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ON DUALLY COMPACT CLOSED CLASSES OF GRAPHS AND BFS-CONSTRUCTIBLE GRAPHS

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Dedicated to Wilfried Imrich on the occasion of his 60th birthday.

Abstract

A class $\mathcal{C}$ of graphs is said to be *dually compact closed* if, for every infinite $G \in \mathcal{C}$, each finite subgraph of G is contained in a finite induced subgraph of G which belongs to $\mathcal{C}$. The class of trees and more generally the one of chordal graphs are dually compact closed. One of the main part of this paper is to settle a question of Hahn, Sands, Sauer and Woodrow by showing that the class of bridged graphs is dually compact closed. To prove this result we use the concept of constructible graph. A (finite or infinite) graph G is *constructible* if there exists a well-ordering $\leq$ (called *constructing ordering)* of its vertices such that, for every vertex x which is not the smallest element, there is a vertex $y < x$ which is adjacent to x and to every neighbor z of x with $z < x$. Finite graphs are constructible if and only if they are dismantlable. The case is different, however, with infinite graphs. A graph G for which every breadth-first search of G produces a particular constructing ordering of its vertices is called a BFS-*constructible* graph. We show that the class of BFS-constructible graphs is a variety (i.e., it is closed under weak retracts and strong products), that it is a subclass of the class of weakly modular graphs, and that it contains the class of bridged graphs and that of Helly graphs (bridged graphs being very special instances of BFS-constructible graphs). Finally we show that the class of interval-finite pseudo-median graphs (and thus the one of median graphs) and the class of Helly graphs are dually compact closed, and that moreover every finite subgraph of an interval-finite pseudo-median graph (resp. a Helly graph) G is contained in a finite isometric pseudo-median (resp. Helly) subgraph of G. We also give two sufficient conditions so that a bridged graph has a similar property.

Keywords: infinite graph; dismantlable graph; constructible graph; BFS-cons-tructible graph; variety; weak-retract; strong product; bridged graph; Helly graph; weakly-modular graph; dually compact closed class.

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1. Introduction

A class $\mathcal{C}$ of graphs is said to be *compact closed* if, whenever a graph G is such that each of its finite subgraphs is contained in a finite induced subgraph of G which belongs to the class $\mathcal{C}$, then the graph G itself belongs to $\mathcal{C}$. In this paper we shall deal with the dual concept. We will say that a class $\mathcal{C}$ of graphs is *dually compact closed* if, for every infinite $G \in \mathcal{C}$, each finite subgraph of G is contained in a finite induced subgraph of G which belongs to $\mathcal{C}$.

The class of trees is clearly dually compact closed and more generally the class of all chordal graphs (a graph is *chordal* if it contains no induced cycles of length greater than three) is dually compact closed because every induced subgraph of a chordal graph is chordal.

In 1981, Hahn, Sands, Sauer and Woodrow [4] proposed the following problem: *Is every cycle of a bridged graph G contained in a finite induced bridged subgraph of G?* We recall that a graph is *bridged* if it contains no isometric cycles of length greater that three. This problem is obviously true if the class of bridged graphs is dually compact closed. In fact, later Hahn, Sauer and Woodrow suggested to determine whether the class of bridged graphs is dually compact closed. With Laviolette they gave a partial answer to this problem by proving the following result.

Theorem 1.1 ([3])**.** *The class of bridged graphs of diameter two is dually compact closed. More precisely every finite subgraph of a bridged graph G of diameter two is contained in a finite induced subgraph of G which is bridged and has diameter two.*

Recently Chastand, Laviolette and Polat [2] gave an affirmative answer to the problem of Hahn, Sauer and Woodrow. In Section 4 of this paper we will recall the proof of this result which uses the concept of constructible graphs. Roughly, a graph G is said to be *dismantlable* if its vertices can be removed one after the other in such a way that a vertex x can be taken off the currently remaining subgraph G_x of G if there exits a vertex y in G_x which is adjacent to x and to all neighbors of x in G_x. On the other hand a graph G is said to be *constructible* if it can be built vertex after vertex so that a vertex x can be added to the currently constructed induced subgraph G_x of G if there exists a vertex y of G_x which is adjacent in G to x and to all neighbors of x belonging to G_x. These opposite concepts, which coincide for finite graphs, are quite different for infinite graphs. Moreover the concept of constructibility seems to be more interesting in the infinite case. For some graphs, breadth-first search always gives an ordering of their vertices that can be induced by constructibility. Some of these graphs are called BFS-*constructible.* They all are weakly modular graphs and their class is a variety which in particular contains bridged graphs and Helly graphs, and that we conjecture to be generated by bridged graphs on account of the very special properties of these graphs.

Several other subclasses of the class of weakly modular graphs are dually compact closed but, as we will see, the class of Helly graphs (as well as that of interval-finite pseudo-median graphs) has the following very interesting refinement of the dually compact closed property: every finite subgraph of a Helly graph G is contained in a finite isometric Helly subgraph of G. We still do not know if the class of bridged graphs (and even of chordal graphs) has an analogous property, that is more precisely, if every finite subgraph of a bridged graph G is contained in a finite isometric (and thus bridged) subgraph of G. Until now we only have partial results. The most interesting one is that this property holds for bridged graphs which contains no infinite complete subgraphs.

Some of the results of this paper have already been published. For the main ones we recall their proofs when they are short and give outlines of proofs otherwise.

2. Preliminaries

The graphs we consider are undirected, without loops and multiple edges. A complete graph will be simply called a *simplex.* If $x \in V(G)$, the set $N_G(x) := \{y \in V(G) : \{x, y\} \in E(G)\}$ is the *neighborhood* of x in G. For

$A \subseteq V(G)$ we denote by $G[A]$ the subgraph of G induced by A, and we set $G - A := G[V(G) - A]$.
A *path* $P = \langle x_0, \ldots, x_n \rangle$ is a graph with $V(P) = \{x_0, \ldots, x_n\}$, $x_i \neq x_j$ if $i \neq j$, and $E(P) = \{\{x_i, x_{i+1}\} : 0 \leq i < n\}$. A *ray* or *one-way infinite path* $\langle x_0, x_1, \ldots \rangle$ and a *double ray* or *two-way infinite path* $\langle \ldots, x_{-1}, x_0, x_1, \ldots \rangle$ are defined similarly. A path $P = \langle x_0, \ldots, x_n \rangle$ is called an (x_0, x_n)*-path*, x_0 and x_n are its *endvertices*, while the other vertices are called its *internal* vertices, $n = |E(P)|$ is the length of P.

The usual *distance* in a connected graph G between two vertices x and y, that is the length of an (x, y)*-geodesic* (i.e., shortest (x, y)-path) in G, is denoted by $d_G(x, y)$. A subgraph H of G is *isometric* in G if $d_H(x, y) = d_G(x, y)$ for all vertices x and y of H. If x is a vertex of G and r a non-negative integer, the set $B_G(x, r) := \{y \in V(G) : d_G(x, y) \leq r\}$ is the *ball of center x and radius r in G*, and the set $S_G(x, r) := \{y \in V(G) : d_G(x, y) = r\}$ is the *sphere of center x and radius r in G*. The smallest integer r such that $V(G) \subseteq B_G(x, r)$ for some vertex x is the *radius* of G.

3. Constructible Graphs

If x and y are two vertices of a graph G, then we say that x is *dominated* by y in G if $B_G(x, 1) \subseteq B_G(y, 1)$, i.e., if y is adjacent to x and to all neighbors of x in G. We will first recall the definition of a dismantlable graph.

Definition 3.1. A graph G is said to be *dismantlable* if there is a well-order $\preceq$ on $V(G)$ such that every vertex x which is not the greatest element of $(V(G), \preceq)$, if such a greatest element exists, is dominated by some vertex $y \neq x$ in the subgraph of G induced by the set $\{z \in V(G) : x \preceq z\}$. The well-order $\preceq$ on $V(G)$, and the enumeration of the vertices of G induced by $\preceq$, will be called a *dismantling order* and a *dismantling enumeration*, respectively.

Definition 3.2. A graph G is said to be *constructible* if there is a well-order $\leq$ on $V(G)$ such that every vertex x which is not the smallest element of $(V(G), \leq)$ is dominated by some vertex $y \neq x$ in the subgraph of G induced by the set $\{z \in V(G) : z \leq x\}$. The well-order $\leq$ on $V(G)$, and the enumeration of the vertices of G induced by $\leq$, will be called a *constructing order* and a *constructing enumeration*, respectively.

Remark 3.3. (1) Clearly a finite graph G is dismantlable if and only if it is constructible. In fact, in this case, a constructing order on $V(G)$ is the dual of a dismantling order on this set. This may not be true if G is infinite. There are constructible graphs which are not dismantlable. For example a double ray $D = \langle \dots, -1, 0, 1, \dots \rangle$ is not dismantlable since no vertex of D is dominated, but it is constructible : $0, 1, -1, 2, -2, \dots$ is a constructing enumeration of $V(D)$. On the other hand there are dismantlable graphs which are not constructible. For example let $\langle a_0, a_1, \dots \rangle$ and $\langle b, c, d \rangle$ be two disjoint paths, and let G be the graph obtained by joining the vertices b and d to a_n for every non-negative integer n. This graph G is dismantlable since $a_0, a_1, \dots, b, c, d$ is a dismantling order on $V(G)$. It is not constructible because if $\leq$ was a constructing order on $V(G)$, if n was such that $a_n < a_p$ for every $p \neq n$, and if x was the greatest vertex of the cycle $\langle a_n, b, c, d, x_n \rangle$ with respect to $\leq$, then x would not be dominated in $G[\{y \in V(G) : y \leq x\}]$, contrary to the definition of a constructing order.

(2) Let $\leq$ be a constructing order on the vertex set of a graph G with u as the smallest element. Any self-map Δ of $V(G)$ such that $\Delta(u) = u$ and, for each vertex $x \in V(G - u)$, $\Delta(x)$ is a vertex of G which dominates x in $G[\{y \in V(G) : y \leq x\}]$, will be called a *domination function* associated with $\leq$. Because a well-order contains no infinite descending chain, for every domination function Δ and every $x \in V(G)$, there exits a non-negative integer n such that $\Delta^n(x) = u$.

The following result will be the corner stone of the solution to the problem of Hahn, Sauer and Woodrow [5].

Theorem 3.4 ([2, Theorem 5.1])**.** *The class of all constructible graphs is dually compact closed.*

Proof. Let G be a constructible graph, and let A be a finite set of vertices of G. Let $\leq$ be a constructing order on $V(G)$ with some vertex u as the smallest element, and let Δ be a domination function associated with $\leq$. By Remark 3.3(2), for every $a \in A$, there exists a non-negative integer $n(a)$ such that $\Delta^{n(a)}(a) = u$. Let $H := G[\bigcup_{a \in A}\{\Delta^i(a) : 0 \leq i \leq n(a)\}]$. This graph H is finite and contains $G[A]$. Furthermore the restriction of $\leq$ to $V(H)$ is obviously a constructing order on $V(H)$, which proves the result. ■

For different classes of graphs, a useful tool for obtaining constructing orders is the concept of breadth-first search (BFS). We recall that a BFS of a given graph G with n vertices produces an enumeration $x_1, \dots, x_n$ of the vertices of G in the following way. We number with 1 some vertex of G and put it at the head of an empty queue. At the i-th step we number and add at the end of the current queue all still unnumbered neighbors of the head x_i of the queue, then we remove x_i.

Definition 3.5. Let G be a connected graph. A well-order $\leq$ on $V(G)$ is called a BFS-*order* if there exists a family $(A_x)_{x \in V(G)}$ of subsets of $V(G)$ such that, for every $x \in V(G)$:

(i) $x \in A_x$;

(ii) if $x \leq y$, then A_x is an initial segment of A_y with respect to the induced order;

(iii) $A_x = A_{(x)} \cup N_G(x)$ where $A_{(x)} := \{x\}$ if x is the least element of $(V(G), \leq)$, and otherwise $A_{(x)} := \bigcup_{y<x} A_y$.

The vertex x will be called the *father* of each element of $A_x - A_{(x)}$. We will denote by ϕ, and call *father function*, the self-map of $V(G)$ such that $\phi(u) = u$ if u is the smallest element of $(V(G), \leq)$, and $\phi(x)$ is the father of x for every $x \in V(G - u)$.

Lemma 3.6 ([13, Lemma 3.6])**.** *There exists a* BFS-*order on the vertex set of any connected graph.*

As is shown in [2], a BFS-order is not necessarily a constructing order, and a constructing order is not necessarily a BSF-order. Furthermore, there exist constructible graphs such that none of their constructing orders is a BFS-order. On the contrary, for some classes of constructible graphs, any BFS-order is a constructing order.

4. BFS-Constructible Graphs and Bridged Graphs

Let $\leq$ be constructing order on the vertex set of a graph G with u as the smallest element. A domination function Δ associated with $\leq$ is said to be *descending* if $d_G(u, \Delta(x)) < d_G(u, x)$ for every $x \neq u$.

Definition 4.1. A connected graph G is said to be BSF-*constructible* if every BFS-order $\leq$ on $V(G)$ is a constructing order for which there exists an associated descending domination function. More generally a graph G is BFS-*constructible* if each component of G is BFS-constructible.

We recall that a graph G is *weakly modular* if it satisfies the following two conditions:

Triangle condition: for any three vertices x_0, x_1, x_2 with $1 = d_G(x_1, x_2) < d_G(x_0, x_1) = d_G(x_0, x_2)$, there exists a common neighbor u of x_1 and x_2 such that $d_G(x_0, u) = d_G(x_0, x_1) - 1$.

Quadrangle condition: for any four vertices x_0, x_1, x_2, x_3 with $d_G(x_1, x_3) = d_G(x_2, x_3) = 1$ and $d_G(x_0, x_1) = d_G(x_0, x_2) = d_G(x_0, x_3) - 1$, there exists a common neighbor u of x_1 and x_2 such that $d_G(x_0, u) = d_G(x_0, x_1) - 1$.

The class of weakly modular graphs contains important subclasses. Among them are the class of *bridged graphs* and thus that of *chordal graphs* (note that a bridged graph is *hereditary*, that is every isometric subgraph of a bridged graph is bridged), and the class of *pseudo-modular graphs* containing itself the important classes of *median graphs*, *pseudo-median graphs* and *Helly graphs*.

Note that there exists no relation between constructible graphs and weakly modular graphs, as is shown by the following two examples. Let C be a cycle of length four. Then C is clearly weakly modular but not constructible. Now let x and y be two new vertices, and let G be the graph obtained by joining x to all vertices of C, and y to two adjacent vertices of C. Then G is constructible but not weakly modular. As we will now see, the case is different for BFS-constructible graphs.

Theorem 4.2. *Every* BFS-*constructible graph is weakly modular.*

Proof. Let G be a BFS-constructible graph.

(a) *Triangle property.*

Let a, b, c be three vertices of G such that b and c are adjacent and a is equidistant from b and c and at distance $n > 1$ from these vertices. Let $\langle x_0, \ldots, x_n \rangle$ be an (a, b)-geodesic with $x_0 = a$ and $x_n = b$. Let $\leq$ be a BFS-order on $V(G)$ such that x_i is the smallest element of the set $S_G(a, i)$ for all i with $0 \leq i \leq n$. Then $b < c$. Since G is BFS-constructible, the vertex c is dominated in $G[\{y \in V(G) : y \leq c\}]$ by a vertex u such that $d_G(a, u) < d_G(a, c)$. Hence u is adjacent to both b and c and $d_G(a, u) = n-1$.

(b) *Quadrangle property.*

Let a, b, c, d be four vertices of G such that b and c are both adjacent to d and $d_G(a, b) = d_G(a, c) =: n = d_G(a, d) - 1$. Assume that no common neighbor of b and c is at distance $n-1$ from a. Then, by (a), b and c cannot be adjacent.
Let $\langle x_0, \ldots, x_n \rangle$ be an (a, b)-geodesic with $x_0 = a$ and $x_n = b$, and let $\leq$ be a BFS-order on $V(G)$ defined as in (a). Since $d_G(a, d) = n + 1$, it follows that $b < d$ and $c < d$. Then, since G is BFS-constructible, there exists a vertex u which dominates d in $G[\{y \in V(G) : y \leq d\}]$ with $d_G(a, u) = n$. Since b and c are not adjacent, u must be distinct from these two vertices.

By (a) there exist b' and c' such that b' is adjacent to u and b, c' is adjacent to u and c, and $d_G(a, b') = d_G(a, c') = n - 1$. By the assumption we must have $b' \neq c'$.

(b.1) We claim that b' and c' are not adjacent. Suppose that they are adjacent. By (a) there is a common neighbor v of b' and c' such that $d_G(a, v) = n - 2$. Then there exists a BFS-order $\leq_b$ on $V(G)$ with b as the smallest element such that $b \leq_b d \leq_b b' \leq_b u \leq_b c \leq_b v \leq_b c'$. Since G is BFS-constructible there must exists a vertex w which is adjacent to all vertices b, u, c, v, c'. Therefore $d_G(a, w) = n - 1$, and this contradicts the assumption and thus proves the claim.

(b.2) Therefore b' and c' are not adjacent. By the same argument as before, there is a common neighbor v of b' and c' such that $d_G(a, v) = n-1$. Also there exist b'' and c'' such that b'' is adjacent to v and b', c'' is adjacent to v and c', and $d_G(a, b'') = d_G(a, c'') = n-2$. If $b'' \neq c''$, then, with the same argument as in (b.1), we would prove the existence of common neighbor w of the vertices b', c', v, b'', c'' such that $d_G(a, w) = n - 2$. Therefore we can suppose that $b'' = c'' =: w$.

Then there exists a BFS-order $\leq_{b'}$ on $V(G)$ with b' as the smallest element such that $b' \leq_{b'} u \leq_{b'} b \leq_{b'} v \leq_{b'} w \leq_{b'} c \leq_{b'} c'$. Since G is BFS-constructible there must exists a vertex y which is adjacent to all vertices b', u, v, w, c, c'. Therefore $d_G(a, y) = n - 1$, contrary to the preceding claim with y instead of c', because b' is adjacent to b and u, y is adjacent to c and u, and b' and y' are adjacent. Consequently the assumption is false. ∎

If G and H are two graphs, a map $f : V(G) \to V(H)$ is a *contraction* (*weak homomorphism* in [7]) if f preserves or contracts the edges, i.e., if $f(x) = f(y)$ or $\{f(x), f(y)\} \in E(H)$ whenever $\{x, y\} \in E(G)$. Note that the contractions between two graphs G and F correspond to the non-expansive maps between the associated metric spaces $(V(G), d_G)$ and $(V(H), d_H)$. Graphs and contractions form a category in which the product, denoted by $\boxtimes$, is what is usually called the *strong product* of graphs. A graph H is a *retract* (*weak retract* in [7]) of a graph G if there are contractions f of G onto H and g of H into G such that $f \circ g = id_H$; f is called a *retraction* and g a *co-retraction* of f.

Theorem 4.3. *The class of all* BFS-*constructible graphs is a variety, i.e., it is closed under retracts and products.*

Proof. (a) *Closure under retracts.*
Let G be a BFS-constructible graph, and H a retract of G. Without loss of generality we will suppose that H is a subgraph of G and that the retraction f of G onto H is such that its restriction to $V(H)$ is the identity. Let $\leq_H$ be a BFS-order on $V(H)$ with u as the smallest element, and let ϕ_H be the father function. We can easily construct a BFS-order $\leq_G$ on $V(G)$ with u as the smallest element such that, for each $v \in V(G)$, if $B(v) := N_G(v) - \bigcup_{w <_G v} N_G(w)$ if $v \neq u$ and $B(v) := N_G(v)$ if $v = u$, then for all $x, y \in B(v)$ with $x \neq y$:

1. $x \in V(H)$ and $y \in V(G - H)$ implies $x <_G y$;
2. $f(x) <_H f(y)$ implies $x <_G y$.

In particular 1 implies that $x \leq_G y$ if and only if $x \leq_H y$. For $x \in V(H)$ put $H_x := H[\{y \in V(H) : y \leq_H x\}]$ and $G_x := G[\{y \in V(G) : y \leq_G x\}]$. Clearly $H_x = H \cap G_x$ for every $x \in V(G)$. Since $\leq_G$ is a BFS-order, and G is BFS-constructible, there is a descending domination function Δ_G which is associated with $\leq_G$. Put $\Delta_H := f \circ \Delta_G$ and let $x \in V(H)$. Because f is a retraction and $\leq_G$ is descending, $d_H(u, \Delta_H(x)) \leq d_G(u, \Delta_G(x)) < d_G(u, x) = d_H(u, x)$. Then $\Delta_H(x) <_H x$ by the properties of a BFS-order. Hence $\Delta_H(x)$ dominates x in H_x because $\Delta_G(x)$ dominates x in G_x and f is a retraction. This proves that Δ_H is a descending domination function which is associated with $\leq_H$, and thus that H is BFS-constructible.

(b) *Closure under products.*
Let $(G_i)_{i \in I}$ be a family of graphs. We will show that $\boxtimes_{i \in I} G_i$ is BFS-constructible. Without loss of generality we can suppose that each G_i is connected. For each $i \in I$ denote by π_i be the i-th projection of $\boxtimes_{i \in I} G_i$ onto G_i. Let $u \in V(G)$, and let G be the component of $\boxtimes_{i \in I} G_i$ which contains u. Let $\leq$ be a BFS-order on $V(G)$ such that u is the smallest element. We have to show that it admits a descending domination function. Let ϕ be the

father function associated with $\leq$, and for each $x \in V(G)$ let $n(x)$ be the smallest non-negative integer such that $\phi^{n(x)}(x) = u$.

For each $i \in I$ put $u_i := \pi_i(u)$. Note that $d_G(x, y) = \max_{i \in I} d_{G_i}(\pi_i(x), \pi_i(y))$ for any two vertices x, y of G. For each vertex x of G let $I(x) := \{i \in I : d_{G_i}(u_i, \pi_i(x)) = d_G(u, x)\}$. If x and y are adjacent vertices of G with $d_G(u, x) = d_G(u, y) + 1$, then $I(y) \subseteq I(x)$ and $y < x$.

(b.1) Let $a \in V(G - u)$ and $i \in I(a)$. We will construct a BFS-order $\leq_i^a$ on $V(G_i)$ with u_i as the smallest element. We first construct a family $(A_{x_i})_{x_i \in V(G_i)}$ of subsets of $V(G_i)$ such that:

1. for every $x_i \in V(G_i)$, $x_i \in A_{x_i}$ and the set A_{x_i} is well-ordered by a relation $\leq_{x_i}$;
2. for every $S \subseteq V(G_i)$ the set $A_S := \bigcup_{x_i \in S} A_{x_i}$ is well-ordered by a relation $\leq_S$ such that the poset $(A_{x_i}, \leq_{x_i})$ is an initial segment of $(A_S, \leq_S)$ for each $x_i \in S$;
3. if $S \neq V(G_i)$, then $A_S - S$ is non-empty.

Let $\leq_{u_i}$ be the relation on $A_{u_i} := \{u_i\} \cup N_{G_i}(u_i)$ such that, for all distinct elements x_i, y_i of A_{u_i}, $x_i <_{u_i} y_i$ if $x_i = u_i$ or if $\min\{x \in V(G) : \pi_i(x) = x_i\} < \min\{y \in V(G) : \pi_i(y) = y_i\}$.

Let S be the set of the vertices x_i of G_i such that A_{x_i} and $\leq_{x_i}$ have already been constructed. Suppose $S \neq V(G)$. By 3, $A_S - S$ is non-empty. Let x_i be the smallest element of $(A_S - S, \leq_S)$. Put $A_{x_i} := A_S \cup N_{G_i}(x_i)$ and let $\preccurlyeq_{x_i}$ be the relation on $N_{G_i}(x_i) - A_S$ such that:

- if $x_i = \pi_i(\phi^k(a))$ for some k with $0 < k \leq n(a)$), then $y_i \prec_{x_i} z_i$ if $\min\{y \in N_G(\phi^k(a)) : \pi_i(y) = y_i\} < \min\{z \in N_G(\phi^k(a)) : \pi_i(z) = z_i\}$;
- if $x_i \neq \pi_i(\phi^k(a))$ for every $k \leq n(a)$, then $y_i \prec_{x_i} z_i$ if $\min\{y \in V(G) : \pi_i(y) = y_i\} < \min\{z \in V(G) : \pi_i(z) = z_i\}$.

Let $(A_{x_i}, \leq_{x_i})$ be the ordered sum of $(A_S, \leq_S)$ with $(N_{G_i}(x_i) - A_S, \prec_{x_i})$.

If $S = V(G)$, then we are done because the well-order $\leq_S$ and the family $(A_{x_i})_{x_i \in V(G_i)}$ clearly satisfy the condition of Definition 3.5.

Put $\leq_i^a := \leq_{V(G_i)}$. By the construction, if ϕ_i denotes the father function associated with $\leq_i^a$, then $\pi_i(\phi^k(a)) = \phi_i^k(\pi_i(a))$ for every k with $0 \leq k \leq n(a)$. It follows that, for every $x < a$, $\pi_i(x) \leq_i^a \pi_i(a)$ and if x and a are adjacent then $\pi_i(x)$ and $\pi_i(a)$ are adjacent too. Since G_i is BFS-constructible, there exists a descending domination function Δ_i^a which is associated with $\leq_i^a$.

(b.2) Now we construct a self-map Δ on $V(G)$ as follows: $\Delta(u) := u$ and, for every $x > u$, $\Delta(x)$ is such that

$$\pi_i(\Delta(x)) := \begin{cases} \pi_i(x) & \text{if } i \notin I(x), \\ \Delta_i^x(\pi_i(x)) & \text{if } i \in I(x). \end{cases}$$

Clearly $d_G(u, \Delta(x)) < d_G(u, x)$ since $d_G(u, \Delta(x)) = d_{G_i}(u_i, \Delta_i^x(\pi_i(x))) < d_{G_i}(u_i, \pi_i(x)) = d_G(u, x)$ for every $i \in I(x)$.

Let y be a neighbor of x with $y < x$. For every $i \subseteq I$, $\pi_i(y)$ and $\pi_i(x)$ either coincide or are adjacent. Moreover, if $i \in I(x)$, then $\pi_i(y)$ and $\Delta_i^x(\pi_i(x))$ either coincide or are adjacent since $\Delta_i^x(\pi_i(x))$ dominates $\pi_i(x)$ in $G_i[\{x_i \in V(G_i) : x_i \leq_i^x \pi_i(x)\}] = G_i[\{\pi_i(z) : z \leq x\}]$. Hence, for every $i \in I$, $\pi_i(y)$ and $\pi_i(\Delta(x))$ either coincide or are adjacent. This proves that y and $\Delta(x)$ either coincide or are adjacent, and thus, that $\Delta(x)$ dominates x in $G[\{y \in V(G) : y \leq x\}]$. Therefore Δ is a domination function which is associated with $\leq$, and which is descending because, as we saw, $d_G(u, \Delta(x)) < d_G(u, x)$ for every $x \neq u$. Consequently G, and thus $\boxtimes_{i \in I} G_i$, are BFS-constructible. ■

Due to the definition of the father function, the following result is substantially the equivalence (i) $\Leftrightarrow$ (ii) of [2, Theorem 4.3].

Theorem 4.4 ([2, Theorem 4.3]). *A graph G is bridged if and only if it is* BFS-*constructible and the father function of any* BFS-*order on $V(G)$ is an associated descending domination function.*

Note that if $\leq$ is a BFS-order on $V(G)$, ϕ its associated father function, and Δ an associated domination function, then clearly $\phi(x) \leq \Delta(x)$ for every $x \in V(G)$. Theorem 4.4 and this remark give to bridged graphs a so special place among BFS-constructible graphs that we are induced to make the following conjecture.

Conjecture 4.5. The variety of BFS-constructible graphs is generated by bridged graphs, i.e., every BFS-constructible graph is a retract of a product of bridged graphs.

As we will now see, the variety of BFS-constructible graphs contains another important subclass of the class of weakly modular graphs.

Definition 4.6. A *Helly graph* is a graph G for which any (finite or infinite) family of pairwise non-disjoint balls has a nonempty intersection.

Lemma 4.7 ([9])**.** *The class of Helly graphs is the variety which is generated by paths.*

Proposition 4.8. *Every Helly graph G is* BFS-*constructible.*

Proof. By Lemma 4.7 Helly graphs are the retracts of products of paths. Since paths are bridged graphs, and thus BFS-constructible graphs, and because the class of BFS-constructible graphs is a variety, it follows that Helly graphs are BFS-constructible. ∎

We will now recall a consequence of Theorems 4.4 and 3.4 and of [1, Corollary 2.6] that we will need to give the solution to Hahn, Sauer and Woodrow's problem about bridged graphs.

Proposition 4.9 ([2, Theorem 4.3])**.** *A connected graph is bridged if and only if it is constructible and has no induced cycles of length* 4 *or* 5*.*

Note that, since a finite graph is dismantlable if and only if it is constructible, this equivalence extends to infinite graphs the result of Anstee and Farber [1, Corollary 2.6].

Theorem 4.10 ([2, Theorem 5.1])**.** *The class of bridged graphs is dually compact closed.*

Proof. Let H be a finite subgraph of a bridged graph G. W.l.o.g. we can suppose that G is connected. Hence, by Proposition 4.9, G is constructible. Therefore, by Theorem 3.4, H is contained in a finite induced subgraph K of G which is constructible. Since G is bridged and K is an induced subgraph of G, K contains no induced cycles of length 4 or 5. Therefore K is bridged by Proposition 4.9. ∎

5. Finite Isometric Subgraphs of a Graph

As we will now see, there are other important subclasses of the class of weakly modular graphs which are dually compact closed and which even have a much more interesting property.

The (*geodesic*) *interval* $I_G(x,y)$ of two vertices x and y of a graph G is the set of vertices of all (x,y)-geodesics in G. We will say that a graph is *interval-finite* if all its intervals are finite. A set A of vertices of a graph G

is (*geodesically*) *convex* if it contains the interval $I_G(x,y)$ for all $x,y \in A$. The (*geodesic*) *convex hull* $co_G(A)$ of a set A of vertices of a graph G is the smallest convex set of G containing A.

A *median* of a triple $\{x,y,z\}$ of vertices of a graph G is any element of the intersection $I_G(x,y) \cap I_G(y,z) \cap I_G(z,x)$. A graph G is *modular* (resp. *median* if every triple $\{x,y,z\}$ of vertices of G admits at least (resp. exactly) one median.

A *pseudo-median* of a triple $\{x,y,z\}$ of vertices of G is a triple $\{x',y',z'\}$ of pairwise adjacent vertices such that $\{x',y'\} \subseteq I_G(x,y)$, $\{y',z'\} \subseteq I_G(y,z)$ and $\{z',x'\} \subseteq I_G(z,x)$. A graph G is *pseudo-modular* if every triple $\{x,y,z\}$ of vertices of G admits a median or a pseudo-median. If, for every triple of vertices, the median or the pseudo-median is unique, then G is said to be *pseudo-median*. A graph is then modular (resp. median) if and only it is pseudo-modular (resp. pseudo-median) and contains no triangle (i.e., K_3).

Proposition 5.1 ([11, Corollary 5.3])**.** *The convex hull of any finite set of vertices of an interval-finite pseudo-median graph is finite.*

Since every interval of a median graph is finite, this result generalizes a result of Tardif [18, Lemma 2.1.3(2)]. By [11, Proposition 5.4], particular instances of interval-finite pseudo-median graphs are the pseudo-median graphs which contain no infinite simplices. Because the subgraph of a (pseudo-) median graph induced by a convex set is clearly (pseudo-) median, we obtain immediately:

Theorem 5.2. *The class of median graphs and the one of interval-finite pseudo-median graphs are dually compact closed. More precisely every finite subgraph of an interval-finite pseudo-median graph G is contained in a finite subgraph of G whose vertex set is convex in $V(G)$ (and thus which is an isometric subgraph of G).*

We still do not know if the class of all pseudo-median graphs is dually compact closed. We will now consider Helly graphs. We will need several properties of Helly graphs to prove that the class of these graphs is dually compact closed. The first one is the *Extension Property.*

Lemma 5.3 ([16] and [8])**.** *A graph G is a Helly graph if and only if, for every graph H and every $A \subseteq V(H)$ and for each map f of A into $V(G)$ such that $d_G(f(x),f(y)) \leq d_H(x,y)$ for all $x,y \in A$, there exists a contraction of H into G which coincides with f on A* (Extension Property).

The second property that we need is the existence of an injective hull for any finite metric space over $\mathbb{N}$ and in particular for every finite graph. This result was obtained by Isbell [6], by Pouzet [14, 15] (see also [8, Theorem IV-1.2.3]) and also by Pesch [10].

Lemma 5.4 ([6], [14, 15] and [10]). *For every finite metric space* (E, d) *over* $\mathbb{N}$ *there exists, up to isomorphism, a unique finite minimal Helly graph* G (injective hull) *such that* (E, d) *is an isometric subspace of* $(V(G), d_G)$.

Lemma 5.5. *Every dominated vertex of the injective hull of a metric space* (E, d) *belongs to* E.

Proof. If x is a dominated vertex of a graph H, then $H - x$ is a retract of H. Hence $H - x$ is a Helly graph if H is a Helly graph. The result is then due to the minimality of the injective hull. ■

Lemma 5.6. *The endvertices of any path of maximal length in a Helly graph are dominated.*

Proof. Let x_0 and x_1 be the endvertices of a path of maximal length in a Helly graph G, and let $k := d_G(x_0, x_1)$. Let $i \in \{0, 1\}$. The balls $B_G(x_i, k-1)$, $B_G(x_{1-i}, k-1)$ and $B_G(u, 1)$ for all $u \in N_G(x_{1-i})$, are pairwise non-disjoint. Hence $B_G(x_i, k-1) \cap \bigcap_{u \in B_G(x_{1-i}, 1)} B_G(u, 1)$ is non-empty since G is a Helly graph. Therefore each element of this intersection dominates x_{1-i}. ■

Theorem 5.7. *The class of Helly graphs is dually compact closed. More precisely, every finite subgraph* H *of a Helly graph* G *is contained in a finite isometric Helly subgraph of* G.

Proof. By Lemma 5.4, the metric space $(V(H), d_G)$ has an injective hull, say K. Then H is an isometric subgraph of K. By Lemmas 5.5 and 5.6, any two vertices of K belong to a path joining two vertices of H. Therefore, by the extension property (Lemma 5.3) and the uniqueness of K, this graph K can be considered as a subgraph of G, and thus as an isometric subgraph of G. ■

We recall that Hahn, Laviolette, Sauer and Woodrow 1.1 proved that *every finite subgraph* H *of a bridged graph* G *of diameter two is contained in a finite induced subgraph* K *of* G *which is bridged and has diameter two.* Note that, in this result, the fact that the finite bridged subgraph K has diameter two implies that K is an isometric subgraph of G. This brings us to the question of whether K can always be chosen to be isometric. Clearly an isometric subgraph of a bridged graph is bridged, hence this problem is a natural enhancement of the one of Hahn, Sauer and Woodrow. However this seems to be very difficult. From this point of view, we will now show some refinements of Theorem 4.10, first by generalizing the result of Hahn et al. [3] by considering bridged graphs of radius 2.

Theorem 5.8 ([2, Theorem 5.6]). *Let G be a bridged graph of radius 2 such that $G[N_G(u)]$ contains no infinite simplices for some vertex u of G with $V(G) = B_G(u, 2)$. Then every finite subgraph H of G is contained in a finite isometric (and hence bridged) subgraph of G.*

The proof of this theorem, which is rather long and technical and will not be recalled here, does not seem to be easily extendable to bridged graphs of radius greater that 2. In order to give an outline of the proof of a much more interesting theorem we will need the following result.

Lemma 5.9 ([13, Theorem 3.8]). *Every bounded bridged graph without infinite simplices and containing only finitely many dominated vertices is finite.*

Theorem 5.10 ([13, Theorem 3.11]). *Let G be a bridged graph containing no infinite simplices. Then every finite subgraph of G is contained in a finite isometric subgraph of G.*

Outline of the proof. Without loss of generality we can assume that G is connected. Let X be a finite subset of $V(G)$, $x \in X$ and $r := \max_{y \in X} d_G(x, y)$. Since every ball of a bridged graph is convex, $co_G(X) \subseteq B_G(x, r)$ because $X \subseteq B_G(x, r)$. Therefore the subgraph $K := G[co_G(X)]$ is an isometric bounded subgraph of G which contains X. This implies in particular that K is bridged, since it is an isometric subgraph of a bridged graph. By [13, Lemma 6.5] and because K contains no infinite simplices, every interval of K is finite. Therefore we can prove that the set of isometric subgraphs of K containing X and ordered by the subgraph relation is inductive. Hence Zorn Lemma implies that there exists a minimal isometric subgraph H of K which contains X.

Suppose that H contains a dominated vertex x which does not belong to X. Then $H - x$ would be an isometric (hence bridged) subgraph of H

containing X, contrary to the minimality of H. Therefore every dominated vertex of H belongs to X. Hence H is a bounded bridged graph without infinite simplices which contains only finitely many dominated vertices. It follows that H is finite by Lemma 5.9. ∎

6. Open Problems

In addition to Conjecture 4.5 and on account of the results of last section, the following questions arise naturally.

Question 6.1. Is the class of weakly modular graphs dually compact closed, and if not so, which subclasses of the class of weakly modular graphs is dually compact closed?

Question 6.2. Is every finite subgraph of a bridged (resp. chordal) graph G contained in a finite isometric subgraph of G?

By Theorem 5.10, this problem is equivalent to:

Question 6.3. Is every finite subgraph of a bridged (resp. chordal) graph G contained in an isometric subgraph of G without infinite simplices?

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