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UNDIRECTED AND DIRECTED GRAPHS WITH NEAR POLYNOMIAL GROWTH

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To Professor Wilfried Imrich on the occasion of his 60th birthday

Abstract

The growth function of a graph with respect to a vertex is near polynomial if there exists a polynomial bounding it above for infinitely many positive integers. In the paper vertex-symmetric undirected graphs and vertex-symmetric directed graphs with coinciding in- and out-degrees are described in the case their growth functions are near polynomial.

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1. Let Γ be a directed graph with vertex set $V(\Gamma)$ and edge set $E(\Gamma) \subseteq (V(\Gamma)\times V(\Gamma))\setminus \mathrm{diag}(V(\Gamma)\times V(\Gamma))$ (in this paper we consider graphs without loops or multiple edges). For an edge (x, y) of Γ, x is the initial vertex and y is the terminal vertex of (x, y). For $x \in V(\Gamma)$, put $\Gamma^0(x) = \{x\}$ and, inductively,

$$\Gamma^n(x) = \Gamma^{n-1}(x) \cup \{x' : (x'', x') \in E(\Gamma) \text{ for some } x'' \in \Gamma^{n-1}(x)\},$$

$$\Gamma^{-n}(x) = \Gamma^{-n+1}(x) \cup \{x' : (x', x'') \in E(\Gamma) \text{ for some } x'' \in \Gamma^{-n+1}(x)\},$$

where n runs over the set of positive integers. The growth function of Γ with respect to the vertex x is defined by $R_{\Gamma,x}(\alpha) = |\Gamma^{[\alpha]}(x)|$ for any non-negative real number α ($[\alpha]$ is the integer satisfying $[\alpha] \leq \alpha < [\alpha]+1$). The graph Γ has polynomial growth with respect to the vertex x if there exist non-negative integers c and d such that $R_{\Gamma,x}(n) \leq c \cdot n^d$ for all non-negative integers n. The graph Γ has near polynomial growth with respect to the vertex x if there exist non-negative integers c and d such that $R_{\Gamma,x}(n_i) \leq c \cdot n_i^d$ for some sequence $n_1 < n_2 < \ldots$ of positive integers and any positive integer i.

For a directed graph Γ, denote by $\bar{\Gamma}$ the underlying undirected graph (i.e., the undirected graph with the vertex set $V(\bar{\Gamma}) = V(\Gamma)$ and the edge set $E(\bar{\Gamma}) = \{\{y', y''\} \subseteq V(\Gamma) : (y', y'') \in E(\Gamma) \text{ or } (y'', y') \in E(\Gamma)\}$).

For an undirected connected graph Δ, a vertex x of Δ, and a non-negative integer n, denote by $\Delta^n(x)$ the ball of radius n with center x (with respect to the natural metric on the vertex set). Recall, that the growth function of Δ with respect to the vertex x is defined by $R_{\Delta,x}(\alpha) = |\Delta^{[\alpha]}(x)|$ for any non-negative real number α. The graph Δ has polynomial growth if, for $x \in V(\Delta)$, there exist non-negative integers c and d such that $R_{\Delta,x}(n) \leq c \cdot n^d$ for all non-negative integers n. The graph Δ has near polynomial growth if, for $x \in V(\Delta)$, there exist non-negative integers c and d such that $R_{\Delta,x}(n_i) \leq c \cdot n_i^d$ for some sequence $n_1 < n_2 < \ldots$ of positive integers and any positive integer i.

Undirected connected vertex-symmetric (i.e., admitting a vertex-transitive group of automorphisms) graphs with polynomial growth were described in [1]. It was shown, in particular, that for any such graph Δ there exists an imprimitivity system σ of $Aut(\Delta)$ on $V(\Delta)$ with finite blocks such that the group $Aut(\Delta/\sigma)$ contains a finitely generated nilpotent subgroup of finite index. Note that for any undirected connected vertex-symmetric graph with polynomial growth Δ there exists a non-negative integer d, called the growth degree of Δ, such that $1/c \cdot n^d \leq R_{\Delta,x}(n) \leq c \cdot n^d$ for $x \in V(\Delta)$, some positive integer c and all positive integers n (see [1]). The proof in [1] depends on [2]. Using [3] instead of [2] in the arguments from [1] it is possible to get a description of undirected connected vertex-symmetric graphs with near polynomial growth, which implies that any such graph has polynomial growth, see Theorem 1 below.

Directed graphs with vertex-transitive groups of automorphisms and polynomial growth are interesting for the theory of dynamical systems, see, for example, [4]. These graphs describe dynamics of generic points of integrable multivalued mappings. In fact a starting point for my investigation of such graphs was a question asked of me by A.P. Veselov (after my talk in the Moscow State University in the beginning of 90s) on their structure in the case of coinciding in-degree and out-degree (in the other case the structure can be deduced from [5], see the Proposition below). Shortly after that the question was answered in [6]. It was proved that for any such graph Γ the underlying undirected graph $\bar{\Gamma}$ also has polynomial growth and is known by the above mentioned result from [1]. The proof was not published. In the present paper it is proved that the same conclusion holds under an *a priori* weaker hypothesis that Γ has near polynomial (instead of polynomial) growth, see Theorem 2 below. The complete determination of the structure of Γ is equivalent to the determination of orbitals of vertex- transitive groups of automorphisms of the graph $\bar{\Gamma}$.

2. Theorem 1. *Let Δ be an undirected connected vertex-symmetric graph with near polynomial growth. Then Δ has polynomial growth, and* (*see* [1, Theorem 1]) *there exists an imprimitivity system σ of $Aut(\Delta)$ on $V(\Delta)$ with finite blocks such that the group $Aut(\Delta/\sigma)$ contains a finitely generated nilpotent subgroup of finite index and the stabilizer in $Aut(\Delta/\sigma)$ of a vertex of the graph Δ/σ is finite.*

Proof. Arguments from [3] for a locally finite Cayley graph of a group can be generalized to be applied to any connected vertex-symmetric locally finite graph Δ' and to give an associated arcwise connected, locally connected homogeneous metric space $Y_{\Delta'}$ on which the group $Aut(\Delta')$ acts by isometries (see [7]). Moreover, in the case of near polynomial growth of Δ' the arguments from [3, Section 6] can be easily generalized to prove that $Y_{\Delta'}$ can be chosen locally compact and finite dimensional. Arguing as in [1, p. 414], we conclude (using the solution of Hilbert's fifth problem) that the group of isometries of $Y_{\Delta'}$ is a Lie group with a finite number of connected components. For any element g from the kernel of the action of $Aut(\Delta')$ on $Y_{\Delta'}$ by isometries, there exists an increasing sequence $t_1 < t_2 < \ldots$ of positive integers such that for a fixed vertex x of Δ'

$$\max\{d_{\Delta'}(y, g(y))/t_i : y \in (\Delta')^{t_i}(x)\} \to 0 \text{ as } i \to \infty$$

(where $d_{\Delta'}(.,.)$ is the usual metric on $V(\Delta')$).

Now Theorem 1 can be proved by a rather direct generalization of arguments from [1], excluding arguments from [1, §5] which should be replaced by the arguments given above.

3. The following terminology concerning a directed graph Γ will be used. A sequence $(x_0, \ldots, x_s)$ of vertices of Γ is a path of Γ if either $s = 0$ or $s > 0$ and, for each $0 \le i < s$, $(x_i, x_{i+1}) \in E(\Gamma)$ or $(x_{i+1}, x_i) \in E(\Gamma)$. A path $(x_0, \ldots, x_s)$ of Γ is a directed path if either $s = 0$ or $s > 0$ and, for each $0 \le i < s$, $(x_i, x_{i+1}) \in E(\Gamma)$.

Let $X = (x_0, \ldots, x_s)$ be a path of Γ. Then s is the length of X. Denote by X^{-1} the path $(x_s, \ldots, x_0)$. For a path $Y = (y_0, \ldots, y_t)$ of Γ with $x_s = y_0$, XY is the path $(x_0, \ldots, x_s = y_0, \ldots, y_t)$. Any path of Γ can be written as $X_1 Y_1 \ldots X_k Y_k$ where $X_1, Y_1^{-1}, \ldots, X_k, Y_k^{-1}$ are directed paths of Γ.

For $G \le Aut(\Gamma)$ and $x \in V(\Gamma)$, denote by G_x the stabilizer of x in G.

The graph Γ is vertex-symmetric if $Aut(\Gamma)$ is vertex-transitive.

Denote by T the directed graph such that, firstly, $\bar{\mathrm{T}}$ is the tree with one vertex v of degree 2 and all other vertices of degree 3, and, secondly, the out-degree of every vertex of T is equal to 2. The graph Γ is out-hyperbolic with respect to a vertex x if there exist an injection $\varphi : V(\mathrm{T}) \to V(\Gamma)$ with $\varphi(v) = x$ and a positive integer t such that for any edge (v', v'') of T there exists a directed path of Γ from $\varphi(v')$ to $\varphi(v'')$ of length not greater than t. The graph Γ is in-hyperbolic with respect to a vertex x if the graph obtained from Γ by the inversion of the direction is out-hyperbolic with respect to the vertex x. If the graph Γ is out-hyperbolic with respect to a vertex x, then, obviously, Γ is not a graph with near polynomial growth with respect to the vertex x.

At last, if the graph Γ is vertex-symmetric and out-hyperbolic (in-hyperbolic) with respect to some vertex, then it is out-hyperbolic (respectively, in-hyperbolic) with respect to any vertex and is called simply out-hyperbolic (respectively, in-hyperbolic).

Proposition. *Let Γ be a directed graph for which the graph $\bar{\Gamma}$ is connected and locally finite, and G be a vertex-transitive group of automorphisms of Γ. Then:*

(1) *If $|\{g(y) : g \in G_x\}| > |\{g(x) : g \in G_y\}|$ for some vertices x, y of Γ such that there exists a direct path of Γ from x to y (from y to x), then Γ is out-hyperbolic (respectively, in-hyperbolic).*

(2) *If $|\Gamma^n(x)| > |\Gamma^{-n}(x)|$ for a vertex x of Γ and an integer n, then Γ is out-hyperbolic in the case $n > 0$, and Γ is in-hyperbolic in the case $n < 0$.*

(3) *If Γ is not out-hyperbolic and $|\Gamma^n(x)| \ne |\Gamma^{-n}(x)|$ for a vertex x of Γ and an integer n, then (Γ is in-hyperbolic by (2) and) $|\Gamma^{n'}(x')| < |\Gamma^{-n'}(x')|$ for any vertex x' of Γ and any positive integer n'.*

Proof. A proof of (1) can be derived from the proof of Theorem 2 in [5]. To prove (2) and (3), we need some properties of paired orbits. Recall that for any vertex x of Γ and any G_x-orbit X on $V(\Gamma)$ the paired G_x-orbit on $V(\Gamma)$, denoted by X^*, is defined by $X^* = \{y : g(y) = x \text{ and } g(x) \in X \text{ for some } g \in G\}$. The map $X \mapsto X^*$ is a bijection on the set of G_x-orbits on $V(\Gamma)$, and $(X^*)^* = X$ for any G_x-orbit. For an integer n, if X is a G_x-orbit on $\Gamma^n(x)$ (note that the set $\Gamma^n(x)$ is G_x-invariant), then, obviously, X^* is contained in $\Gamma^{-n}(x)$. Now 1) can be reformulated in the following way. If x is a vertex of Γ, n is a positive integer, and X is a G_x-orbit on $\Gamma^n(x)$ such that $|X| > |X^*|$ (such that $|X| < |X^*|$), then Γ is out-hyperbolic (respectively, in-hyperbolic). Finally, suppose that there exists a G_x-orbit on $V(\Gamma)$ such that $|X| \neq |X^*|$. Then, by [5, Theorem 1], the closure $\bar{G}$ of G in the group $Aut(\Gamma)$ equipped with the natural compact-open topology is not unimodular. Since, by the connectedness of $\bar{\Gamma}$, the group $\bar{G}$ is generated by the compact subgroup $\bar{G}_x$ and all elements of $\bar{G}$ mapping the vertex x to vertices from $\bar{\Gamma}^1(x)$, it follows that the value of the modular function of $\bar{G}$ on some element of $\bar{G}$ mapping the vertex x to some vertex $y \in \bar{\Gamma}^1(x)$ differs from 1. Since G_x-orbits and $\bar{G}_x$-orbits on $V(\Gamma)$ coinside, we get $|Y| \neq |Y^*|$ where Y is the G_x-orbit containing y.

Now to prove (2) note that $|\Gamma^n(x)| > |\Gamma^{-n}(x)|$ for a vertex x of Γ and an integer n implies $|X| > |X^*|$ for some G_x-orbit X on $\Gamma^n(x)$. Thus 2) follows by the above.

Turning to (3), note that, since Γ is not out-hyperbolic, the above implies $|X| \leq |X^*|$ for any G_x-orbit X on $\Gamma^{n'}(x)$ where n' is an arbitrary positive integer. Now $|\Gamma^n(x)| \neq |\Gamma^{-n}(x)|$ implies that $|X| < |X^*|$ for some G_x-orbit X on $\Gamma^{|n|}(x)$. As it was mentioned above, it follows $|Y| < |Y^*|$ for some G_x-orbit Y on $\Gamma^1(x)$. Thus $|\Gamma^{n'}(x)| < |\Gamma^{-n'}(x)|$ for any positive integer n', and (3) follows by vertex-transitivity of G.

4. Theorem 2. *Let Γ be a directed vertex-symmetric graph with near polynomial growth. Suppose the graph $\bar{\Gamma}$ is connected, and $|\Gamma^1(x)| = |\Gamma^{-1}(x)|$ for $x \in V(\Gamma)$. Then the graph $\bar{\Gamma}$ has polynomial growth, and (see Theorem 1) there exists an imprimitivity system τ of $Aut(\Gamma)$ on $V(\Gamma)$ with finite blocks such that the group $Aut(\Gamma/\tau)$ contains a finitely generated nilpotent subgroup of finite index and the stabilizer in $Aut(\Gamma/\tau)$ of a vertex of the graph Γ/τ is finite.*

Proof. Since the graph Γ is vertex-symmetric, the growth function $R_{\Gamma,x}$ is independent of $x \in V(\Gamma)$. We denote it by R. By hypothesis and Proposition, there exist positive integers c and d such that

$$R(n_i) = |\Gamma^{n_i}(x)| = |\Gamma^{-n_i}(x)| \leq c \cdot n_i^d$$

for some sequence $n_1 < n_2 < \ldots$ of positive integers and any positive integer i. By Theorem 1, to prove Theorem 2 it is sufficient to show that there exist positive integers c' and d' such that

$$|\bar{\Gamma}^{n'_i}(x)| \leq c' \cdot {n'_i}^{d'}$$

for some sequence $n'_1 < n'_2 < \ldots$ of positive integers and any positive integer i.

Following arguments are very close to ones from [8] (where, however, they are formulated in other terms).

Fix a real number λ and a positive integer a such that

$$1 < \lambda < \min\{2^{1/d}, 3/2\}, \tag{1}$$

$$a \geq (\log_\lambda 2 + 2 + 4\log_\lambda 1/(\lambda - 1))(\log_\lambda 2 - d)^{-1} \tag{2}$$

(note that $a > 1$), and put

$$b = a \cdot d + \log_\lambda 2 + 1 + 4\log_\lambda 1/(\lambda - 1). \tag{3}$$

Without loss of generality we will suppose that

$$n_1 \geq 2^a.$$

For each positive integer j, put

$$m_j = n_j^{1/a} \geq 2,$$

$$E_j = \{i \in \{1, 2, \ldots, [b \cdot \log_2 m_j]\} : R(\lambda^{i+1}) > 2R(\lambda^i)\},$$

$$F_j = \{[\log_\lambda(m_j)], [\log_\lambda(m_j)] + 1, \ldots, [b \cdot \log_2 m_j]\} \setminus E_j.$$

Then

$$R(\lambda^{[b \cdot \log_2 m_j]+1}) \geq 2^{|E_j|}$$

and, by (1) – (3),

$$\lambda^{[b \cdot \log_2 m_j]+1} \leq n_j = m_j^a. \tag{4}$$

Thus
$$c \cdot m_j^{a \cdot d} \geq 2^{|E_j|}.$$
Now
$$|F_j| \geq [b \cdot \log_2 m_j] - [\log_\lambda m_j] - |E_j|$$
$$\geq [b \cdot \log_2 m_j] - [\log_\lambda m_j] - \log_2 c - a \cdot d \cdot \log_2 m_j$$
$$\geq \log_2 m_j (b - \log_\lambda 2 - a \cdot d - (\log_2 c + 1)(\log_2 m_j)^{-1}).$$

By (3), this implies that, for all sufficiently large j, say for all $j \geq j'$,
$$|F_j| > 4 \log_\lambda 1/(\lambda - 1) \log_2 m_j.$$

Thus, for each $j \geq j'$ and for $q_j = [\log_2 m_j] + 1$, there exists a subset $K_j = \{k(j,1), \ldots, k(j, 2q_j)\}$ of F_j such that
$$k(j,1) > \log_\lambda m_j + \log_\lambda 1/(\lambda - 1)$$
and
$$k(j, r+1) - k(j, r) > \log_\lambda 1/(\lambda - 1)$$
for all $1 \leq r < 2q_j$. Put, in addition,
$$k(j, 0) = \log_\lambda m_j$$
for each $j \geq j'$.

We show that, for any $j \geq j'$ and $0 \leq r \leq 2q_j - 2$, if $X_1 Y_1 X_2 Y_2$ is a path of Γ such that $X_1, Y_1^{-1}, X_2, Y_2^{-1}$ are directed paths of Γ of length not greater than $\lambda^{k(j,r)}$, then there exists a path XY of Γ with the same initial and terminal vertices as $X_1 Y_1 X_2 Y_2$ and such that X, Y^{-1} are directed paths of Γ of length not greater than $\lambda^{k(j,r+2)}$. Let x', y, x'' be the terminal vertices of X_1, Y_1, X_2, respectively. If
$$\Gamma^{[\lambda^{k(j,r+1)}]}(x') \cap \Gamma^{[\lambda^{k(j,r+1)}]}(x'') = \emptyset,$$
then
$$\Gamma^{[\lambda^{k(j,r+1)}]}(x') \cup \Gamma^{[\lambda^{k(j,r+1)}]}(x'') \subseteq \Gamma^{[\lambda^{k(j,r+1)} + \lambda^{k(j,r)}]}(y)$$
implies
$$R(\lambda^{k(j,r+1)+1})/R(\lambda^{k(j,r+1)}) > R(\lambda^{k(j,r+1)} + \lambda^{k(j,r)})/R(\lambda^{k(j,r+1)}) \geq 2,$$

a contradiction. Thus there exist a path Z_1Z_2 of Γ such that Z_1 is a directed path of Γ with the initial vertex x' and of the length not greater than $\lambda^{k(j,r+1)}$, and Z_2^{-1} is a directed path of Γ with the initial vertex x'' and of the length not greater than $\lambda^{k(j,r+1)}$. Put $X = X_1Z_1$, $Y = Z_2Y_2$. Then the path XY of Γ has the same initial and terminal vertices as the path $X_1Y_1X_2Y_2$, and X, Y^{-1} are directed paths of Γ of length not greater than

$$\lambda^{k(j,r+1)} + \lambda^{k(j,r)} < \lambda^{k(j,r+1)+1} < \lambda^{k(j,r+2)}.$$

Thus XY is the required path.

As a consequence we have that, for any $j \geq j'$ and $0 \leq r \leq 2q_j - 2$, if $X_1Y_1 \ldots X_{2k}Y_{2k}$ is a path of Γ such that $X_1, Y_1^{-1}, \ldots, X_{2k}, Y_{2k}^{-1}$ are directed paths of length not greater than $\lambda^{k(j,r)}$, then there exists a path $X_1'Y_1' \ldots X_k'Y_k'$ of Γ with the same initial and terminal vertices as $X_1Y_1 \ldots X_{2k}Y_{2k}$ and such that $X_1', Y_1'^{-1}, \ldots, X_k', Y_k'^{-1}$ are directed paths of Γ of length not greater than $\lambda^{k(j,r+2)}$.

Thus, for any $j \geq j'$, if $X_1Y_1 \ldots X_sY_s$, where $s = 2^{q_j}$, is a path of Γ such that $X_1, Y_1^{-1}, \ldots, X_s, Y_s^{-1}$ are directed paths of length not greater than $\lambda^{k(j,0)} = m_j$, then there exists a path $X'Y'$ of Γ with the same initial and terminal vertices as $X_1Y_1 \ldots X_sY_s$ and such that $X_1', Y_1'^{-1}$ are directed paths of Γ of length not greater than $\lambda^{k(j,2q_j)} < n_j$ (see (4)). Since any path of Γ of length not greater than $m_j = \lambda^{k(j,0)}$ can be written as $X_1Y_1 \ldots X_sY_s$, where $s = 2^{q_j} > m_j$ and $X_1, Y_1^{-1}, \ldots, X_s, Y_s^{-1}$ are directed paths of length not greater than $\lambda^{k(j,0)} = m_j$, it follows

$$\bar{\Gamma}^{m_j}(x) \subseteq \cup_{y \in \Gamma^{n_j}(x)} \Gamma^{-n_j}(y)$$

for all $j \geq j'$. Thus

$$|\bar{\Gamma}^{m_j}(x)| \leq R(n_j)^2 \leq c^2 \cdot n_j^{2d} = c^2 \cdot m_j^{2d \cdot a}$$

for all $j \geq j'$. As it was noted in the beginning of the proof, the theorem follows.

Remark 1. Let Γ be a directed vertex-symmetric graph with near polynomial growth, which is not in-hyperbolic. Then $|\Gamma^1(x)| = |\Gamma^{-1}(x)|$ for $x \in V(\Gamma)$ by the Proposition. Thus Theorem 2 can be applied to components of Γ (which are isomorphic).

Remark 2. It can be deduced from Theorem 2 (modifying, for example, arguments from [8]) that if d is the growth degree of the graph $\bar{\Gamma}$ from

Theorem 2, then there exists a positive integer c_1 such that $1/c_1 \cdot n^d \leq |\Gamma^n(x)| = |\Gamma^{-n}(x)| \leq c_1 \cdot n^d$ for all positive integers n.

Remark 3. In [8], it was proved that a finitely generated semigroup with cancellations has polynomial growth if and only if its group of all left quotients (exists and) contains a nilpotent subgroup of finite index. A modification of arguments from [8] (compare the proof of Theorem 2 above) and using [3] instead of [2] implies that here "polynomial growth" can be replaced by "near polynomial growth". (In notation from [8], S has near polynomial growth if there exist positive integers c and d such that $\gamma(n_i) \leq c \cdot n_i^d$ for some sequence $n_1 < n_2 < \ldots$ of positive integers and any positive integer i.)

Remark 4. Obviously, Theorem 1 follows from Theorem 2 (applied to the directed graph $\vec{\Delta}$ with the vertex set $V(\vec{\Delta}) = V(\Delta)$ and the edge set $E(\vec{\Delta}) = \{(y', y'') : \{y', y''\} \in E(\Delta)\}$). But Theorem 1 was used in the proof of Theorem 2.

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