

## HAMILTON CYCLES IN SPLIT GRAPHS WITH LARGE MINIMUM DEGREE

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### Abstract

A graph  $G$  is called a split graph if the vertex-set  $V$  of  $G$  can be partitioned into two subsets  $V_1$  and  $V_2$  such that the subgraphs of  $G$  induced by  $V_1$  and  $V_2$  are empty and complete, respectively. In this paper, we characterize hamiltonian graphs in the class of split graphs with minimum degree  $\delta$  at least  $|V_1| - 2$ .

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### 1. INTRODUCTION

All graphs considered in this paper are finite undirected graphs without loops or multiple edges. If  $G$  is a graph, then  $V(G)$  and  $E(G)$  (or  $V$  and  $E$  in short) will denote its vertex-set and its edge-set, respectively. The set of all neighbours of a subset  $S \subseteq V(G)$  is denoted by  $N_G(S)$  (or  $N(S)$  in short). For a vertex  $v \in V(G)$ , the degree of  $v$ , denoted by  $\deg(v)$ , is  $|N_G(v)|$ . The minimum degree of a graph  $G = (V, E)$ , denoted by  $\delta(G)$  or  $\delta$  in short, is the number  $\min\{\deg(v) \mid v \in V\}$ . The subgraph of  $G$

induced by  $W \subseteq V(G)$  is denoted by  $G[W]$ . Unless otherwise indicated, our graph-theoretic terminology will follow [1].

One of the fundamental problems in graph theory is the hamiltonian problem. Although this is an old one, the amount of papers dealing with this subject does not decrease nowadays (see [3], [8], [11]). Most of these works give sufficient conditions for the existence of a Hamilton cycle in graphs. Only a few of them deal with necessary ones. The history of development of the problem shows that there is a very little hope that an useful and simple characterization of all hamiltonian graphs exists. However, this does not exclude the availability of such a characterization of hamiltonian graphs in some particular classes of graphs, e.g., in [14] hamiltonian self-complementary graphs have been characterized by Rao and in [10] hamiltonian threshold graphs have been characterized by Harary and Peled.

A graph  $G = (V, E)$  is called a split graph if there exists a partition  $V = V_1 \cup V_2$  such that  $G[V_1]$  and  $G[V_2]$  are empty and complete graphs, respectively. We will denote such a graph by  $S(V_1 \cup V_2, E)$ . The notion of split graphs was introduced in [6] by Foldes and Hammer. These graphs have been paid attention because of their connection with many combinatorial problems (see [5], [7], [13]).

In this paper, we consider the hamiltonian problem for split graphs. It is clear that if  $|V_1| > |V_2|$  then a split graph  $G = S(V_1 \cup V_2, E)$  has no Hamilton cycles. So without loss of generality we may consider the hamiltonian problem only for split graphs  $G = S(V_1 \cup V_2, E)$  with  $|V_1| \leq |V_2|$ . The main result here is Theorem 1 below. The condition for the existence of a Hamilton cycle in a split graph obtained here is similar to Hall's condition for the existence of a complete matching in a bipartite graph [9].

**Theorem 1.** *Let  $G = S(V_1 \cup V_2, E)$  be a split graph with  $|V_1| = m \leq n = |V_2|$  and the minimum degree  $\delta(G) \geq m - 2$ . Then  $G$  has a Hamilton cycle if and only if  $|N(S)| > |S|$  for any  $\emptyset \neq S \subseteq V_1$  with  $m - 2 \leq |S| \leq \min\{m, n - 1\}$ , except the following graphs for which the sufficiency does not hold:*

- (i)  $m = 3 < n$  and  $G$  is the graph  $G_n^3$ ,
- (ii)  $m = 4 < n$  and  $G$  is a spanning subgraph of  $D_n^4$  or  $G_n^4$ ,
- (iii)  $m = 4 \leq n$  and  $G - u$  is the graph  $G_n^3$  for some  $u \in V_1$ ,
- (iv)  $m = 5 < n$  and  $G$  is the graph  $F_n^5$  or a spanning subgraph of  $G_n^5$ ,
- (v)  $6 \leq m < n$  and  $G$  is a spanning subgraph of  $G_n^m$ .

The graphs  $G_n^m$ ,  $D_n^4$  and  $F_n^5$  will be defined in Section 2. It will be also proved there that these graphs are split graphs  $S(V_1 \cup V_2, E)$  satisfying  $|V_1| = m < n = |V_2|$ ,  $\delta(G) \geq m - 2$  and  $|N(S)| > |S|$  for any  $\emptyset \neq S \subseteq V_1$  with  $|S| \geq m - 2$ , but they have no Hamilton cycles. Every graph in (iii) also has no Hamilton cycles.

We note that there are in literature some papers dealing with the hamiltonian problem for split graphs [2], [12]. But the conditions obtained there for the existence of a Hamilton cycle in split graphs are only necessary, but not sufficient. In [2] the authors also asked if the conditions obtained there can be sharpened to a necessary and sufficient one.

From Theorem 1 we have the following corollary.

**Corollary 2.** *Let  $G = B(V_1 \cup V_2, E)$  be a bipartite graph with bipartition  $V = V_1 \cup V_2$ , where  $|V_1| = m \leq n = |V_2|$  and  $\delta(V_1) = \min\{\deg(v) \mid v \in V_1\} \geq m - 2$ . Then  $G$  has a Hamilton cycle if and only if  $m = n$  and  $|N(S)| > |S|$  for any  $\emptyset \neq S \subseteq V_1$  with  $|S| = m - 2$  or  $m - 1$ , unless  $m = 4$  and  $G - u$  is the graph  $BG_4^3$  for some  $u \in V_1$ .*

The graph  $BG_4^3$  is obtained from  $G_4^3$  by deleting all edges, the both endvertices of which are in  $V_2$ . The sufficiency does not hold for these exceptional graphs.

Thus, we have got in this paper a characterization of hamiltonian split graphs  $G = S(V_1 \cup V_2, E)$  with  $\delta(G) \geq |V_1| - 2$ . We note that although many sufficient conditions for the existence of a Hamilton cycle in a graph are known (see [3]), almost all they involve the order  $|V|$  of  $G$  and all they are not necessary. Meanwhile, our condition is also necessary and involves only the cardinality  $|V_1|$  of the subset  $V_1$ . Therefore, it is not a consequence of former conditions.

## 2. PRELIMINARIES

Let  $C$  be a cycle in a graph  $G = (V, E)$ . By  $\vec{C}$  we denote the cycle  $C$  with a given orientation, and by  $\overleftarrow{C}$  the cycle  $C$  with the reverse orientation. If  $u, v \in V(C)$ , then  $u\vec{C}v$  denotes the consecutive vertices of  $C$  from  $u$  to  $v$  in the direction specified by  $\vec{C}$ . The same vertices in the reverse order are given by  $v\overleftarrow{C}u$ . We will consider  $u\vec{C}v$  and  $v\overleftarrow{C}u$  both as paths and as vertex sets. If  $u \in V(C)$ , then  $u^+$  denotes the successor of  $u$  on  $\vec{C}$ , and  $u^-$  denotes

its predecessor. Similar notation as described above for cycles is used for paths.

If  $W \subseteq V(G)$  and  $v \in W$ , then  $v$  is called a  $W$ -vertex. Also, by  $N_{G,W}(u)$  or  $N_W(u)$  in short we denote the set  $W \cap N_G(u)$ .

**Lemma 3.** *If a split graph  $G = S(V_1 \cup V_2, E)$  has a Hamilton cycle, then  $|N(S)| > |S|$  for any  $\emptyset \neq S \subseteq V_1$  with  $|S| \leq \min\{m, n-1\}$ .*

**Proof.** Suppose that  $G = S(V_1 \cup V_2, E)$  has a Hamilton cycle  $C$ . Let  $u_1, \dots, u_k$  ( $1 \leq k \leq \min\{m, n-1\}$ ) be the vertices of  $\emptyset \neq S \subseteq V_1$ , occurring on  $\vec{C}$  in the order of their indices. It is not difficult to see that  $u_i^+ \neq u_j^+$  if  $i \neq j$ . Since  $|S| = k < n$ , there exists  $i \in \{1, \dots, k\}$  such that there are at least two  $V_2$ -vertices in  $u_i^+ \vec{C} u_{i+1}^-$  (indices mod  $k$ ). Therefore,  $u_i^+ \neq u_{i+1}^-$ . So  $N(S) \supseteq \{u_i^+, \dots, u_k^+, u_{i+1}^-\}$  and  $|N(S)| \geq k+1 > k = |S|$ . ■

**Lemma 4.** *Let  $G = S(V_1 \cup V_2, E)$  be a split graph with  $|V_1| < |V_2|$ . Then  $G$  has a Hamilton cycle if and only if  $|N_G(V_1)| > |V_1|$  and the subgraph  $G' = G[V_1 \cup N_G(V_1)]$  has a Hamilton cycle.*

**Proof.** Suppose that  $G$  has a Hamilton cycle  $C$ . By Lemma 3,  $|N_G(V_1)| > |V_1|$ . If  $v \in V_2 - N_G(V_1)$ , then both  $v^-$  and  $v^+$  are in  $V_2$ . So  $v^- v^+ \in E(G-v)$ . This means that  $C' = C - v + v^- v^+$  is a Hamilton cycle of  $G-v$ . By backward induction on  $|V_2 - N_G(V_1)|$  we can show that  $G'$  has a Hamilton cycle.

Conversely, let  $|N_G(V_1)| > |V_1|$  and  $G' = G[V_1 \cup N_G(V_1)]$  have a Hamilton cycle  $C'$ . Further, let  $V_2 - N_G(V_1) = \{y_1, \dots, y_l\}$ . Since  $|N_G(V_1)| > |V_1|$ , there exists  $v \in N_G(V_1)$  such that both  $v$  and  $v^+$  (with respect to  $C'$ ) are in  $N_G(V_1) \subseteq V_2$ . It follows that  $C = v y_1 \dots y_l v^+ \vec{C}' v$  is a Hamilton cycle of  $G$ . ■

In Table 1 we define the split graphs  $G_n^m$ ,  $D_n^4$  and  $F_n^5$ . The conditions that  $m$  and  $n$  must be satisfied for the corresponding graph are indicated in parentheses under its name in Column 1. The subsets  $V_1$  and  $V_2$  of the vertex-set  $V$  for each of these graphs are indicated in Column 2. Finally, in Column 3, we present the edges of the corresponding graph.

**Lemma 5.** (a) *Let  $G = S(V_1 \cup V_2, E)$  with  $|V_1| = m$  and  $|V_2| = n$  be one of the split graphs  $G_n^m$ ,  $D_n^4$  and  $F_n^5$ . Then  $m < n$ ,  $\delta(G) \geq m-2$  and  $|N(S)| > |S|$  for any  $\emptyset \neq S \subseteq V_1$  with  $|S| \geq m-2$ , but  $G$  has no Hamilton cycles.*

(b) Every graph  $G = S(V_1 \cup V_2, E)$ , for which  $G - u$  is the graph  $G_n^3$  for some  $u \in V_1$ , also has no Hamilton cycles.

Table 1. The graphs  $G_n^m$ ,  $D_n^4$  and  $F_n^5$ .

The graph $G = (V, E)$	The vertex-set $V = V_1 \cup V_2$	The edge-set $E = E_1 \cup E_2 \cup E_3$
$G_n^m$ ( $3 \leq m < n$ )	$V_1 = \{u_1, \dots, u_m\},$ $V_2 = \{v_1, \dots, v_n\}.$	$E_1 = \{u_1v_1, u_2v_2, u_3v_3\},$ $E_2 = \{u_iv_j \mid i = 1, \dots, m;$ $j = 4, \dots, m+1\},$ $E_3 = \{v_iv_j \mid i \neq j; i, j = 1, \dots, n\}.$
$D_n^4$ ( $4 < n$ )	$V_1 = \{u_1, \dots, u_4\},$ $V_2 = \{v_1, \dots, v_n\}.$	$E_1 = \{u_1v_2, u_2v_1, u_iv_i \mid$ $i = 1, 2, 3, 4\},$ $E_2 = \{u_iv_5 \mid i = 1, 2, 3, 4\},$ $E_3 = \{v_iv_j \mid i \neq j; i, j = 1, \dots, n\}.$
$F_n^5$ ( $6 < n$ )	$V_1 = \{u_1, \dots, u_5\},$ $V_2 = \{v_1, \dots, v_n\}.$	$E_1 = \{u_iv_i \mid i = 1, \dots, 5\},$ $E_2 = \{u_iv_j \mid i = 1, \dots, 5; j = 6, 7\},$ $E_3 = \{v_iv_j \mid i \neq j; i, j = 1, \dots, n\}.$

**Proof.** (a) It is not difficult to verify that for each of these graphs the inequalities  $m < n$ ,  $\delta(G) \geq m - 2$  and  $|N(S)| > |S|$  for any  $\emptyset \neq S \subseteq V_1$  with  $|S| \geq m - 2$  are true.

Suppose that  $G = G_n^m$  has a Hamilton cycle  $C$ . Set  $R = \{v_4, \dots, v_{m+1}\}$ , where  $v_4, \dots, v_{m+1}$  occur on  $\vec{C}$  in the order of their indices. Then  $G - R$  can be covered by  $m - 2 = |R|$  vertex-disjoint paths  $P_1 = v_4^+ \vec{C} v_5^-$ ,  $P_2 = v_5^+ \vec{C} v_6^-$ , ...,  $P_{m-2} = v_{m+1}^+ \vec{C} v_4^-$ . On the other hand, it is not difficult to see that  $G - R$  need at least  $m - 1$  vertex-disjoint paths to cover it, a contradiction.

The proofs of the fact that  $D_n^4$  and  $F_n^5$  are non-hamiltonian are left to the reader.

(b) Let  $G$  have a Hamilton cycle  $C$ . Then both  $u^-$  and  $u^+$  are in  $V_2$ . So  $u^-u^+ \in E(G - u)$  and therefore  $C' = u^-u^+\overrightarrow{C}u^-$  is a Hamilton cycle of  $G - u$ , contradicting the fact that  $G_n^3$  is non-hamiltonian by (a). ■

By Lemma 4, we assume from now on that all considered split graphs  $S(V_1 \cup V_2, E)$  have  $N(V_1) = V_2$ .

**Lemma 6.** *Let  $G = S(V_1 \cup V_2, E)$  with  $|V_1| = m$  and  $|V_2| = n$  be a maximal non-hamiltonian split graph with  $\delta(G) \geq m - k$  ( $0 \leq k \leq m$ ). Then for any  $v \in V_2$ , either  $|N_{V_1}(v)| \leq k$  or  $|N_{V_1}(v)| = m$ .*

**Proof.** Suppose that there exists  $v \in V_2$  such that  $k < |N_{V_1}(v)| < m$ . Since  $|N_{V_1}(v)| < m$ , there exists  $u \in V_1$  such that  $uv \notin E$ . Therefore,  $G + uv$  has a Hamilton cycle  $C$ , which must contain the edge  $uv$  because  $G$  is maximal non-hamiltonian. Without loss of generality we may assume  $u^- = v$ . Let  $x_1, \dots, x_t$  be the neighbours in  $G$  of  $u$ , occurring on  $u\overrightarrow{C}v$  in the order of their indices. Then  $t \geq m - k$  by the assumption and  $x_i^-$  is not adjacent to  $v$  in  $G$  for every  $i = 1, \dots, t$  because otherwise,  $C' = u\overrightarrow{C}x_i^-v\overleftarrow{C}x_iu$  is a Hamilton cycle of  $G$ , a contradiction. So  $x_1^-, \dots, x_t^-$  are in  $V_1$  because all  $V_2$ -vertices are adjacent to  $v$ . Hence,  $|N_{V_1}(v)| \leq m - t \leq m - (m - k) = k$ , contradicting  $|N_{V_1}(v)| > k$ . ■

**Proposition 7.** *Let  $G = S(V_1 \cup V_2, E)$  be a split graph with  $|V_1| < |V_2|$  and  $|N_{V_1}(v)| \leq 2$  for each  $v \in V_2$ . Then  $G$  has a Hamilton cycle if and only if  $|N(S)| > |S|$  for any  $\emptyset \neq S \subseteq V_1$ .*

**Proof.** The necessity follows from Lemma 3. Now we prove the sufficiency by induction on  $|V_1|$ . If  $|V_1| = 1$ , then  $G$  trivially has a Hamilton cycle. Suppose that the sufficiency has been proved when  $|V_1| < t$  and  $G$  is a split graph such that  $|V_1| = t < |V_2|$ ,  $|N_{V_1}(v)| \leq 2$  for any  $v \in V_2$  and  $|N(S)| > |S|$  for any  $\emptyset \neq S \subseteq V_1$ . By the induction hypothesis, for any  $u \in V_1$ , the graph  $G_u = G - u$  has a Hamilton cycle  $C$ .

First assume that there exists  $v_1 \in V_2$  such that  $|N_{V_1}(v_1)| = 1$ , say  $N_{V_1}(v_1) = \{u\}$ . Since  $|N_G(u)| > |\{u\}| = 1$ , there exists  $v_2 \in N(u)$  with  $v_2 \neq v_1$ . If  $v_1^+ = v_2$ , then  $C' = v_1uv_2\overrightarrow{C}v_1$  is a Hamilton cycle of  $G$ . So we assume that  $v_1^+ \neq v_2$ . Since  $N_{V_1}(v_1) = \{u\}$ ,  $|N_{V_1}(v_2)| \leq 2$  and  $v_2 \in N(u)$ , either both  $v_1^-$  and  $v_2^-$  or both  $v_1^+$  and  $v_2^+$  are in  $V_2$ , say  $v_1^+$  and  $v_2^+$ . Then  $C' = v_1uv_2\overleftarrow{C}v_1^+v_2^+\overrightarrow{C}v_1$  is a Hamilton cycle of  $G$ .

Now assume that for any  $v \in V_2$ ,  $|N_{V_1}(v)| = 2$ . If for any  $u \in V_1$ ,  $|N(u)| \leq 2$ , then

$$2|V_1| \geq \sum_{u \in V_1} |N(u)| = \sum_{v \in V_2} |N_{V_1}(v)| = 2|V_2|,$$

contradicting  $|V_1| < |V_2|$ . Thus, there exists  $u \in V_1$  such that  $|N_G(u)| \geq 3$ . Let  $v_1, v_2, v_3 \in N(u)$ . Since  $|N_{V_1}(v_i)| = 2$  for each  $i = 1, 2, 3$ , either  $\{v_1^-, v_2^-, v_3^-\}$  or  $\{v_1^+, v_2^+, v_3^+\}$  contains two  $V_2$ -vertices, say  $v_1^+$  and  $v_2^+$ . If  $v_1^+ = v_2$ , then  $C' = v_1 u v_2 \vec{C} v_1$  is a Hamilton cycle of  $G$ . Otherwise,  $C' = v_1 u v_2 \overleftarrow{C} v_1^+ v_2^+ \vec{C} v_1$  is a Hamilton cycle of  $G$ . ■

**Proposition 8.** *Let  $G = S(V_1 \cup V_2, E)$  be a split graph with  $|V_1| < |V_2|$  and  $\delta(G) \geq |V_1|$ . Then  $G$  has a Hamilton cycle if and only if  $|N(V_1)| > |V_1|$ .*

**Proof.** The necessity follows from Lemma 3. Now we prove the sufficiency. It is not difficult to verify that  $|V(G)| \geq 3$ ,  $\alpha(G) = |V_1| \leq \kappa(G)$ , where  $\alpha(G)$  and  $\kappa(G)$  are the independence number and the connectivity of  $G$ , respectively. By [4]  $G$  has a Hamilton cycle. ■

A graph  $G$  is called to have the property  $(\bullet)$  if the following conditions are satisfied:

1.  $G$  is a split graph  $S(V_1 \cup V_2, E)$  with  $m = |V_1| < |V_2| = n$  and  $\delta(G) \geq |V_1| - k$  where  $1 \leq k \leq 2$ ;
2.  $G$  is a maximal non-hamiltonian, but for any  $u \in V_1$  the graph  $G_u = G - u$ , which is the split graph  $S(W_1 \cup V_2, E_u)$  with  $W_1 = V_1 - u$ ,  $E_u = E - \{uv \in E \mid v \in V_2\}$ , has a Hamilton cycle  $C$ ;
3. For any  $\emptyset \neq S \subseteq V_1$  with  $|S| \geq m - k$ ,  $|N(S)| > |S|$ ;
4.  $G$  has a  $V_1$ -vertex  $u$  such that  $u$  has a neighbour  $v_1$  with  $v_1^+ \in V_2$  and a neighbour  $v_2$  with  $v_2^- \in V_2$  (with respect to  $C$ ). The vertex  $v_1$  may coincide with  $v_2$ .

We note that if the vertices  $v_1 \neq v_2$ , then  $v_2 \notin \{v_1^-, v_1^+\}$  because otherwise  $C' = v_1 u v_2 \vec{C} v_1$  or  $C' = v_2 u v_1 \vec{C} v_2$  is a Hamilton cycle of  $G$ , a contradiction.

Let  $G$  be a graph with the property  $(\bullet)$  and  $u, v_1$  and  $v_2$  be the vertices of  $G$  chosen as in its definition above. Set

$$B_i = \{v \in V_2 \mid |N_{V_1}(v)| = i\},$$

$$A_i = \bigcup_{v \in B_i} N_{V_1}(v).$$

Many assertions below can be proved easily by contradiction. So we omit their detailed proofs and give in parentheses only a Hamilton cycle  $C'$  of  $G$  if we assume the contrary.

**Claim 2.1.**  $B_m \neq \emptyset$ .

( $C'$  exists by Lemma 6 and Proposition 7 if  $B_m = \emptyset$ .)

**Claim 2.2.**  $v_1^+$  and  $v_2^-$  are not in  $B_m$ .

( $C' = v_1 u v_1^+ \vec{C} v_1$  if  $v_1^+ \in B_m$ .)

**Claim 2.3.** For any  $x \in B_m$ ,  $x^+$  and  $x^-$  are in  $W_1$ .

( $C' = v_1 u x \overleftarrow{C} v_1^+ x^+ \vec{C} v_1$  if  $x^+ \in V_2$ .)

By Claim 2.1 and Claim 2.3 there exists a positive integer  $t$  such that  $C$  possesses  $t$  disjoint paths  $P_1 = x_1 \vec{C} y_1, \dots, P_t = x_t \vec{C} y_t$ , which occur in  $v_1^+ \vec{C} v_1^-$  in the order of their indices and have the following properties:

- (a) Vertices of  $W_1$  and  $B_m$  occur alternatively in  $P_i$  for every  $i = 1, \dots, t$ ,
- (b) The endvertices  $x_i$  and  $y_i$  of  $P_i$  are in  $W_1$  for every  $i = 1, \dots, t$ ,
- (c)  $x_i^-$  and  $y_i^+$  are not in  $B_m$  for every  $i = 1, \dots, t$ ,
- (d) Every vertex of  $B_m$  is in one of  $P_1, \dots, P_t$ .

Let  $l_i$  be the number of  $B_m$ -vertices in  $P_i$ . Then it is clear that the number of  $W_1$ -vertices in  $P_i$  is  $l_i + 1$ . By (d),  $l_1 + \dots + l_t = |B_m|$ . So in total, the number of  $W_1$ -vertices in all paths  $P_1, \dots, P_t$  is  $|B_m| + t$ . It follows that  $|B_m| + t \leq |W_1| = m - 1$ . Thus, we have proved the following.

**Claim 2.4.**  $|B_m| \leq m - 1 - t$ .

Set  $Q_1 = v_1^+ \vec{C} v_2^-$  and  $Q_2 = v_2^+ \vec{C} v_1^-$ . Thus, if  $v_1 = v_2$ , then  $Q_1 = Q_2$ . But if  $v_1 \neq v_2$ , then  $Q_1$  and  $Q_2$  are disjoint and each of them has at least one vertex because  $v_2 \notin \{v_1^-, v_1^+\}$  as we have noted before. Let among  $P_1, \dots, P_t$  there be  $l$  paths in  $Q_1$  ( $0 \leq l \leq t$ ). Since  $P_1, \dots, P_t$  occur in  $v_1^+ \vec{C} v_1^-$  in the order of their indices, these  $l$  paths in  $Q_1$  are  $P_1, \dots, P_l$ . Then the following assertions are also true.



**Claim 2.5.** All  $W_1$ -neighbours of  $v_1^+$  and  $v_2^-$  are in  $Q_1$ .

( $C' = v_1 u v_2 \vec{C} u_1 v_1^+ \vec{C} v_2^- u_1^+ \vec{C} v_1$  if  $u_1$  is a  $W_1$ -neighbour in  $Q_2$  of  $v_1^+$ .)

**Claim 2.6.** If among  $P_1, \dots, P_t$  there are  $l \geq 1$  paths in  $Q_1$  and  $w$  is a  $W_1$ -neighbour in some  $P_i$  of  $v_1^+$  (resp.  $v_2^-$ ), then  $w = x_1$  (resp.  $w = y_l$ ).

Suppose the otherwise that  $w \neq x_1$ . If  $w^- \in B_m$ , then  $C' = v_1 u w^- \overleftarrow{C} v_1^+ w \vec{C} v_1$  is a Hamilton cycle of  $G$ , a contradiction. So  $w^- \notin B_m$ . It follows  $w = x_i$  for some  $i \geq 2$ . Then  $C' = v_1 u x_1^+ \overleftarrow{C} v_1^+ x_i \overleftarrow{C} x_1^{++} x_i^+ \vec{C} v_1$  is a Hamilton cycle of  $G$  because both  $x_1^+$  and  $x_i^+$  are in  $B_m$ , a contradiction again. By symmetry, we can show the assertion for  $v_2^-$ .

**Claim 2.7.** If among  $P_1, \dots, P_t$  there are  $l \geq 1$  paths in  $Q_1$  and  $v_1^+$  (resp.  $v_2^-$ ) has a  $W_1$ -neighbour in some  $P_i$ , then  $v_1^{++} \in W_1$  (resp.  $v_2^{--} \in W_1$ ).

Suppose the otherwise that  $v_1^{++} \in V_2$ . Let  $w$  be a  $W_1$ -neighbour of  $v_1^+$  in  $P_i$ . By Claim 2.6,  $w = x_1$ . Therefore,  $C' = v_1 u x_1^+ \vec{C} v_2^- v_1^+ x_1 \overleftarrow{C} v_1^{++} v_2 \vec{C} v_1$  is a Hamilton cycle of  $G$ , a contradiction. By symmetry, we can show the assertion for  $v_2^-$ .

**Proposition 9.** Let  $G = S(V_1 \cup V_2, E)$  be a split graph with  $m = |V_1| < |V_2| = n$  and  $\delta(G) \geq m - 1$ . Then  $G$  has a Hamilton cycle if and only if  $|N(S)| > |S|$  for any  $\emptyset \neq S \subseteq V_1$  with  $|S| \geq m - 1$ , except the graph  $G_n^3$  for which the sufficiency does not hold.

**Proof.** The necessity follows from Lemma 3. Now, we prove the sufficiency. Let  $G = S(V_1 \cup V_2, E)$  be a maximal non-hamiltonian split graph satisfying  $|N(S)| > |S|$  for any  $\emptyset \neq S \subseteq V_1$  with  $|S| \geq m - 1$ . By Lemma 6,  $B_2 = B_3 = \dots = B_{m-1} = \emptyset$ . Set  $A = V_1 \setminus A_1$ .

For any  $u \in V_1$  denote by  $G_u$  the graph  $G - u$ . Thus,  $G_u = S(W_1 \cup V_2, E_u)$  with  $W_1 = V_1 - u$ . By Proposition 8,  $G_u$  has a Hamilton cycle  $C$ . The following assertions are easily proved by contradiction. (We again indicate in parentheses a Hamilton cycle  $C'$  of  $G$  if we assume the contrary.)

**Claim 2.8.**  $B_1 \neq \emptyset$ .

( $C'$  exists by Proposition 8 if  $B_1 = \emptyset$ .)

**Claim 2.9.** Each  $u \in A_1$  has only one  $B_1$ -neighbour.

( $C' = v_1 u v_2 \overleftarrow{C} v_1^+ v_2^+ \overrightarrow{C} v_1$  if  $v_1$  and  $v_2$  are two different  $B_1$ -neighbours of  $u$ .)

Let  $u \in A_1$  and  $v$  be the  $B_1$ -neighbour of  $u$ . Then both  $v^-$  and  $v^+$  are in  $V_2$ . Thus  $G$  is a graph having the property  $(\bullet)$  with  $k = 1$ . By Claim 2.1  $B_m \neq \emptyset$ . By Claim 2.9, all neighbours of  $u$  but  $v$  are in  $B_m$ . It follows that  $|B_m| = |N(u)| - 1 \geq (m - 1) - 1 = m - 2$ . Together with Claim 2.4 we have  $m - 2 \leq |B_m| \leq m - 1 - t$ , where  $t$  is the number of paths  $P_i = x_i \overrightarrow{C} y_i$  defined for a graph with property  $(\bullet)$  as above. So  $|B_m| = m - 2$ ,  $t = 1$  because  $t$  is a positive integer. Therefore, the number of  $W_1$ -vertices in  $P_1 = x_1 \overrightarrow{C} y_1$  is  $m - 1 = |W_1|$ . This means that all vertices of  $W_1$  are in  $P_1$ . So if  $R = y_1^+ \overrightarrow{C} x_1^-$ , then  $V(R) = B_1$ . By Claim 2.2 both  $v^-$  and  $v^+$  are in  $B_1$ . Hence, all  $v^-$ ,  $v$  and  $v^+$  are in  $R$ . Let  $w$  be the unique  $W_1$ -neighbour of  $v^+$ , then  $w = x_1 = v^{++}$  by Claim 2.6 and Claim 2.7. By symmetry, if  $w$  is the unique  $W_1$ -neighbour of  $v^-$ , then  $w = y_1 = v^{--}$ . This means that  $v^+ = x_1^-$  and  $v^- = y_1^+$  and therefore  $R = v^- v v^+$  with all  $v^-$ ,  $v$ ,  $v^+$  in  $B_1$ . Since  $|B_m| = m - 2$  and  $\delta(G) \geq m - 1$ , the set  $A$  must be empty. Thus,  $B_1 = \{v^-, v, v^+\}$ ,  $A = \emptyset$ . Using Claim 2.9 it is not difficult to see that  $G$  must be  $G_4^3$ .  $\blacksquare$

### 3. PROOF OF THE RESULTS

First we prove the following two propositions 10 and 11.

**Proposition 10.** *Let  $G = S(V_1 \cup V_2, E)$  be a split graph with  $m = |V_1| < |V_2| = n$  and  $\delta(G) \geq m - 2$ . Then  $G$  has a Hamilton cycle if and only if  $|N(S)| > |S|$  for any  $\emptyset \neq S \subseteq V_1$  with  $|S| \geq m - 2$ , except the following graphs for which the sufficiency does not hold :*

- (i)  $m = 3$  and  $G$  is the graph  $G_n^3$ ,
- (ii)  $m = 4$  and  $G$  is a spanning subgraph of  $D_n^4$  or  $G_n^4$ ,
- (iii)  $m = 4$  and  $G - u$  is the graph  $G_n^3$  for some  $u \in V_1$ ,
- (iv)  $m = 5$  and  $G$  is the graph  $F_n^5$  or a spanning subgraph of  $G_n^5$ ,
- (v)  $m \geq 6$  and  $G$  is a spanning subgraph of  $G_n^m$ .

**Proof.** The necessity follows from Lemma 3. Now, we prove the sufficiency. If  $m = 1$  or  $2$ , then by Proposition 7,  $G$  has a Hamilton cycle. For any  $3 \leq m < n$ , let  $G = S(V_1 \cup V_2, E)$  be a maximal non-hamiltonian split graph satisfying  $\delta(G) \geq m - 2$  and  $|N(S)| > |S|$  for any  $\emptyset \neq S \subseteq V_1$  with  $|S| \geq m - 2$ . By Lemma 6,  $B_3 = B_4 = \dots = B_{m-1} = \emptyset$ . Set  $A = V_1 \setminus (A_1 \cup A_2)$ .

For any  $u \in V_1$ , denote by  $G_u$  the graph  $G - u = S(W_1 \cup V_2, E_u)$ , where  $W_1 = V_1 - u$ . By Proposition 9, either  $|W_1| = m - 1 = 3$  and  $G_u$  is the graph  $G_4^3$  or  $G_u$  has a Hamilton cycle. In the former case,  $G$  is a graph in (iii). So we assume from now on that

**Claim 3.1.** For any  $u \in V_1$ ,  $G_u$  has a Hamilton cycle  $C$  with a fixed orientation  $\vec{C}$ .

Below we omit the detailed proofs of many assertions which can be easily proved by contradiction. In these cases, as before, we indicate in parentheses a Hamilton cycle  $C'$  of  $G$  if we assume the contrary.

**Claim 3.2.** Each  $u \in A_1$  has only one  $B_1$ -neighbour.

( $C' = v_1 u v_2 \overleftarrow{C} v_1^+ v_2^+ \overrightarrow{C} v_1$  if  $v_1$  and  $v_2$  are two different  $B_1$ -neighbours of  $u$ .)

**Claim 3.3.**  $A_1 \cap A_2 = \emptyset$ .

Suppose the otherwise that  $u \in A_1 \cap A_2$ . Let  $v_1$  and  $v_2$  be a  $B_1$ -neighbour and a  $B_2$ -neighbour of  $u$ , respectively. Then  $v_1^-$  and  $v_1^+$  are in  $V_2$  and at least one of  $v_2^-$  and  $v_2^+$  is in  $V_2$ , say  $v_2^+$ . So  $C' = v_1 u v_2 \overleftarrow{C} v_1^+ v_2^+ \overrightarrow{C} v_1$  is a Hamilton cycle of  $G$ , a contradiction.

**Claim 3.4.** Each  $u \in A_2$  has at most two  $B_2$ -neighbours.

Suppose the otherwise that some  $u \in A_2$  has three  $B_2$ -neighbours  $v_1, v_2$  and  $v_3$ . Then either  $\{v_1^-, v_2^-, v_3^-\}$  or  $\{v_1^+, v_2^+, v_3^+\}$  has at least two vertices in  $V_2$ , say  $v_1^+ \in V_2$  and  $v_2^+ \in V_2$ . Then  $C' = v_1 u v_2 \overleftarrow{C} v_1^+ v_2^+ \overrightarrow{C} v_1$  is a Hamilton cycle of  $G$ , a contradiction.

**Claim 3.5.** At least one of  $B_1$  and  $B_2$  is not empty.

( $C'$  exists by Proposition 8 if both  $B_1$  and  $B_2$  are empty.)

Now we consider separately two cases.

*Case 1.*  $B_1 \neq \emptyset$ .

Let  $v \in B_1$  and  $u$  be the  $V_1$ -neighbour of  $v$ . By Claim 3.1,  $G_u$  has a Hamilton cycle  $C$  with a fixed orientation  $\vec{C}$ . Since  $v \in B_1$ , both  $v^-$  and  $v^+$  are in  $V_2$ . Thus,  $G$  is a graph having the property  $(\bullet)$  with  $k = 2$ .

(Here  $v_1 = v_2 = v$  and therefore  $Q_1 = v_1^+ \vec{C} v_2^- = v_2^+ \vec{C} v_1^- = Q_2$ .) By Claim 3.2 and Claim 3.3, all neighbours of  $u$  but  $v$  are in  $B_m$ . It follows that  $|B_m| = |N(u)| - 1 \geq (m - 2) - 1 = m - 3$ . Together with Claim 2.4 we have  $m - 3 \leq |B_m| \leq m - 1 - t$ , where  $t$  is the number of paths  $P_i = x_i \vec{C} y_i$  defined for a graph with the property  $(\bullet)$  as in Section 2. So  $|B_m| = m - 2$  or  $m - 3$  because  $t$  is a positive integer.

If  $|B_m| = m - 2$ , then  $t = 1$  and the number of  $W_1$ -vertices in  $P_1$  is  $m - 1 = |W_1|$ . It follows that all vertices of  $W_1$  are in  $P_1$ . So if  $R = y_1^+ \vec{C} x_1^-$ , then  $V(R) = B_1 \cup B_2$ . In particular,  $v^-, v$  and  $v^+$  are in  $R$  by Claim 2.2. Let  $w$  be a  $W_1$ -neighbour of  $v^+$ . Then  $w$  is in  $P_1$ . By Claim 2.6,  $w = x_1$  and therefore  $v^+ \in B_1$ . By Claim 2.7,  $v^+ = x_1^-$ . By symmetry,  $v^- \in B_1$  and  $v^- = y_1^+$ . Therefore,  $R = v^- v v^+$  with all  $v^-, v$  and  $v^+$  in  $B_1$  and  $B_2 = \emptyset$ . Using Claim 3.2 it is not difficult to see that  $G$  is  $G_n^m$  in this subcase.

If  $|B_m| = m - 3$ , then  $t = 1$  or  $2$  and  $A = \emptyset$  because  $\delta(G) \geq m - 2$ .

First assume that  $|B_m| = m - 3$ ,  $t = 1$  and  $A = \emptyset$ . Then  $P_1$  contains  $m - 2$   $W_1$ -vertices. Therefore,  $R = y_1^+ \vec{C} x_1^-$  contains exactly one  $W_1$ -vertex, say  $u_1$ . All the other vertices of  $R$  are in  $B_1 \cup B_2$ . By symmetry, without loss of generality we may assume that  $u_1$  is in  $v^{++} \vec{C} x_1^-$ . From Claim 3.2, Claim 3.3 and the fact that both  $u_1^-$  and  $u_1^+$  are in  $R$ , we see that  $u_1 \in A_2$ . If  $v^+ \neq u_1^-$ , then  $v^+$  is not adjacent to  $u_1$  because otherwise  $u_1$  has three neighbours in  $B_1 \cup B_2$ , contradicting Claim 3.3 and Claim 3.4. Thus  $v^+$  has a  $W_1$ -neighbour in  $P_1$ . Now by Claim 2.7,  $v^{++} \in W_1$ , contradicting the fact that there are no  $W_1$ -vertices in  $v^+ \vec{C} u_1^-$ . Thus  $v^+ = u_1^-$ . So  $v^+ \in B_2$  and  $v^+$  has another  $W_1$ -neighbour in  $P_1$ , namely,  $x_1$  by Claim 2.6. Now if  $u_1^+ \neq x_1^-$ , then  $C' = v u x_1^+ \vec{C} v^- x_1^- x_1 v^+ \vec{C} x_1^- v$  is a Hamilton cycle of  $G$ , a contradiction. It follows that  $u_1^+ = x_1^-$ . Further, consider  $v^-$ . If  $w$  is a  $W_1$ -neighbour of  $v^-$ , then  $w \neq u_1$  because otherwise  $u_1$  has three neighbours in  $B_1 \cup B_2$ . So  $w$  is in  $P_1$ . By Claim 2.6 and Claim 2.7,  $v^- = y_1^+$  and  $v^- \in B_1$ . Thus,  $R = v^- v v^+ u_1 x_1^-$ ,  $B_1 = \{v^-, v\}$ ,  $B_2 = \{v^+, x_1^-\}$ ,  $N_{W_1}(v^+) = N_{W_1}(x_1^-) = \{u_1, x_1\}$  and  $A = \emptyset$ . Using Claims 3.2–3.4, it is not difficult to see that  $G$  is  $D_5^4$  in this subcase.

Now assume that  $|B_m| = m - 3$ ,  $t = 2$  and  $A = \emptyset$ . Since the total number of  $W_1$ -vertices in  $P_1$  and  $P_2$  is  $m - 1 = |W_1|$ , every vertex of  $W_1$  is in  $P_1$  or  $P_2$ . Set  $R_1 = y_1^+ \vec{C} x_2^-$  and  $R_2 = y_2^+ \vec{C} x_1^-$ . Then  $R_1$  has at least one vertex,  $R_2$  contains  $v^-, v, v^+$  and  $V(R_1 \cup R_2) = B_1 \cup B_2$ . It follows that all  $W_1$ -neighbours of  $v^+$  are in  $P_1$  or  $P_2$ . So by Claim 2.6, the only  $W_1$ -neighbour of  $v^+$  is  $x_1$ . This means that  $v^+ \in B_1$ . By Claim 2.7,  $v^+ = x_1^-$ .

By symmetry, we can show that  $v^- \in B_1$  and  $v^- = y_2^+$ . Thus,  $R_2 = v^- v v^+$  with all  $v^-, v, v^+$  in  $B_1$ .

Consider  $R_1$ . If  $y_1^{++} \notin R_1$ , then  $y_1^+ = x_2^-$  and  $R_1 = y_1^+$ . So  $y_1^+ \in B_2$ . Thus,  $B_1 = \{v^-, v, v^+\}$ ,  $B_2 = \{y_1^+\}$  and  $A = \emptyset$ . Since  $y_1^+ \in B_2$ ,  $y_1^+ u$ ,  $y_1^+ x_1$  and  $y_1^+ y_2$  are not edges of  $G$ . From this, Claim 3.2 and Claim 3.3 it is not difficult to see that  $G$  is a proper spanning subgraph of  $G_6^5$ , contradicting the choice of  $G$  because  $G_6^5$  is non-hamiltonian by Lemma 5. Thus  $y_1^{++} \in R_1$ . If  $y_1^{++} \neq x_2^-$ , then  $y_1^{++}$  has a  $W_1$ -neighbour  $u_1$ . By symmetry, without loss of generality we may assume that  $u_1$  is in  $P_1$ . If  $u_1 \neq y_1$ , then  $C' = v u u_1^+ \overrightarrow{C} y_1^{++} u_1 \overleftarrow{C} v^+ y_1^{++} \overrightarrow{C} v$  is a Hamilton cycle of  $G$ . If  $u_1 = y_1$ , then  $C' = v u y_1^- \overleftarrow{C} v^+ y_1^+ y_1 y_1^{++} \overleftarrow{C} v$  is a Hamilton cycle of  $G$ . These contradictions show that  $y_1^{++} = x_2^-$  and therefore  $R_1 = y_1^+ x_2^-$ . If  $y_1^+ \in B_2$ , then  $y_1^+$  has a  $W_1$ -neighbour  $u_1 \neq y_1$ . If  $u_1$  is in  $P_1$ , then  $C' = v u u_1^+ \overrightarrow{C} y_1^+ u_1 \overleftarrow{C} v^+ x_2^- \overrightarrow{C} v$  is a Hamilton cycle of  $G$ , a contradiction. If  $u_1$  is in  $P_2$ , then  $u_1 \neq y_2$  because otherwise  $y_2 \in A_2$  and therefore  $v^- \in B_2$ , contradicting the fact that  $v^- \in B_1$  as shown in the preceding paragraph. It follows that  $u_1^+ \in B_m$  and  $C' = v u u_1^+ \overrightarrow{C} v^- x_2^- \overrightarrow{C} u_1 y_1^+ \overleftarrow{C} v$  is a Hamilton cycle of  $G$ , a contradiction. So  $y_1^+ \in B_1$ . Similarly,  $x_2^- \in B_1$ . Thus,  $B_1 = \{v^-, v, v^+, y_1^+, x_2^-\}$ ,  $B_2 = \emptyset$  and  $A = \emptyset$ . Using Claim 3.2 it is not difficult to show that  $G$  is  $F_7^5$  in this subcase.

*Case 2.*  $B_1 = \emptyset$ .

By Claim 3.5,  $B_2 \neq \emptyset$ . Then  $A_2 \neq \emptyset$ . Let  $u \in A_2$  and  $C$  be a Hamilton cycle of  $G_u$ . We again divide this case into two subcases.

*Subcase 2.1.* Every  $A_2$ -vertex has exactly one  $B_2$ -neighbour.

Let  $v$  be the only  $B_2$ -neighbour of  $u$ . Then at least one of  $v^-$  and  $v^+$  is in  $V_2$ , say  $v^+$ . Then  $v^+$  must be in  $B_2$  and therefore it has a  $W_1$ -neighbour  $u_1$  such that  $u_1^- \neq v^+$ . If  $u_1^- \in B_m$ , then  $C' = v u u_1^- \overleftarrow{C} v^+ u_1 \overrightarrow{C} v$  is a Hamilton cycle of  $G$ , a contradiction. So  $u_1^- \in B_2$ . It follows that  $u_1$  is an  $A_2$ -vertex which has two  $B_2$ -neighbours, namely,  $v^+$  and  $u_1^-$ . This contradicts the assumption of this subcase. Thus, Subcase 2.1 cannot occur.

*Subcase 2.2.* There exists an  $A_2$ -vertex  $u$  such that  $u$  has two  $B_2$ -neighbours  $v_1$  and  $v_2$ .

Since  $v_1$  has only one  $W_1$ -neighbour in  $G_u$ , one of  $v_1^-$  and  $v_1^+$  is in  $V_2$ . Similarly, one of  $v_2^-$  and  $v_2^+$  is in  $V_2$ . The following assertions are easily proved by contradiction.

**Claim 3.6.**  $v_1^+ \neq v_2$  and  $v_2^+ \neq v_1$ .

( $C' = v_1uv_2\overrightarrow{C}v_1$  if  $v_1^+ = v_2$ .)

**Claim 3.7.**  $v_1^+$  is not adjacent to  $v_2^+$  and  $v_1^-$  is not adjacent to  $v_2^-$ .

( $C' = v_1uv_2\overleftarrow{C}v_1^+v_2^+\overrightarrow{C}v_1$  if  $v_1^+$  is adjacent to  $v_2^+$ .)

In particular,  $v_1^+$  and  $v_2^+$  (resp.  $v_1^-$  and  $v_2^-$ ) cannot be in  $V_2$  simultaneously. It follows that either  $v_1^+$  and  $v_2^-$  are in  $V_2$  or  $v_1^-$  and  $v_2^+$  are in  $V_2$ . For definiteness, assume that  $v_1^+$  and  $v_2^-$  are in  $V_2$ . Then  $v_1^-$  and  $v_2^+$  must be in  $W_1$ . Thus  $G$  has the property  $(\bullet)$  with  $k = 2$ . Set  $Q_1 = v_1^+\overrightarrow{C}v_2^-$  and  $Q_2 = v_2^+\overrightarrow{C}v_1^-$ . If  $v_1^+ = v_2^-$ , then  $Q_1 = v_1^+$ . So  $Q_1$  contains no  $W_1$ -vertices. Therefore, all  $W_1$ -neighbours of  $v_1^+$  are in  $Q_2$ , contradicting Claim 2.5. Thus,

**Claim 3.8.**  $v_1^+ \neq v_2^-$ .

By Claim 3.4, all neighbours of  $u$  but  $v_1$  and  $v_2$  are in  $B_m$ . So we have  $|B_m| = |N(u)| - 2 \geq (m - 2) - 2 = m - 4$ . Together with Claim 2.4 we have  $m - 4 \leq |B_m| \leq m - 1 - t$ , where  $t$  is the number of paths  $P_i = x_i\overrightarrow{C}y_i$  defined for a graph with the property  $(\bullet)$  as in Section 2. It follows that the ordered pair  $(|B_m|, t)$  is equal to one of  $(m - 2, 1)$ ,  $(m - 3, 1)$ ,  $(m - 3, 2)$ ,  $(m - 4, 1)$ ,  $(m - 4, 2)$  and  $(m - 4, 3)$ .

If  $(|B_m|, t)$  is one of  $(m - 2, 1)$ ,  $(m - 3, 2)$  and  $(m - 4, 3)$ , then the number of  $W_1$ -vertices in  $P_1 \cup \dots \cup P_t$  is  $m - 1 = |W_1|$ . So all  $W_1$ -vertices are in  $P_1 \cup \dots \cup P_t$ . By Claim 2.5 and Claim 2.6,  $v_1^+$  has at most one  $W_1$ -neighbour, contradicting  $v_1^+ \in B_2$ .

If  $(|B_m|, t)$  is one of  $(m - 3, 1)$  and  $(m - 4, 2)$ , then the number of  $W_1$ -vertices in  $P_1 \cup \dots \cup P_t$  is  $m - 2$  and there is exactly one  $W_1$ -vertex outside  $P_1 \cup \dots \cup P_t$ , say  $u_1$ . If  $u_1$  is in  $Q_2$  or  $u_1$  is in  $Q_1$  but all  $P_1, \dots, P_t$  are in  $Q_2$ , then again by Claim 2.5 and Claim 2.6  $v_1^+$  has at most one  $W_1$ -neighbour, contradicting  $v_1^+ \in B_2$ . If  $u_1$  and some  $P_i$  are in  $Q_1$ , then at least one of inequalities  $v_1^+ \neq u_1^-$  and  $u_1^+ \neq v_2^-$  is true. By symmetry, without loss of generality we may assume that  $u_1^+ \neq v_2^-$ . If  $v_2^-$  is adjacent to  $u_1$ , then  $u_1$  has three  $B_2$ -neighbours, namely,  $u_1^-, u_1^+$  and  $v_2^-$ . This contradicts Claim 3.4. It follows by Claim 2.5 and Claim 2.6 that  $v_2^-$  has only one  $W_1$ -neighbour, contradicting  $v_2^- \in B_2$ .

If  $(|B_m|, t)$  is  $(m - 4, 1)$ , then the number of  $W_1$ -vertices in  $P_1$  is  $m - 3$  and there are exactly two  $W_1$ -vertices outside  $P_1$ , say  $u_1$  and  $u_2$ . If  $P_1$  is in

$Q_2$  and  $\{v_1^-, v_2^+\} \not\subseteq V(P_1)$ , then  $Q_1$  has at most one  $W_1$ -vertex. By Claim 2.5,  $v_1^+$  has at most one  $W_1$ -neighbour, a contradiction. If  $P_1$  is in  $Q_2$  and  $\{v_1^-, v_2^+\} \subseteq V(P_1)$ , then both  $u_1$  and  $u_2$  are in  $Q_1$ . For definiteness without loss of generality we may assume that  $u_1$  is in  $v_1^+ \vec{C} u_2^-$ . Then  $u_1^+ \neq v_2^-$ . If  $v_2^-$  is adjacent to  $u_1$ , then  $u_1$  has three  $B_2$ -neighbours, namely,  $u_1^-, u_1^+$  and  $v_2^-$ . This contradicts Claim 3.4. It follows that  $v_2^-$  has at most one  $W_1$ -neighbour by Claim 2.5, contradicting again  $v_2^- \in B_2$ . Finally, let  $P_1$  be in  $Q_1$ . Then at most one  $W_1$ -vertex from  $\{u_1, u_2\}$  may be in  $Q_1$  because  $v_1^-$  and  $v_2^+$  are in  $W_1$ . If none of  $u_1$  and  $u_2$  is in  $Q_1$ , then by Claim 2.5 and Claim 2.6,  $v_1^+$  has only one  $W_1$ -neighbour, contradicting  $v_1^+ \in B_2$ . If there is one  $W_1$ -vertex from  $\{u_1, u_2\}$  in  $Q_1$ , then by arguments similar to those for the last situation in the preceding paragraph we can get a contradiction.

Thus, Subcase 2.2 also cannot occur.  $\blacksquare$

**Proposition 11.** *Let  $G = S(V_1 \cup V_2, E)$  be a split graph with  $|V_1| = |V_2| = m$  and  $\delta(G) \geq m - 2$ . Then  $G$  has a Hamilton cycle if and only if  $|N(S)| > |S|$  for any  $\emptyset \neq S \subseteq V_1$  with  $|S| = m - 2$  or  $m - 1$ , unless  $m = 4$  and  $G - u$  is the graph  $G_4^3$  for some  $u \in V_1$ .*

**Proof.** The necessity follows from Lemma 3. Now we prove the sufficiency. Let  $G = S(V_1 \cup V_2, E)$  be a maximal non-hamiltonian split graph satisfying  $|V_1| = |V_2| = m$ ,  $\delta(G) \geq m - 2$  and  $|N(S)| > |S|$  for any  $\emptyset \neq S \subseteq V_1$  with  $|S| = m - 2$  or  $m - 1$ . By Lemma 6,  $B_3 = B_4 = \dots = B_{m-1} = \emptyset$ . The following assertions are true.

**Claim 3.9.**  $B_1 = \emptyset$ .

Suppose the otherwise that  $B_1 \neq \emptyset$ . Let  $v \in B_1$  and  $N_{V_1}(v) = \{u\}$ . Then  $|N_G(V_1 - u)| \leq |V_2 - v| = |V_2| - 1 = |V_1| - 1 = |V_1 - u|$ , contradicting  $|N_G(V_1 - u)| > |V_1 - u|$ .

**Claim 3.10.**  $B_2 \neq \emptyset$ .

Suppose the otherwise that  $B_2 = \emptyset$ . Since  $B_1$  is also empty by Claim 3.9, we have  $V_2 = B_m$ . Therefore,  $G$  contains the complete bipartite graph  $K_{m,m}$  with the bipartition  $V = V_1 \cup V_2$ . So  $G$  has a Hamilton cycle, a contradiction.

For any  $u \in V_1$ , by Proposition 9,  $G_u = G - u = S(W_1 \cup V_2, E_u)$  where  $W_1 = V_1 - u$  has a Hamilton cycle  $C$ , unless  $|W_1| = m - 1 = 3$  and  $G_u$  is the graph  $G_4^3$ .

First assume that for any  $u \in V_1$  the graph  $G_u$  has a Hamilton cycle  $C$  with a fixed orientation  $\vec{C}$ . Let  $w_1, \dots, w_{m-1}$  be the vertices of  $W_1$ , occurring on  $\vec{C}$  in the order of their indices. Since  $|V_2| = m$ , it is not difficult to see that the following assertion is true.

**Claim 3.11.** There exists exactly one of  $T_1 = w_1 \vec{C} w_2, \dots, T_{m-1} = w_{m-1} \vec{C} w_1$ , which contains exactly two  $V_2$ -vertices. Each of the others  $T_i$  contains exactly one  $V_2$ -vertex.

Now we consider separately two cases.

*Case 1.* There exists  $u \in A_2$  which has two different  $B_2$ -neighbours  $v_1$  and  $v_2$ .

Then  $v_1^+ \neq v_2$  because otherwise  $C' = v_1 u v_2 \vec{C} v_1$  is a Hamilton cycle of  $G$ . Since  $v_1$  (resp.  $v_2$ ) has only one  $W_1$ -neighbour, one of  $v_1^-$  and  $v_1^+$  (resp.  $v_2^-$  and  $v_2^+$ ) is in  $V_2$ . For definiteness, without loss of generality we may assume that  $v_1^+ \in V_2$ . Then  $v_2^+$  cannot be in  $V_2$  because otherwise  $C' = v_1 u v_2 \vec{C} v_1^+ v_2^+ \vec{C} v_1$  is a Hamilton cycle of  $G$ , a contradiction. So  $v_2^- \in V_2$ . If  $v_1^+ \vec{C} v_2^-$  has no  $W_1$ -vertices, then  $v_1 \vec{C} v_2$  has at least three  $V_2$ -vertices. This contradicts Claim 3.11 because  $v_1 \vec{C} v_2$  must be contained in some  $T_i$ . If  $v_1^+ \vec{C} v_2^-$  has a  $W_1$ -vertex, then  $v_1^+ \neq v_2^-$  and there exist two different  $T_i$  and  $T_j$  such that  $T_i$  contains  $\{v_1, v_1^+\}$  and  $T_j$  contains  $\{v_2^-, v_2\}$ . This contradicts Claim 3.11 again.

*Case 2.* Every  $A_2$ -vertex has exactly one  $B_2$ -neighbour.

By arguments similar to those used for Subcase 2.1 of Proposition 10, we can get a contradiction in this case. So Case 2 also cannot occur.

Thus, there exists  $u \in V_1$  such that  $G_u$  does not have a Hamilton cycle. So  $|W_1| = m - 1 = 3 \Leftrightarrow m = 4$  and  $G_u$  is  $G_4^3$ . ■

**Proof of Theorem 1.** The necessity follows from Lemma 3 and the sufficiency follows from Propositions 10 and 11. ■

**Proof of Corollary 2.** If a bipartite graph  $G = B(V_1 \cup V_2, E)$  with  $|V_1| = m$  and  $|V_2| = n$  has a Hamilton cycle  $C$ , then vertices of  $V_1$  and  $V_2$  occur on  $C$  alternatively. It follows that  $m$  must be equal to  $n$ . So we may assume further that  $|V_1| = |V_2| = m$ . Let  $G' = S(V_1 \cup V_2, E')$  be the split graph obtained from  $G$  by adding to  $E$  all edges joining any two different



vertices of  $V_2$ . It is not difficult to show that  $G$  has a Hamilton cycle if and only if  $G'$  does. Now Corollary 2 follows from Proposition 11. ■

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