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RADIO k-COLORINGS OF PATHS

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Abstract

For a connected graph G of diameter d and an integer k with $1 \leq k \leq d$, a radio k-coloring of G is an assignment c of colors (positive integers) to the vertices of G such that

$$d(u,v) + |c(u) - c(v)| \geq 1 + k$$

for every two distinct vertices u and v of G, where $d(u,v)$ is the distance between u and v. The value $\mathrm{rc}_k(c)$ of a radio k-coloring c of G is the maximum color assigned to a vertex of G. The radio k-chromatic number $\mathrm{rc}_k(G)$ of G is the minimum value of $\mathrm{rc}_k(c)$ taken over all radio k-colorings c of G. In this paper, radio k-colorings of paths are studied. For the path P_n of order $n \geq 9$ and n odd, a new improved bound for $\mathrm{rc}_{n-2}(P_n)$ is presented. For $n \geq 4$, it is shown that $\mathrm{rc}_{n-3}(P_n) \leq$

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$\binom{n-2}{2} + 2$. Upper and lower bounds are also presented for $\mathrm{rc}_k(P_n)$ in terms of k when $1 \le k \le n-1$. The upper bound is shown to be sharp when $1 \le k \le 4$ and n is sufficiently large.

Keywords: radio k-coloring, radio k-chromatic number.

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1. Introduction to Radio k-Colorings of Graphs

In the United States, the Federal Communications Commission (FCC) requires (see [8]) that two FM radio stations that are located sufficiently close to each other broadcast on channels that are sufficiently far apart. The problem of obtaining an optimal assignment of channels for a specified set of radio stations according to some prescribed restrictions on the distances between the stations as well as other factors is referred to as the Channel Assignment Problem. This problem has been modeled mathematically in a variety of ways. All of the papers [1–7], for example, deal with this topic. In particular, this problem led to the introduction of radio k-colorings of graphs in [2].

Specifically, for a connected graph G of diameter d and an integer k with $1 \le k \le d$, a *radio k-coloring* of G is an assignment c of colors (positive integers) to the vertices of G such that

$$d(u,v) + |c(u) - c(v)| \ge 1 + k$$

for every two distinct vertices u and v of G. The *value* $\mathrm{rc}_k(c)$ of a radio k-coloring c of G is the maximum color assigned to a vertex of G. The *radio k-chromatic number* $\mathrm{rc}_k(G)$ of G is $\min\{\mathrm{rc}_k(c)\}$ taken over all radio k-colorings c of G. A radio k-coloring c of a connected graph G with $\mathrm{rc}_k(c) = \mathrm{rc}_k(G)$ is called a *minimum radio k-coloring* of G. The study of this concept has been primarily restricted to extreme values of k, namely, those in the set $\{1, 2, d-1, d\}$. The parameter $\mathrm{rc}_1(G)$ is the standard chromatic number $\chi(G)$ of a graph G. Consequently, radio k-colorings provide a generalization of ordinary colorings of graphs. The radio d-chromatic number was studied in [1, 2] and was also called the *radio number* of G. Radio d-colorings were also referred to as *radio labelings*. Thus in a radio labeling of a connected graph of diameter d, the labels (colors) assigned to adjacent vertices must differ by at least d, the labels assigned to two vertices whose distance is 2 must differ by at least $d-1$, and so on, up to vertices whose distance is d, that is, *antipodal vertices*, whose labels are only required to be different.

According to FCC regulations, however, if the distance between two radio stations is sufficiently great, then there is no restriction on the channels on which they can broadcast. Applying this to radio k-colorings of graphs, we see, from a practical point of view, that it is useful to study such colorings for integers k with $3 \leq k \leq d-2$ as well. Consequently, it is most appropriate to consider classes of graphs having arbitrarily large diameters. Probably the simplest class with this property are the paths. Thus in this paper we study radio k-colorings of the paths P_n of order n, where $1 \leq k \leq n-1$, where we see that even for this class of graphs, the problem is highly nontrivial. In Section 2 we consider the case $k = n-2$, in Section 3 we study $k = n-3$, and in Section 4 we develop upper bounds for $\mathrm{rc}_k(P_n)$, where $2 \leq k \leq n-4$, in terms of k.

The next two observations from [2] concerning graphs in general will be useful to us and are therefore stated here as well.

Observation 1.1. *Let G be a connected graph with diameter d and let k be an integer such that $1 \leq k \leq d$. If c is a minimum radio k-coloring of G with $\mathrm{rc}_k(c) = \ell$, then*

(a) *there exist vertices u and v such that $c(u) = 1$ and $c(v) = \ell$,*
(b) *for each integer $\ell' > \ell$, there exists a radio k-coloring c' of G with $\mathrm{rc}_k(c') = \ell'$.*

For a radio k-coloring c of G, the *complementary coloring* $\overline{c}$ of c is defined by

$$\overline{c}(v) = (\mathrm{rc}_k(c) + 1) - c(v)$$

for all $v \in V(G)$.

Observation 1.2. *Let G be a connected graph having diameter d. If c is a radio k-coloring of G, where $1 \leq k \leq d$, then so too is $\overline{c}$ and $\mathrm{rc}_k(c) = \mathrm{rc}_k(\overline{c})$.*

There is another observation we will have cause to use.

Observation 1.3. *Let G be a connected graph containing a connected subgraph G', where G and G' have diameters d and d', respectively. If k is a positive integer with $k \leq \min(d, d')$, then $\mathrm{rc}_k(G') \leq \mathrm{rc}_k(G)$.*

Proof. Let c be a minimum radio k-coloring of G, and let u and v be vertices in G'. Then $d_G(u,v) \le d_{G'}(u,v)$. Since $d_G(u,v) + |c(u) - c(v)| \ge 1 + k$, it follows that $d_{G'}(u,v) + |c(u) - c(v)| \ge 1 + k$. Hence c is a radio k-coloring of G' as well. Thus the value of c in G' is at most the value of c in G; that is, $\mathrm{rc}_k(G') \le \mathrm{rc}_k(G)$. ■

2. Radio Antipodal Colorings of Paths

For connected graphs G of diameter d, radio k-colorings of G were investigated for $k = d - 1$ in [4, 3]. A radio $(d-1)$-coloring of G is then a coloring c of G for which

$$d(u,v) + |c(u) - c(v)| \ge d$$

for every two distinct vertices u and v of G. Thus in a radio $(d-1)$-coloring of G, it is possible for two vertices u and v to be colored the same, but only if u and v are antipodal. For this reason, radio $(d-1)$-colorings have been referred to as *radio antipodal colorings* or, more simply, as *antipodal colorings*. The *value* $\mathrm{ac}(c)$ then of an antipodal coloring c of G is the maximum color assigned to a vertex of G. The *antipodal chromatic number* $\mathrm{ac}(G)$ of G is $\min\{\mathrm{ac}(c)\}$, taken over all antipodal colorings c of G. An antipodal coloring c of G is a *minimum antipodal coloring* if $\mathrm{ac}(c) = \mathrm{ac}(G)$.

An upper bound for the antipodal chromatic number of paths was established in [4]. It was shown that if P_n is a path of order n, then

$$\mathrm{ac}(P_n) \le \binom{n-1}{2} + 1 \tag{1}$$

and that equality holds in (1) for $1 \le n \le 6$. Moreover, it was conjectured that equality holds in (1) for all positive integers n. We show that this conjecture is false, at least if n is odd and $n \ge 9$, by presenting an improved upper bound for $\mathrm{ac}(P_n)$ when n is odd and $n \ge 9$.

Theorem 2.1. *If P_n is a path of odd order $n \ge 7$, then*

$$\mathrm{ac}(P_n) \le \binom{n-1}{2} - \frac{n-1}{2} + 4.$$

Proof. Let $n = 2p+1$, where $p \geq 3$, and let $P_{2p+1} : v_1, v_2, \cdots, v_{2p+1}$. Define a coloring c of P_{2p+1} by

$$\begin{aligned} c(v_i) &= 1 + (p-i)(n-2) \;\text{ for } 1 \leq i \leq p, \\ c(v_{p+1}) &= \binom{n-1}{2} - \frac{n-1}{2} + 4 = p(n-2) - p + 4, \\ c(v_{p+j}) &= (p+1) + (p-j)(n-2) \;\text{ for } 2 \leq j \leq p, \\ c(v_{2p+1}) &= (p-1)(n-2) + 3. \end{aligned}$$

Note that $c(v_p) = 1$, $c(v_{p+1}) = c(v_{p+2}) + n$, $c(v_{2p}) = p+1$, $c(v_{2p+1}) = c(v_1) + 2$, $c(v_{2p+1}) > c(v_{2p})$, and

$$c(v_1) > c(v_2) > \cdots > c(v_p) \text{ and } c(v_{p+1}) > c(v_{p+2}) > \cdots > c(v_{2p}).$$

For each of P_9, P_{11}, and P_{13}, the coloring c just defined is shown in Figure 1.

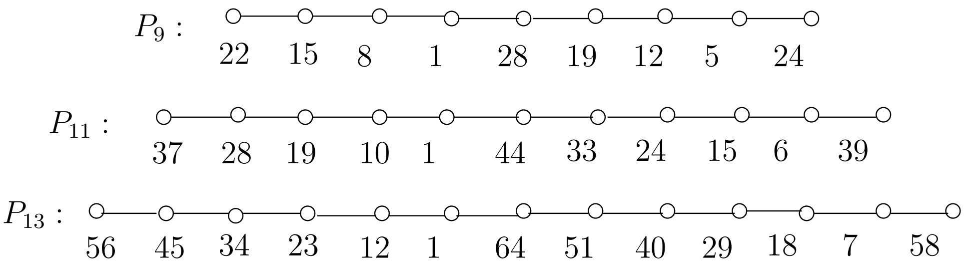

Figure 1. Antipodal colorings of P_9, P_{11}, and P_{13}

We show that c is an antipodal coloring of P_{2p+1}. Let u and v be distinct vertices of P_{2p+1}. We consider two cases.

Case 1. $u \neq v_{2p+1}$ *and* $v \neq v_{2p+1}$. If $u, v \in \{v_1, v_2, \cdots, v_p\}$ or $u, v \in \{v_{p+1}, v_{p+2}, \cdots, v_{2p}\}$, then $|c(u) - c(v)| \geq n-2$ by the definition of c. Thus $d(u,v) + |c(u) - c(v)| \geq n - 1 = \operatorname{diam} P_n$. So assume that $u \in \{v_1, v_2, \cdots, v_p\}$ and $v \in \{v_{p+1}, v_{p+2}, \cdots, v_{2p}\}$. Then $u = v_i$ for some i with $1 \leq i \leq p$. The only possibilities for a vertex v for which $|c(u) - c(v)| < n-2$ are $v = v_{p+i}$ or $v = v_{p+i+1}$ (where the latter situation occurs only if $1 \leq i < p$). If $v = v_{p+i}$, then $d(u,v) = p$ and $|c(u) - c(v)| = p + 2$ if $i = 1$ and $|c(u) - c(v)| = p$ if $2 \leq i \leq p$; while if $v = v_{p+i+1}$, then $d(u,v) = p+1$ and $|c(u)-c(v)| = p-1$. In either case, $d(u,v) + |c(u) - c(v)| \geq 2p = n - 1 = \operatorname{diam} P_n$.

Case 2. *One of u and v is v_{2p+1}, say $v = v_{2p+1}$*. If $u \in \{v_1, v_2, \cdots, v_p\}$, then the only possibility for a vertex u for which $|c(u) - c(v)| < n - 2$ is v_1, in which case

$$d(u,v) + |c(u) - c(v)| = (n-1) + 2 = n + 1 > \operatorname{diam} P_n.$$

If $u \in \{v_{p+1}, v_{p+2}, \cdots, v_{2p}\}$, then the only possibilities for a vertex u for which $|c(u) - c(v)| < n - 2$ are $u = v_{p+1}$ or $u = v_{p+2}$. If $u = v_{p+1}$, then $d(u,v) = p$ and $|c(u) - c(v)| = p$; while if $u = v_{p+2}$, then $d(u,v) = p - 1$ and $|c(u) - c(v)| = p + 1$. In either case, $d(u,v) + |c(u) - c(v)| = 2p = n - 1 = \operatorname{diam} P_n$.

Hence c is a antipodal coloring of P_{2p+1}. Since $\max\{c(v_i) : 1 \leq i \leq 2p+1\} = c(v_{p+1}) = \binom{n-1}{2} - \frac{n-1}{2} + 4$, it follows that $\mathrm{ac}(c) = \binom{n-1}{2} - \frac{n-1}{2} + 4$. Therefore, $\mathrm{ac}(P_n) \leq \binom{n-1}{2} - \frac{n-1}{2} + 4$. ■

For $n = 7$, we have $\mathrm{ac}(P_7) \leq 16$, which agrees with the upper bound stated in (1). However, when n is odd and $n \geq 9$, the upper bound $\binom{n-1}{2} - \frac{n-1}{2} + 4$ marks an improvement over the previous upper bound of $\binom{n-1}{2} + 1$. We are, however, unable to comment on the sharpness of this new bound. For n even, though, $\binom{n-1}{2} + 1$ remains the best upper bound for $\mathrm{ac}(P_n)$ known to us.

3. Nearly Antipodal Colorings of Paths

For a connected graph G of diameter $d \geq 3$, a radio $(d-2)$-coloring of G is a radio coloring c of G for which

$$d(u,v) + |c(u) - c(v)| \geq d - 1$$

for every two distinct vertices u and v of G. Thus in a radio $(d-2)$-coloring of G, two vertices u and v are colored the same only if $\operatorname{diam} G - 1 \leq d(u,v) \leq \operatorname{diam} G$. We refer to a radio $(d-2)$-coloring as a *nearly radio antipodal coloring* or, more simply, as a *nearly antipodal coloring*. Consequently, the *value* $\mathrm{ac}'(c)$ of a nearly antipodal coloring c of G is the maximum color assigned to a vertex of G. The *nearly antipodal chromatic number* $\mathrm{ac}'(G)$ of G is $\min\{\mathrm{ac}'(c)\}$ taken over all nearly antipodal colorings c of G. Therefore, if G

is a connected graph of diameter 1 or 2, then $\mathrm{ac}'(G) = 1$; while if $\operatorname{diam} G = 3$, then $\mathrm{ac}'(G)$ is the chromatic number of G. Thus nearly antipodal colorings are most interesting for connected graphs of diameter 4 or more. Consequently, we investigate nearly antipodal chromatic number of paths. Figure 2 shows nearly antipodal colorings of the path P_n for $n = 5, 6, 7, 8$, which has diameter $n - 1$. These colorings show that $\mathrm{ac}'(P_5) \leq 5$, $\mathrm{ac}'(P_6) \leq 7$, $\mathrm{ac}'(P_7) \leq 11$, and $\mathrm{ac}'(P_8) \leq 16$. In fact, there is equality in each case.

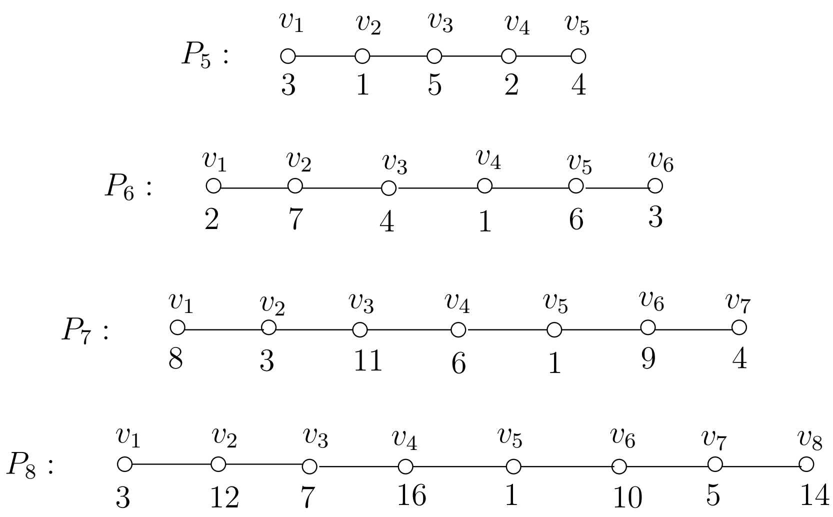

Figure 2. Nearly antipodal colorings of P_n for $5 \leq n \leq 8$

Example 3.1. $\mathrm{ac}'(P_5) = 5$*,* $\mathrm{ac}'(P_6) = 7$*,* $\mathrm{ac}'(P_7) = 11$*, and* $\mathrm{ac}'(P_8) = 16$*.*

Proof. We only verify that $\mathrm{ac}'(P_5) = 5$ and $\mathrm{ac}'(P_6) = 7$, beginning with the first of these. Assume, to the contrary, that there is a nearly antipodal coloring c of P_5 with $\mathrm{ac}'(c) = 4$. Since the complementary coloring $\overline{c}$ of c is nearly antipodal as well, we may assume that $c(v_3) = 1$ or $c(v_3) = 2$. Suppose, first, that $c(v_3) = 1$. Then one of v_2 and v_4 is 3 and the other 4, say $c(v_2) = 3$. However, then, there is no color for v_1, which is impossible.

Next we show that $\mathrm{ac}'(P_6) = 7$. Assume, to the contrary, that there is a nearly antipodal coloring c of P_6 with $\mathrm{ac}'(c) = 6$. By symmetry and replacing c by the complementary coloring $\overline{c}$, if necessary, we may assume that $1 \leq c(v_3) \leq 3$. If $c(v_3) \geq 2$, then v_2 and v_4 must be colored at least 5.

However, $d(v_2, v_4) = 2$ and so $|c(v_2) - c(v_4)| \geq 2$, which is impossible. Hence $c(v_3) = 1$. This implies that one of v_2 and v_4 is colored 4 and the other 6. However, both neighbors of the vertex colored 4 must be colored 1, which is impossible.

By similar arguments, it can be shown that $\mathrm{ac}'(P_7) = 11$ and $\mathrm{ac}'(P_8) = 16$. ■

We now present an upper bound for the nearly antipodal chromatic number of paths.

Theorem 3.2. *If P_n is a path of order $n \geq 1$, then*

$$\mathrm{ac}'(P_n) \leq \binom{n-2}{2} + 2.$$

Proof. Let $P_n : v_1, v_2, \cdots, v_n$. The result is immediate if $1 \leq n \leq 4$. So assume that $n \geq 5$. We consider two cases, according to whether n is odd or n is even.

Case 1. *n is odd.* Then $n = 2p + 1$ for some integer $p \geq 2$. Define a coloring c of P_{2p+1} by

$$\begin{aligned}
c(v_i) &= 1 + (p - i)(n - 3) \ \text{ for } 1 \leq i \leq p, \\
c(v_{p+1}) &= \binom{n-2}{2} + 2 = p(n-4) + 3, \\
c(v_{p+j}) &= p + (p - j)(n - 3) \ \text{ for } 2 \leq j \leq p, \\
c(v_{2p+1}) &= \frac{(n-3)^2}{2} + 2.
\end{aligned}$$

Note that $c(v_p) = 1$, $c(v_{p+1}) = c(v_{p+2}) + (n - 2)$, $c(v_{2p}) = p$, $c(v_{2p+1}) = c(v_1) + 1$, $c(v_{2p+1}) > c(v_{2p})$,

$$c(v_1) > c(v_2) > \cdots > c(v_p), \text{ and } c(v_{p+1}) > c(v_{p+2}) > \cdots > c(v_{2p}).$$

For $n = 5, 7, 9, 11$, the coloring of P_n is shown in Figure 3. An argument similar to the one used in the proof of Theorem 2.1 shows that the coloring c is a nearly antipodal coloring of P_n. Since $\max\{c(v_i) : \ 1 \leq i \leq 2p + 1\}$

$= c(v_{p+1}) = \binom{n-2}{2}+2$, it follows that $\mathrm{ac}'(c) = \binom{n-2}{2}+2$. Therefore, $\mathrm{ac}'(P_n) \leq \mathrm{ac}'(c) = \binom{n-2}{2} + 2$.

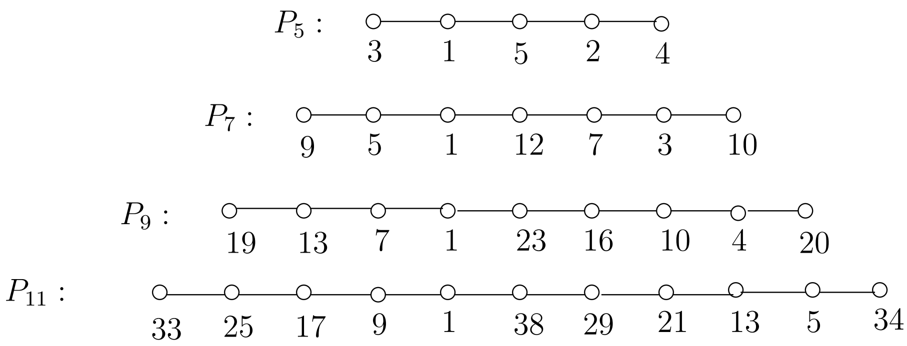

Figure 3. Nearly antipodal colorings of P_n for $n = 5, 7, 9, 11$

Case 2. *n is even.* Then $n = 2p$ for some integer $p \geq 3$. Define a coloring c of P_{2p} by

$$\begin{aligned} c(v_i) &= 1 + (p-i)(n-3) \ \text{ for } 1 \leq i \leq p, \\ c(v_{p+j}) &= p + (p-1-j)(n-3) \ \text{ for } 1 \leq j \leq p-1, \\ c(v_{2p}) &= 2 + (p-1)(n-3) = \binom{n-2}{2} + 2. \end{aligned}$$

Note that $c(v_p) = 1$, $c(v_{2p-1}) = p$, $c(v_{2p-1}) < c(v_{2p})$, $c(v_1) > c(v_2) > \cdots > c(v_{p-1})$, and $c(v_{p+1}) > c(v_{p+2}) > \cdots > c(v_{2p-1})$. For $n = 6, 8, 10, 12$, the coloring of P_n is shown in Figure 4.

We show that the coloring c is a nearly antipodal coloring of P_{2p}. Let u and v be distinct vertices of P_{2p}. If $u, v \in \{v_1, v_2, \cdots, v_p\}$, then $|c(u)-c(v)| \geq n-3$ by the definition of c. Thus $d(u,v) + |c(u) - c(v)| \geq n-2 = \operatorname{diam} P_n - 1$. Let $u, v \in \{v_{p+1}, v_{p+2}, \cdots, v_{2p}\}$. If $\{u, v\} = \{v_{p+1}, v_{2p}\}$, then $|c(u) - c(v)| = p-1$ and $d(u,v) = p-1 \geq 2$, implying that $d(u,v) + |c(u) - c(v)| \geq n-2 = \operatorname{diam} P_n - 1$. If $\{u,v\} \neq \{v_{p+1}, v_{2p}\}$, then $|c(u) - c(v)| \geq n-3$ and so $d(u,v) + |c(u) - c(v)| \geq n-2 = \operatorname{diam} P_n - 1$. Thus assume that $u \in \{v_1, v_2, \cdots, v_p\}$ and $v \in \{v_{p+1}, v_{p+2}, \cdots, v_{2p}\}$. Then $u = v_i$ for some i with $1 \leq i \leq p$. We consider two subcases.

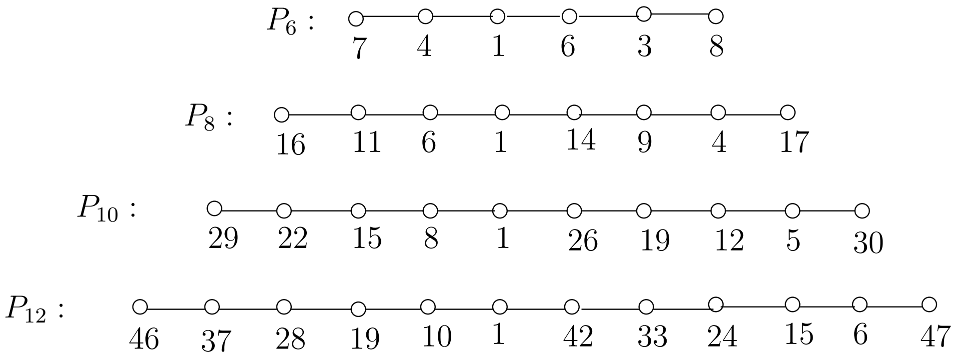

Figure 4. Nearly antipodal colorings of P_n for $n = 6, 8, 10, 12$

Subcase 2.1. $u = v_1$. Then the only possibilities for a vertex v such that $|c(u) - c(v)| < n - 3$ are $v = v_{p+1}$ or $v = v_{2p}$. If $v = v_{p+1}$, then $d(u, v) = p$ and $|c(u) - c(v)| = p - 2$; while if $v = v_{2p}$, then $d(u, v) = 2p - 1$ and $|c(u) - c(v)| = 1$. In either case, $d(u, v) + |c(u) - c(v)| \geq 2p - 2 = n - 2 = \operatorname{diam} P_n - 1$.

Subcase 2.2. $u = v_i$ *for some integer* i *with* $2 \leq i \leq p$. Then the only possibilities for a vertex v such that $|c(u) - c(v)| < n - 3$ are $v = v_{p+i-1}$ and $v = v_{p+i}$. If $v = v_{p+i-1}$, then $d(u, v) = p - 1$ and $|c(u) - c(v)| = p - 1$; while if $v = v_{p+i}$, then $d(u, v) = p$ and $|c(u) - c(v)| = p - 2$. In either case, $d(u, v) + |c(u) - c(v)| \geq 2p - 2 = n - 2 = \operatorname{diam} P_n - 1$.

Hence the coloring c is a nearly antipodal coloring of P_n. Since $\max\{c(v_i) : 1 \leq i \leq 2p + 1\} = c(v_{2p}) = \binom{n-2}{2} + 2$, it follows that $\operatorname{ac}'(c) = \binom{n-2}{2} + 2$. Therefore, $\operatorname{ac}'(P_n) \leq \operatorname{ac}'(c) = \binom{n-2}{2} + 2$. ■

4. Bounds for the Radio k-Chromatic Number of a Path

Thus far we have discussed upper bounds for $\operatorname{rc}_k(P_n)$ as a function of n when $n - 3 \leq k \leq n - 1$. We now consider the situation when $1 \leq k \leq n - 1$ in general and provide bounds for $\operatorname{rc}_k(P_n)$ as a function of k, begining with a lower bound for $\operatorname{rc}_k(P_n)$.

Theorem 4.1. *For* $1 \leq k \leq n - 1$,

$$\operatorname{rc}_k(P_n) \quad \geq \quad \begin{cases} \frac{k^2+4}{4} & \text{if } k \text{ is even,} \\ \frac{k^2+3}{4} & \text{if } k \text{ is odd.} \end{cases}$$

Proof. Let $P_n : v_1, v_2, \ldots, v_n$ be the path of order $n \geq 2$ and let c be a minimum radio k-coloring of P_n. Furthermore, let $x_1, x_2, \ldots, x_{k+1}$ be an ordering of the first $k+1$ vertices $v_1, v_2, \ldots, v_{k+1}$ of P_n such that

$$c(x_1) \leq c(x_2) \leq \ldots \leq c(x_{k+1}).$$

Since c is a radio k-coloring of P_n, it follows that

$$|c(x_{i+1}) - c(x_i)| + d(x_{i+1}, x_i) = c(x_{i+1}) - c(x_i) + d(x_{i+1}, x_i) \geq k+1$$

for $1 \leq i \leq k$. Thus

$$\begin{aligned} &\sum_{i=1}^{k}[c(x_{i+1}) - c(x_i) + d(x_{i+1}, x_i)] \\ &= \ c(x_{k+1}) - c(x_1) + \sum_{i=1}^{k} d(x_{i+1}, x_i) \geq k(k+1). \end{aligned}$$

Since $c(x_1) \geq 1$, it follows that

$$c(x_{k+1}) \geq k(k+1) - \sum_{i=1}^{k} d(x_{i+1}, x_i) + 1. \tag{2}$$

We now obtain an upper bound for $\sum_{i=1}^{k} d(x_{i+1}, x_i)$. Let

$$X = \{x_1, x_2, \ldots, x_{k+1}\} = \{v_1, v_2, \ldots, v_{k+1}\}.$$

There are two cases.

Case 1. *k is even.* Then $k = 2\ell$ for some integer $\ell \geq 1$. Observe that (1) $d(v_i, x) \leq 2\ell - i + 1$ for all $x \in X$ and $1 \leq i \leq \ell$ and (2) $d(v_i, x) \leq i - 1$ for all $x \in X$ and $\ell + 1 \leq i \leq 2\ell + 1$. Therefore,

$$\begin{aligned} \sum_{i=1}^{k} d(x_{i+1}, x_i) &\leq 2[2\ell + (2\ell - 1) + \ldots + (\ell + 1)] + \ell \\ &= 2\ell(2\ell + 1) - \ell(\ell + 1) + \ell = k(k+1) - \frac{k^2 + 2k}{4} + \frac{k}{2} \\ &= k(k+1) - \frac{k^2}{4}. \end{aligned}$$

It then follows from (2) that

$$\mathrm{rc}_k(P_n) \;=\; \mathrm{rc}_k(c) \geq c(x_{k+1}) \geq \frac{k^2}{4} + 1 = \frac{k^2+4}{4}.$$

Case 2. *k is odd.* Then $k = 2\ell - 1$ for some integer $\ell \geq 1$. Observe that $d(v_i, x) \leq 2\ell - i$ for all $x \in X$ and $1 \leq i \leq \ell$ and (2) $d(v_i, x) \leq i - 1$ for all $x \in X$ and $\ell + 1 \leq i \leq 2\ell$. Therefore,

$$\begin{aligned}\sum_{i=1}^{k} d(x_{i+1}, x_i) &\leq 2[(2\ell-1) + (2\ell-2) + \ldots + \ell] \\ &= (2\ell-1)2\ell - \ell(\ell-1) = k(k+1) - \frac{k^2-1}{4},\end{aligned}$$

it then follows from (2) that

$$\mathrm{rc}_k(P_n) \;=\; \mathrm{rc}_k(c) \geq c(x_{k+1}) \geq \frac{k^2-1}{4} + 1 = \frac{k^2+3}{4}$$

as desired. ■

Because of the way in which the lower bound for $\mathrm{rc}_k(P_n)$ was derived in Theorem 4.1, it is clear that this bound cannot be sharp. However, we now turn our attention to an upper bound for $\mathrm{rc}_k(P_n)$ in terms of k, which, as we will see, *is* sharp – at least for small values of k.

Theorem 4.2. *For* $1 \leq k \leq n-1$,

$$\mathrm{rc}_k(P_n) \;\leq\; \begin{cases} \frac{(k+1)^2}{2} & \text{if } k \text{ is odd,} \\ \frac{(k+1)^2+1}{2} & \text{if } k \text{ is even.} \end{cases}$$

Proof. Let $P_n : v_1, v_2, \ldots, v_n$. We consider two cases, according to whether k is odd or k is even.

Case 1. *k is odd.* Then $k = 2\ell + 1$ for some integer $\ell \geq 0$. Define a coloring c of P_n by

$$\begin{aligned} c(v_i) &= 1 + (i-1)(2\ell+3) \quad \text{for } 1 \leq i \leq \ell+1, \\ c(v_i) &= \ell + 2 + (i-\ell-2)(2\ell+3) \quad \text{for } \ell+2 \leq i \leq 2\ell+2, \\ c(v_j) &= c(v_i) \quad \text{for } j \equiv i \pmod{2\ell+2}. \end{aligned}$$

We show that c is a radio k-coloring of P_n. Let u and v be distinct vertices of P_n. If $c(u) = c(v)$, then $d(u, v) \geq 2\ell+2 = k+1$ and so $|c(u)-c(v)|+d(u, v) \geq k+1$. Suppose that $c(u) \neq c(v)$ and that $u = v_p$ and $v = v_q$, where say $1 \leq p < q \leq n$. If $d(u, v) \geq k$, then certainly $|c(u) - c(v)| + d(u, v) \geq k + 1$. So we may assume that $d(u, v) \leq k - 1$. In addition, we may assume from the way that c is defined that

$$1 \leq p < q \leq k + 1 = 2\ell + 2 \text{ or } 1 \leq p \leq k + 1 \leq q \leq 2k.$$

Since the arguements are similar, we will only consider the case when $1 \leq p < q \leq k+1 = 2\ell+2$. If either $1 \leq p < q \leq \ell+1$ or $\ell+2 \leq p < q \leq 2\ell+2$, then $|c(u)-c(v)| \geq 2\ell+3 = k+2$. Otherwise, $1 \leq p \leq \ell+1$ and $\ell+2 \leq q \leq 2\ell+2$. Then

$$\begin{aligned} c(u) - c(v) &= c(v_p) - c(v_q) = [1 + (p-1)(2\ell+3)] \\ &\quad -[\ell + 2 + (q - \ell - 2)(2\ell + 3)] \\ &= -\ell - 1 + (2\ell + 3)(p - q + \ell + 1). \end{aligned}$$

If $|c(v_p)-c(v_q)| \geq 2\ell+1 = k$, then certainly $|c(v_p)-c(v_q)|+d(v_p, v_q) \geq k+1$. Hence we may assume that $|c(v_p) - c(v_q)| \leq 2\ell$. However, this implies that either $q = p + \ell$ or $q = p + \ell + 1$. We consider these two possibilities.

Subcase 1.1. $q = p + \ell$. Then $d(v_p, v_q) = \ell$ and

$$\begin{aligned} |c(v_p) - c(v_q)| &= | - \ell - 1 + (2\ell + 3)(p - q + \ell + 1)| \\ &= | - \ell - 1 + (2\ell + 3)| = \ell + 2. \end{aligned}$$

Therefore, $|c(v_p) - c(v_q)| + d(v_p, v_q) = (\ell + 2) + \ell = k + 1$.

Subcase 1.2. $q = p + \ell + 1$. Then $d(v_p, v_q) = \ell + 1$ and

$$\begin{aligned} |c(v_p) - c(v_q)| &= | - \ell - 1 + (2\ell + 3)(p - q + \ell + 1)| \\ &= | - \ell - 1| = \ell + 1. \end{aligned}$$

Therefore, $|c(v_p) - c(v_q)| + d(v_p, v_q) = (\ell + 1) + (\ell + 1) = k + 1$. Hence, c is a radio k-coloring of P_n and so

$$\begin{aligned}\mathrm{rc}_k(P_n) \;\le\; \mathrm{rc}_k(c) &= \max\{c(v_{\ell+1}), c(v_{2\ell+2})\} = c(v_{2\ell+2}) = (\ell+2) + \ell(2\ell+3) \\ &= \left(\frac{k-1}{2}+2\right) + \frac{k-1}{2}(k+2) = \frac{(k+1)^2}{2}\end{aligned}$$

Case 2. *k is even.* Then $k = 2\ell$ for some integer $\ell \ge 1$. Define a coloring c of P_n by

$$\begin{aligned} c(v_i) &= 1 + (i-1)(2\ell+2) \quad \text{for } 1 \le i \le \ell+1, \\ c(v_i) &= \ell + 2 + (i-\ell-2)(2\ell+2) \quad \text{for } \ell+2 \le i \le 2\ell+1, \\ c(v_j) &= c(v_i) \quad \text{for } j \equiv i \pmod{2\ell+1}. \end{aligned}$$

We show that c is a radio k-coloring of P_n. Let u and v be distinct vertices of P_n. If $c(u) = c(v)$, then $d(u,v) \ge 2\ell+2 = k+1$ and so $|c(u)-c(v)|+d(u,v) \ge k+1$. Suppose that $c(u) \ne c(v)$ and that $u = v_p$ and $v = v_q$, where say $1 \le p < q \le n$. If $d(u,v) \ge k$, then certainly $|c(u)-c(v)| + d(u,v) \ge k+1$. So we may assume that $d(u,v) \le k-1$ and

$$1 \le p < q \le k+1 = 2\ell+1 \text{ or } 1 \le p \le k+1 \le q \le 2k.$$

Since the arguments are similar, we will only consider the case when $1 \le p < q \le k+1 = 2\ell+1$. If either $1 \le p < q \le \ell+1$ or $\ell+2 \le p < q \le 2\ell+1$, then $|c(u)-c(v)| \ge 2\ell+2 = k+2$. Otherwise, $1 \le p \le \ell+1$ and $\ell+2 \le q \le 2\ell+1$. Then

$$\begin{aligned} c(u) - c(v) &= c(v_p) - c(v_q) = [1 + (p-1)(2\ell+2)] \\ &\quad -[\ell + 2 + (q-\ell-2)(2\ell+2)] \\ &= -\ell - 1 + (2\ell+2)(p-q+\ell+1). \end{aligned}$$

If $|c(v_p) - c(v_q)| \ge 2\ell = k$, then certainly $|c(v_p) - c(v_q)| + d(v_p, v_q) \ge k+1$. Hence we may assume that $|c(v_p) - c(v_q)| \le 2\ell - 1$. However, this implies that either $q = p + \ell$ or $q = p + \ell + 1$.

Subcase 2.1. $q = p + \ell$. Then $d(v_p, v_q) = \ell$ and

$$\begin{aligned} |c(v_p) - c(v_q)| &= |-\ell - 1 + (2\ell+2)(p-q+\ell+1)| \\ &= |-\ell - 1 + (2\ell+2)| = \ell + 1. \end{aligned}$$

Subcase 2.2. $q = p + \ell + 1$. Then $d(v_p, v_q) = \ell + 1$ and

$$\begin{aligned} |c(v_p) - c(v_q)| &= |-\ell - 1 + (2\ell + 2)(p - q + \ell + 1)| \\ &= |-\ell - 1| = \ell + 1. \end{aligned}$$

Thus, in either case, $|c(v_p) - c(v_q)| = \ell + 1$. Since $d(v_p, v_q) \geq \ell$, it follows that

$$|c(v_p) - c(v_q)| + d(v_p, v_q) \geq (\ell + 1) + \ell = k + 1.$$

Hence, c is a radio k-coloring of P_n and so

$$\begin{aligned} \mathrm{rc}_k(P_n) &\leq \mathrm{rc}_k(c) = \max\{c(v_{\ell+1}), c(v_{2\ell+2})\} = c(v_{\ell+1}) = 1 + \ell(2\ell + 2) \\ &= 1 + \frac{k(k+2)}{2} = \frac{(k+1)^2 + 1}{2}, \end{aligned}$$

as desired. ■

Next, we show that if n is sufficiently large, then the upper bound in Theorem 4.2 is sharp for $1 \leq k \leq 4$.

It is a simple observation that $\mathrm{rc}_1(P_n) = 2$ for all $n \geq 2$ since $\mathrm{rc}_1(G) = \chi(G)$ for every graph G and P_n is a nontrivial connected bipartite graph. Thus $\mathrm{rc}_1(P_n) = (1 + 1)^2/2 = 2$ for $n \geq 2$. Therefore, equality in Theorem 4.2 holds for $k = 1$ and $n \geq 2$.

It is also not difficult to see that $\mathrm{rc}_2(P_3) = \mathrm{rc}_2(P_4) = 4$. Let $P_5 : v_1, v_2, v_3, v_4, v_5$. We now show that $\mathrm{rc}_2(P_5) = 5$. By Theorem 4.2, $\mathrm{rc}_2(P_5) \leq 5$. Assume, to the contrary, that $\mathrm{rc}_2(P_5) \leq 4$, and let c be a radio 2-coloring of P_5 having value 4. First, observe that none of v_2, v_3, or v_4 is colored 2 since otherwise both neighbors of such a vertex must be colored 4, which is impossible. By the same reasoning, none of v_2, v_3, or v_4 is colored 3. Hence all three vertices are colored 1 or 4. But this implies that two of these three vertices are colored the same, which cannot occur. Thus $\mathrm{rc}_2(P_5) = 5$, as claimed. By Observation 1.3 and Theorem 4.2, $\mathrm{rc}_2(P_n) = 5 = [(2+1)^2+1]/2$ for all $n \geq 5$. Therefore, equality in Theorem 4.2 holds for $k = 2$ and $n \geq 5$.

We now turn to $\mathrm{rc}_3(P_n)$, where $n > 3$. First, it is routine to show that $\mathrm{rc}_3(P_4) = 6$ and $\mathrm{rc}_3(P_n) = 7$ for $5 \leq n \leq 7$. We now consider $\mathrm{rc}_3(P_8)$. Let $P_8 : v_1, v_2, \ldots, v_8$. By Theorem 4.2, $\mathrm{rc}_3(P_8) \leq 8$. We show that $\mathrm{rc}_3(P_8) = 8$. Assume, to the contrary, that there a radio 3-coloring of P_8 having value 7. First, observe that no vertex v_i $(2 \leq i \leq 7)$ can be colored 3 since otherwise its neighbors can only be colored 6 or 7, which is impossible. By the same reasoning, no vertex v_i $(2 \le i \le 7)$ can be colored 5. Now no vertex v_i $(3 \le i \le 6)$ can be colored 6 since otherwise one of its neighbors must be colored 3. Also, no vertex v_i $(3 \le i \le 6)$ can be colored 2 for otherwise one of its neighbors must be colored 5. Hence the vertices v_3, v_4, v_5, v_6 can only be colored 1, 4, or 7, implying that two of these vertices are colored the same. Since the distance between these vertices is at most 3, this is impossible. Thus $\mathrm{rc}_3(P_8) = 8$, as claimed. Since $\mathrm{rc}_3(P_8) = 8$, it follows that $\mathrm{rc}_3(P_n) \ge 8$ for all $n \ge 8$. By Observation 1.3 and Theorem 4.2, $\mathrm{rc}_3(P_n) = 8 = (3+1)^2/2$ for all $n \ge 8$. Therefore, equality holds in Theorem 4.2 for $k = 3$ and $n \ge 8$.

By a case-by-case analysis, one can show that $\mathrm{rc}_4(P_{13}) = 13$. It then follows from Observation 1.3 and Theorem 4.2 that $\mathrm{rc}_4(P_n) = [(4+1)^2 + 1]/2 = 13$ for $n \ge 13$. Therefore, equality holds in Theorem 4.2 for $k = 4$ and $n \ge 13$.

Based on the observations that we have just made about $\mathrm{rc}_k(P_n)$ for $1 \le k \le 4$, one might think that we have equality in Theorem 4.2 for all k with $1 \le k \le k - 1$. However, such is not the case.

For $k = n - 1$, a radio k-coloring of P_n is a radio labeling and an upper bound for $\mathrm{rc}_{n-1}(P_n) = rn(P_n)$ was established in [2].

Theorem A. *For every integer $n \ge 1$,*

$$rn(P_n) \le \begin{cases} \binom{n-1}{2} + \frac{n}{2} + 1 & \text{if } n \text{ is even,} \\ \binom{n}{2} + 1 & \text{if } n \text{ is odd.} \end{cases}$$

If n is sufficiently large, then the upper bound for $\mathrm{rc}_{n-1}(P_n)$ in Theorem A is strictly smaller than that in Theorem 4.2. Furthermore, if n is sufficiently large, then the upper bounds for $\mathrm{rc}_k(P_n)$ when $k = n - 2$ and $k = n - 3$, respectively, in Theorems 2.1 and 3.2 are strictly smaller than that in Theorem 4.2 as well.

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