drawn with all its vertices on the boundary of one region, consider a crossing-free drawing of $L(G)$ in which all vertices of the subgraph $L(C_r)$ are on the boundary of one region. Then the vertex u_r corresponding to the block C_r of the graph G can be placed into this region in such a way that the edges joining u_r with the vertices of $L(C_r)$ do not cross the edges of $L(G)$. Assume K_2 is a block of G. Then $L(K_2)$ is the vertex say w_n, in $L(G)$. Let u_n be the corresponding vertex of K_2 in $P(G)$. Then the edge $u_n w_n$ is in $P(G)$. This edge can be placed in some region of $P(G)$ without losing planarity. Thus $P(G)$ is planar. This completes the proof. ∎

Theorem 3. *The plick graph $P(G)$ of a connected graph G is outerplanar if and only if G satisfies the following conditions:*

(1) $\Delta(G) \leq 3$, *and*

(2) *G is a tree.*

Proof. Suppose $P(G)$ is outerplanar. Then $P(G)$ is planar. By Theorem 2, $\Delta(G) \leq 4$. Assume $\Delta(G) = 4$. Then K_4 is the subgraph of $L(G)$. Since

$L(G)$ is a subgraph of $P(G)$, $P(G)$ is not outerplanar, a contradiction. This proves (1). Suppose now G is not a tree. Then the cycle is a subgraph of G. By Theorem A, the wheel is the subgraph of $P(G)$, a contradiction. This proves (2).

Conversely, suppose G satisfies (1) and (2). Then every block of $L(G)$ is either a K_2 or a triangle. The graph $L(G)^+$ is obtained from $L(G)$ by adjoining an endedge at each vertex of $L(G)$. Clearly $L(G)^+$ is outerplanar. By Theorem 1, $P(G)$ is isomorphic to $L(G)^+$. Thus $P(G)$ is outerplanar. This completes the proof. ■

Theorem 4. *The plick graph $P(G)$ of a graph G is minimally nonouterplanar if and only if G satisfies the following conditions:*

(1) $\Delta(G) \leq 3$, *and*

(2) *G is unicyclic.*

Proof. Suppose $P(G)$ is minimally nonouterplanar. By Remark 1, $L(G)$ is a subgraph of $P(G)$. Thus, the subgraph $L(G)$ is planar and, by Theorem B, G does not contain a vertex of degree at least 5. If G contains a vertex of degree 4, then K_4 is the subgraph of $L(G)$. Consider a crossing-free drawing of $L(G)$. This drawing is not outerplanar, because at most three vertices of the subgraph K_4 are on the boundary of one region. In $P(G)$ every vertex of the considered subgraph K_4 is adjacent to some vertex of $P(G)$, which is not a vertex of $L(G)$. These new vertices cannot be placed into regions of the planar drawing of $L(G)$ without losing planarity or minimally nonouterplanarity. Thus, $P(G)$ is not minimally nonouterplanar. We arrive at a contradiction and so $\Delta(G) \leq 3$. This proves (1).

As $P(G)$ is minimally nonouterplanar, by Theorem 3, G contains at least one cycle. Suppose G has at least two cycles. We consider two cases.

Case 1. Suppose each cycle is a block of G. Then $P(G)$ has at least two different subgraphs each of which is a wheel. It is known that every wheel is minimally nonouterplanar. Thus $P(G)$ is not minimally nonouterplanar.

Case 2. Suppose G has a subgraph homeomorphic to $K_4 - x$. Then by Theorem 2, $P(G)$ is nonplanar, a contradiction.

In both cases (1) and (2) we arrive at a contradiction. Therefore G is unicyclic. Conversely, suppose G satisfies (1) and (2). By Theorem 3, $P(G)$ is not outerplanar and by Theorem C, $L(G)$ is outerplanar. Thus

$L(G)$ can be drawn on the plane in such a way that all its vertices lie on the exterior region. By condition (2), G contains exactly one cycle C. It is easy to see that the line graph of the cycle C is again a cycle C' having the same length as C. In every outerplanar drawing of $L(G)$ there is a bounded region having all vertices of C' on its boundary. The corresponding vertex u of the block C in $P(G)$ can be drawn in this region. A cycle C' in $L(G)$ together with the edges uv_i, $i = 1, 2, 3, \ldots, t$; $v_i \in C'$ forms the wheel as the subgraph in $P(G)$. Suppose G has blocks B_i each of which is a K_2. Let w_i be the corresponding vertex of B_i of G in $P(G)$. Then the vertex u_i of $L(G)$ corresponding to a block B_i of G is adjacent with the vertex w_i for each i. These edges can be placed in the exterior region of $L(G)$ without losing minimally nonouterplanarity. Thus $P(G)$ is minimally nonouterplanar graph. This completes the proof. ∎

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REFERENCES

[1] G. Chartrand, D. Geller and S. Hedetniemi, *Graphs with forbidden subgraphs*, J. Combin. Theory **10** (1971) 12–41.

[2] G. Chartrand and F. Harary, *Planar permutation graphs*, Ann. Inst. Henri Poincare Sec B. **3** (1967) 433.

[3] F. Harary, Graph Theory (Addison – Wesley, Reading, Mass, 1969).

[4] V.R. Kulli, *On minimally nonouterplanar graphs*, Proc. Indian Nat. Sci. Acad **41** (1975) 275–280.

[5] V.R. Kulli and B. Basavanagoud, *Traversability of Plick graphs*, preprint.

[6] J. Sedláček, *Some properties of interchange graphs*, in: Theory of graphs and its applications, M. Fiedler, ed. (Academic Press, New York, 1962) 145–150.

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