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ON A SPECIAL CASE OF HADWIGER'S CONJECTURE

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Abstract

Hadwiger's Conjecture seems difficult to attack, even in the very special case of graphs G of independence number $\alpha(G) = 2$. We present some results in this special case.

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1. Introduction

Hadwiger's Conjecture is the major unsolved problem in graph coloring theory. Even for graphs of independence number $\alpha = 2$ a proof has proved elusive so far. This has led to speculation that the conjecture might be false, even in this special case.

The special case was first considered by Duchet and Meyniel [7], but it was W. Mader, who in a private communication a few years ago made clear to us how interesting the special case is.

Graphs G with independence number $\alpha(G) \leq 2$ may at first seem rather restricted, but noticing that this is equivalent to $\overline{G}$ being K_3-free and remembering the wide variety of K_3-free graphs, one may consider $\alpha \leq 2$ as a mild restriction only.

The purpose of this paper is to investigate how far one can go by standard methods in an attempt to solve the special case $\alpha = 2$.

By the determination of the order of magnitude of the Ramsey number $r(3, n)$ by Kim [11] there is a constant $c > 0$ such that there exist graphs G on n vertices, with $\alpha(G) = 2$ and clique number $\omega(G) \leq c\sqrt{n \log n}$. Again this indicates the non-triviality of Hadwiger's Conjecture for $\alpha = 2$: The chromatic number $\chi(G)$ is at least $n/2$ (since every color can be used at most twice), so although we want G to have at least a $K_{\lceil n/2 \rceil}$ as a minor, it may have only a complete graph of order $\sqrt{n \log n}$ as a subgraph.

For such a graph G to have $K_{\lceil n/2 \rceil}$ as a minor one needs to make at least $n/2 - c\sqrt{n \log n}$ contractions into single vertices of connected subgraphs on ≥ 2 vertices. Most of these connected subgraphs will be complete 2-graphs (because there are only n vertices altogether). That is, G will have a large matching with every two matching edges joined by at least one edge. We shall call such a matching *connected*. Thus the problem to find large connected matchings in graphs G with $\alpha(G) = 2$ is closely related to Hadwiger's Conjecture for $\alpha = 2$. The problem of finding a large connected matching in a general graph is NP-hard, as we shall see in Section 7.

Duchet and Meyniel [7] proved that a graph G always has $K_{\lceil n/(2\alpha(G)-1) \rceil}$ as a minor, i.e., a graph G with $\alpha(G) \leq 2$ has $K_{\lceil n/3 \rceil}$ as a minor. (This is easily proved by induction, contracting an induced path of length 2 when possible.) P. Seymour has asked if one can at least prove that there is a positive ϵ such that any graph G with $\alpha(G) \leq 2$ has $K_{\lceil (1/3+\epsilon)n \rceil}$ as a minor.

In the present paper we shall present a large number of properties possessed by a smallest counterexample to Hadwiger's Conjecture for $\alpha = 2$. Moreover, we shall prove that the conjecture is true for several infinite families of $\alpha = 2$ graphs and their inflations. An *inflation* of a graph is obtained by replacing its vertices by complete graphs. Note that inflations of $\alpha = 2$ graphs likewise have $\alpha = 2$.

Our results support the following extended conjecture:

EH. (Extended Hadwiger's Conjecture for $\alpha = 2$). Every graph G having $\alpha(G) = 2$ has a connected matching M such that the contractions of the edges in M to $|M|$ single vertices result in a graph containing a $K_{\lceil |V(G)|/2 \rceil}$.

2. Notation

Let G denote a finite simple undirected graph with vertex set $V(G)$ and edge set $E(G)$. We shall often denote $|V(G)|$ by n. A vertex set is *independent* if no two of its members are adjacent. The cardinality of any largest independent set in G is called the *independence number* of G and is denoted by $\alpha(G)$ or just α when graph G is understood. A graph G is said to be α-*critical* if for every edge $e \in E(G)$, $\alpha(G-e) > \alpha(G)$. An edge $e = xy$ in G is said to be a *dominating edge* if every vertex of G different from x and from y is adjacent to at least one of x and y. A *matching* in G is a set of edges no two of which share a vertex. A matching M in G is said to be *connected* if every pair of edges of M are joined by at least one edge. A matching M in G is said to be *dominating* if every vertex in $G - V(M)$ is adjacent to at least one endvertex of every edge of M. We shall write $x \sim y$ ($x \not\sim y$) when vertices x and y are (are not) adjacent. A graph H obtained from a graph G by deletions (of vertices and/or edges) and/or contractions (of edges) is a *minor* of G. We express this relation between the graphs G and H by $G \succeq H$ (or by $H \preceq G$). As usual, the chromatic number of G is denoted by $\chi(G)$, the vertex connectivity by $\kappa(G)$ and the minimum degree of G by $\delta(G)$. For $X \subseteq V(G)$, we denote by $G[X]$ the subgraph of G induced by X, further, $G - X = G[V(G) - X]$.

3. Hadwiger's Conjecture

H. Hadwiger presented his conjecture in a colloquium of the Eidgenössische Technische Hochschule in Zürich on December 15, 1942. The conjecture resulted from Hadwiger's suggestion that graph coloring should be studied in terms of a combinatorial classification of graphs, rather than in terms of classification based upon embeddings as was the more common approach of the time. The new classification used the maximum k for which G has K_k as a minor and the conjecture simply stated that the chromatic number $\chi(G)$ is at most this number k.

H1. Hadwiger's Conjecture [10]. $\forall G\colon G \succeq K_{\chi(G)}$.

Toft [18] gave a comprehensive survey of H1. We shall consider the conjecture in the special case where the independence number $\alpha(G)$ of G is ≤ 2:

H2. Hadwiger's Conjecture for $\alpha = 2$. $\forall G : \alpha(G) \leq 2 \Rightarrow G \succeq K_{\chi(G)}$.

Since $\alpha(G) \leq 2$ implies that $\chi(G) \geq |V(G)|/2$, it follows immediately that H2 $\Rightarrow$ H3, where H3 is the following conjecture:

H3. Hadwiger's Conjecture for $\alpha = 2$. $\forall G : \alpha(G) \leq 2 \Rightarrow G \succeq K_{\lceil |V(G)|/2 \rceil}$.

As we shall see below, H3 $\Rightarrow$ H2 also; hence we have given H2 and H3 the same name.

Let us suppose in the following that the graph G is a smallest possible counterexample for H2 in terms of the number of vertices (i.e., we assume that H2 is false and that G is a counterexample with a smallest possible $|V(G)|$). In the following we shall obtain properties (1), (2), …, (19) about such a graph G. Of course $\alpha(G) = 2$. Since each color class in a coloring has size at most 2, it follows that $|V(G)| \leq 2\chi(G)$.

If $|V(G)| = 2\chi(G)$ then for an arbitrary $x \in V(G)$ we have that $2\chi(G) - 1 = |V(G-x)| \leq 2\chi(G-x)$ hence $\chi(G) - \frac{1}{2} \leq \chi(G-x) \leq \chi(G)$, i.e., $\chi(G-x) = \chi(G)$. But $G \succeq G-x$ and $G-x \succeq K_{\chi(G-x)} = K_{\chi(G)}$ (by the minimality of G). Hence $G \succeq K_{\chi(G)}$, contradicting the fact that G is a counterexample to H2. Therefore $|V(G)| \leq 2\chi(G) - 1$.

If $\chi(G-x) = \chi(G)$ for a vertex $x \in V(G)$ then by the minimality of G we get a contradiction as above. So

(1) $\forall x \in V(G)\colon \chi(G-x) < \chi(G)$, i.e., G is *vertex-critical*.

Moreover

(2) The complement $\overline{G}$ of G is connected.

Assume, to the contrary, that the complement of G is disconnected. This means that G consists of two disjoint graphs G_1 and G_2 completely joined by edges. Moreover, by the minimality of G, $G_1 \succeq K_{\chi(G_1)}$ and $G_2 \succeq K_{\chi(G_2)}$, hence $G \succeq K_{\chi(G_1)+\chi(G_2)} = K_{\chi(G)}$, contradicting that G is a counterexample to H2. This proves (2).

Assume now that $|V(G)| \le 2\chi(G) - 2$. By a deep theorem of Gallai [8], $\overline{G}$ is disconnected, contradicting (2) above. Hence

(3) $|V(G)| = 2\chi(G) - 1$.

From $G \not\succeq K_{\chi(G)}$ and (3) above, it follows that $G \not\succeq K_{(|V(G)|+1)/2} = K_{\lceil |V(G)|/2 \rceil}$. Hence G is a counterexample also to H3. This proves that H3 $\Rightarrow$ H2. Hence

Theorem 3.1. *The conjectures* H2 *and* H3 *are equivalent. More precisely: any counterexample to* H3 *is a counterexample to* H2*, and any smallest* (*in terms of* $|V(G)|$) *counterexample to* H2 *is a counterexample to* H3.

Let us now assume that G_1 is a counterexample to H2 which has a smallest chromatic number. Then $\chi(G_1) \le \chi(G)$ (by the minimality of G_1) and $|V(G_1)| \ge |V(G)|$ (by the minimality of G). Then from $\alpha(G_1) = 2$, we get $2\chi(G) - 1 = |V(G)| \le |V(G_1)| \le 2\chi(G_1) \le 2\chi(G)$. The only option is that $\chi(G_1) = \chi(G)$. This proves:

Theorem 3.2. *If G is a smallest counterexample to* H2 *in terms of* $|V(G)|$*, then G is a smallest counterexample to* H2 *in terms of* $\chi(G)$*.*

Theorem 3.2 is interesting since it is not known if the statement holds with H2 replaced by H1. This is a problem due to A.A. Zykov (see Toft [18]).

A further property of G can be derived from $\alpha(G - x) \le \alpha(G) = 2$ and $2\chi(G - x) = 2\chi(G) - 2 = |V(G)| - 1 = |V(G - x)|$, namely that $G - x$ has a $(\chi(G) - 1)$-coloring in which each color class has size exactly 2. That means

(4) $\forall x \in V(G)$: $\overline{G} - x$ has a perfect matching, i.e., the complement $\overline{G}$ of G is *factor-critical*.

Choose $xy \in E(G)$ and let H denote the graph obtained from G by contracting xy to a new vertex z. Then H has one vertex less than G. Hence $G \succeq H \succeq K_{\chi(H)}$ by the minimality of G. Therefore $\chi(H) < \chi(G)$. A $(\chi(G) - 1)$-coloring of H gives immediately a $(\chi(G) - 1)$-coloring of $G - xy$

by giving x and y the color of z and retaining all other colors. But in a $(\chi(G)-1)$-coloring of the $2\chi(G)-1$ vertices of $G-xy$ at least one color class must have size ≥ 3, and hence:

(5) $\forall xy \in E(G){:}\, \alpha(G-xy) = 3 > 2 = \alpha(G)$, i.e., G is α-critical.

Finally, let H be a proper minor of G; i.e., $G \succeq H$ and $G \neq H$. If H has fewer vertices than G, then $H \succeq K_{\chi(H)}$, so $\chi(H) < \chi(G)$. If $|V(H)| = |V(G)|$ then $H \subseteq G - xy$ for some edge xy of G since H is a proper minor, so $\chi(H) \leq \chi(G-xy) < \chi(G)$. In any case $\chi(H) < \chi(G)$. So

(6) $\forall H,\ G \succeq H$ and $G \neq H{:}\, \chi(H) < \chi(G)$, i.e., G is *contraction-critical*.

We still assume that G is a smallest counterexample to H2 (and hence to H3 by Theorem 1). By (3) and (6), G is non-complete contraction-critical. A large number of properties follow from this, as listed in the paper by Toft [18]:

(7) $\chi(G) \geq 7$ (Robertson, Seymour and Thomas [16]).

(8) G is 7-vertex-connected (Dirac [6] and Mader [14]).

(9) $\delta(G) \geq \chi(G)$ (Dirac [6]) and G is $\chi(G)$-edge-connected (Toft [17]).

The proof of result (7) above depends on the truth of the Four Color Theorem. However, for the special case $\alpha = 2$, this can of course be established in a much more elementary way. By (3) above, the cases when $\chi \leq 6$ deal with graphs having at most 11 vertices and as we shall see, these are easy to handle.

From (3) and (9) and a well known theorem of Dirac [5], we get (10), which in turn, together with (3), implies (11).

(10) G is Hamiltonian.

(11) G is factor-critical.

Concerning matchings, the following property is easily proved.

(12) G does not contain a non-empty connected dominating matching.

Proof. Suppose $M \neq \emptyset$ is a connected dominating matching in G. Then $G' = G - V(M)$ is smaller than G and hence $G' \succeq K_{\lceil n'/2 \rceil} = K_{\lceil n/2 \rceil - |M|}$. Contracting the edges of M into $|M|$ single vertices, we thus obtain a $K_{\lceil n/2 \rceil}$

as a minor of G, since M is connected and dominating. But G is a counterexample to H2, so this is a contradiction. ■

Let H be an arbitrary graph. By a 2-*path* of H we mean an induced subpath of length 2 in H. Clearly, H does not contain any 2-path if and only if every component of H is a complete graph. If H has a 2-path P and $\alpha(H) = 2$, then contracting P to one vertex results in that vertex being joined to all other vertices. This simple observation leads to the following result.

Theorem 3.3. *Let G be any graph with $\alpha(G) \leq 2$. Let $n = |V(G)|$ and $\omega = \omega(G)$. Then the following statements hold.*

(a) $G \succeq K_{\lceil(\omega+n)/3\rceil}$.

(b) *If $n \geq 2k-1$ and $\omega \geq k-2$, then $G \succeq K_k$.*

Proof. If $\alpha(G) = 1$, part (a) is trivial. So we assume $\alpha(G) = 2$. We proceed to prove statement (a) by induction on n. Let $K = K_\omega$ be a maximum complete subgraph of G and let $H = G - V(K)$. If H contains a 2-path P, then the induction hypothesis implies that $G' = G - V(P)$ has $K_{\lceil(\omega+(n-3))/3\rceil} = K_{\lceil(\omega+n)/3\rceil-1}$ as a minor. (Note that $K \subseteq G'$ and hence $\omega(G') = \omega(G) = \omega$.) Since $\alpha(G) = 2$, this gives $G \succeq K_{\lceil(\omega(G')+n)/3\rceil} = K_{\lceil(\omega(G)+n)/3\rceil}$. If H does not contain any 2-path, then H is either a complete graph or the disjoint union of two complete graphs. In both cases we claim that $\omega \geq \lceil n/2\rceil$. In the first case, this is evident. In the second case, H is the disjoint union of two complete graphs, say H_1 and H_2 and, because of $\alpha(G) = 2$, every vertex of the complete subgraph K is either joined to all vertices of H_1 or to all vertices of H_2. This implies the claim. Consequently, $G \succeq K_\omega \succeq K_{\lceil(\omega+n)/3\rceil}$. Thus statement (a) is proved.

For the proof of (b) suppose, on the contrary, that $G \not\succeq K_k$. Then $\alpha(G) = 2$, and from (a) it follows that $n = 2k-1$ and $\omega = k-2$. Since the non-neighbors of any vertex in G induce a complete graph, this implies that $\delta(G) \geq k$. Then, because the Ramsey number $r(3,3) = 6$, we conclude that $k \geq 6$. Note that in case $k = 5$ we have $n = 9$ and $\delta(G) \geq 5$ implying that one vertex of G has degree at least 6 and thus $\omega \geq 4$.

Now, consider an arbitrary maximum complete subgraph $K = K_{k-2}$ of G and let $H = G - V(K)$. If H does not contain any 2-path, then H is either a complete graph or the disjoint union of two complete graphs and, as in the proof of (a), we conclude that $\omega \geq \lceil n/2\rceil = k$, a contradiction. If H has two vertex disjoint 2-paths, then contraction of both these 2-paths results in a K_k, a contradiction, too. Consequently, H has one, but not two vertex disjoint, 2-paths. Moreover, if P is any 2-path of H, then $H-V(P) = G-V(K)-V(P)$ is either a complete graph or the disjoint union of two complete graphs.

Next, consider an arbitrary 2-path $P = xzy$ of $G-V(K)$. We claim that $G-V(P)$ contains two vertex disjoint maximum complete subgraphs, say K' and K''. This is evident if $G-V(K)-V(P)$ is a complete graph. Otherwise, $G-V(K)-V(P)$ is the disjoint union of two complete graphs, say H_1 and H_2, and, because of $\alpha(G) = 2$, every vertex of K is joined to all vertices of either H_1 or H_2. Since $|V(K)|+|V(H_1)|+|V(H_2)| = 2k-4$ and $\omega = k-2$, this also implies the claim. Thus the claim is proved.

Since $\omega = k-2$, vertex x has a non-neighbor x' in K' and y has a non-neighbor y' in K'. Moreover, $\alpha(G) = 2$ implies that $x' \neq y'$ and that $x'y$ and xy' are edges of G (see Figure 3.1).

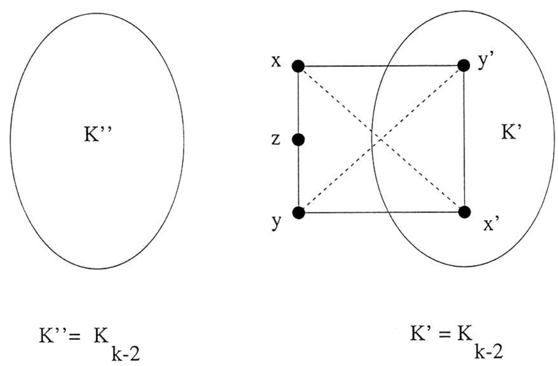


Figure 3.1

Since $x'y'x$ and $y'x'y$ are 2-paths and there are not two disjoint 2-paths in $G-V(K'')$, the vertices x, z and y are each either joined to all vertices of $H = K'-x'-y'$ or to none. Since $\alpha(G) = 2$, we may assume that x is joined to all vertices of H. If y is also joined to all vertices of H, then any two vertices x'' and y'' of H produce two disjoint 2-paths $x'x''x$ and $y'y''y$ of $G-V(K'')$, a contradiction. Note that $|V(H)| = |V(K')|-2 = k-4 \geq 2$. If, on the other hand, vertex y is not joined to any vertex of H, then the degree of y in G is at most $k-3+2 = k-1$, contrary to $\delta(G) \geq k$. Note that y is not joined to all vertices of $K'' = K_{k-2}$, since $\omega = k-2$.

Therefore, we have obtained a contradiction in all cases, and hence statement (b) is proved. ∎

For the minimum counterexample G for H2, we then conclude from Theorem 3.3 and property (3) that

(13) $\omega(G) \leq \chi(G) - 3.$

Since the non-neighbors of any vertex in G induce a complete graph, we conclude from properties (13) and (3) that

(14) $\delta(G) \geq \chi(G) + 1.$

Theorem 3.4. *Let G be any connected graph with $\alpha(G) = 2$ and $\kappa(G) \leq |V(G)|/2$.* Then G contains a non-empty connected dominating matching.

Proof. Let S be a minimum vertex cut in G and suppose that the (necessarily exactly) two components of $G - S$ are A and B respectively. Then both induced subgraphs $G[A]$ and $G[B]$ are complete. Moreover, each vertex in S is either joined to all vertices in A or to all vertices in B.

Let $S_A = \{v \in S | v \sim \text{ every vertex in } A\}$ and $S_B = \{v \in S | v \sim \text{ every vertex in } B\}$. Let $s_A = |S_A|, s_B = |S_B|, s = |S| = \kappa(G), a = |A|$ and $b = |B|$.

Case 1. Suppose $s_A \leq b$ and $s_B \leq a$. Let M_A be a complete matching of S_B into A and let M_B be a complete matching of S_A into B. (Both must exist by Hall's Theorem or Menger's Theorem.) Then $M = M_A \cup M_B$ contains a connected dominating matching in G.

Case 2. Now suppose $s_A > b$ and assume that $a \geq s - b$. Then let M_B be a complete matching of B into S_A. (Again such must exist by Hall's Theorem or Menger's Theorem.) Now let $M_B(B)$ denote the set of all vertices in S_A matched by M_B. Then $S - M_B(B)$ has size $s - b \leq a$. So there exists a complete matching M' of $S - M_B(B)$ into A. Finally, let $M = M_B \cup M'$. Then M is a connected dominating matching in G.

So there remains to consider only the case when $s_A > b$ and $a < s - b$. But then $s > a + b$ and $s + (a + b) = |V(G)|$. Thus $s = \kappa(G) > |V(G)|/2$, a contradiction. ■

The next property then follows from Theorem 3.4 and properties (3) and (12).

(15) $\kappa(G) \geq \chi(G).$

Now let x and y be two arbitrary non-adjacent vertices of G. Let B be the set of common neighbors of x and y, let A be the remaining (private) neighbors of x and let C be the remaining (private) neighbors of y. Then, since $\alpha(G) = 2$, $G[A]$ and $G[C]$ are both complete. Moreover, $B \neq \emptyset$ since G does not contain a $K_{\lceil |V(G)|/2 \rceil}$.

(16) Let b be any member of B. Then b has at least one non-neighbor in A and at least one non-neighbor in C. (Hence, in particular, both A and C are non-empty.)

Proof. By (5), the edge xb is critical, i.e., $\alpha(G - xb) = 3$. The common non-neighbor c of x and b must lie in C. Similarly, edge yb is critical and the common non-neighbor a of y and b must lie in A. ■

(17) Let $a \in A$ and $c \in C$. Then a is adjacent to c if and only if there is a common non-neighbor $b \in B$ of a and c.

Proof. If a and c have a common non-neighbor b, they are joined by an edge, since $\alpha(G) = 2$. Conversely, if a and c are joined by an edge ac, then $\alpha(G - ac) = 3$, implying a common non-neighbor b of a and c. The vertex b must belong to B, since $\alpha(G) = 2$. ■

(18) If sets A, B and C are as above, then $2 \leq |A| \leq k - 4$, $2 \leq |C| \leq k - 4$ and $5 \leq |B| \leq 2k - 7$, where $k = \chi(G)$.

Proof. Since x is joined to all vertices of A, (13) implies that $|A| \leq k - 4$. Similarly, $|C| \leq k - 4$. But then $|B| = 2k - 3 - |A| - |C| \geq 5$.

We know A and C are each non-empty, so suppose $|A| = 1$. Then by (16) there are no edges between A and B. Hence $G[B]$ is complete. Then either B or C induces a complete graph of size at least $k - 2$, contradicting (13). ■

(19) Any two non-adjacent vertices x and y in G are joined by at least five 2-paths. Moreover, every 2-path in G is part of an induced C_5.

Proof. The first part follows immediately from the fact that $|B| \geq 5$ and the second part follows from (16) and (17) above. ■

4. Inflations

Given a graph G, we say that graph $H = \inf(G)$ is an *inflation* of G if each vertex v of G is replaced by a complete graph K^v (or the empty set) and if vertices u and v of G are adjacent, then in H every vertex of K^u is joined to every vertex of K^v. We call the complete graph K^u which replaces vertex u an *atom* of the inflation. Clearly, inflation preserves the property of having $\alpha \le 2$. Hence, H3 implies that any inflation H obtained from a graph G with $\alpha(G) \le 2$ would satisfy $H \succeq K_{\lceil |V(H)|/2 \rceil}$.

We have been unable to prove that every inflation H obtained from a graph G with $\alpha(G) = 2$ satisfies $H \succeq K_{|V(H)|/2}$, even when G itself satisfies H3. However, in Section 6 we shall prove

Theorem 4.1. *In any inflation on n vertices of a graph G with $\alpha(G) \le 2$ and $|V(G)| \le 11$, there exists a dominating connected matching M such that by contracting the edges of M, one obtains a graph containing a $K_{\lceil n/2 \rceil}$.*

We shall also prove Hadwiger's Conjecture for inflations of the following infinite family. For $k \ge 1$, we define a family of graphs C_{3k-1}^{k-1} as follows. Let $C_2^0 = \overline{K_2}$. Now suppose $k \ge 2$. Arrange $3k-1$ vertices in a cycle. Now for each k successive vertices on this cycle, join every pair.

Theorem 4.2. *In each inflation G of C_{3k-1}^{k-1} having $k \ge 1$ and n vertices, there exists a connected dominating matching M such that by contracting the edges of M, one obtains a graph containing a $K_{\lceil n/2 \rceil}$.*

Proof. The proof is by induction on k. For $k = 1$, graph G consists of two disjoint complete graphs and hence $G \supseteq K_{\lceil n/2 \rceil}$ and $M = \emptyset$ suffices. Next suppose $k = 2$. Choose a smallest atom in C_5^1 and label it B_1. Now let the remaining four atoms be labelled B_2, B_3, B_4 and B_5 either clockwise or counterclockwise in such a way that $|B_4| \le |B_3|$. Now let M_1 be a matching of B_1 into B_2 which covers all of B_1 and let M_2 be a matching of B_4 into B_3 which covers all of B_4. Then $M_0 = M_1 \cup M_2$ is a connected dominating matching. The unmatched vertices form two complete graphs, namely B_5 and an induced complete subgraph of $B_2 \cup B_3$. Take the larger of these two subgraphs and call it B_0. Then contracting all the edges of matching M_0 we obtain a minor (containing B_0) which has at least $n/2$ vertices and is complete. So we are done when $k = 2$.

So suppose the theorem is true for all $k' < k$ and consider an inflation of C^{k-1}_{3k-1}. Choose the smallest atom and denote it by B_1. Number the rest of the atoms (clockwise or counterclockwise) by $B_2, \ldots, B_{3k-1}$ so that $|V(B_{2k})| \leq |V(B_{k+1})|$.

Now let M_1 be a matching of all of B_1 into B_k and let M_2 be a matching of all of B_{2k} into B_{k+1}. Observe that $M_0 = M_1 \cup M_2$ is then a connected dominating matching. Next let D_k denote the complete graph spanned by all unmatched vertices of $B_k \cup B_{k+1}$. (See Figure 4.1.)

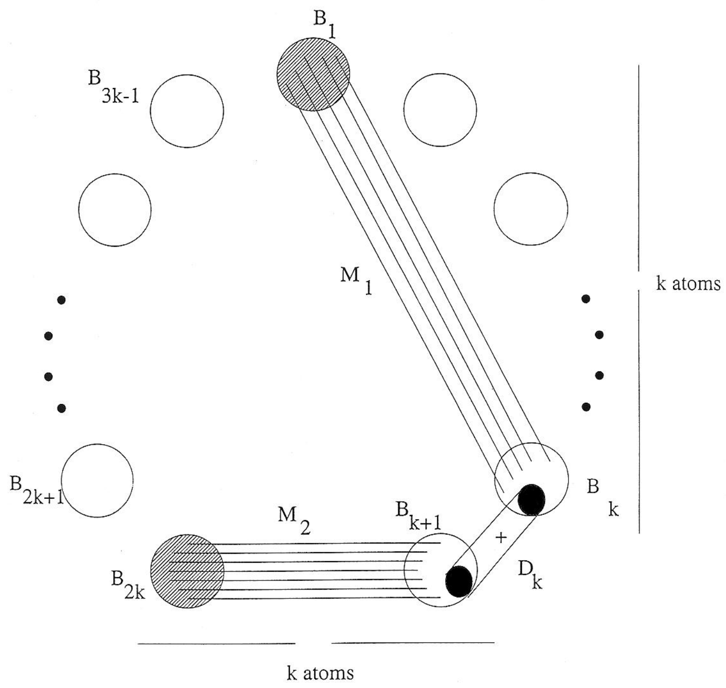


Figure 4.1

Then replacing $B_k \cup B_{k+1}$ with D_k and deleting B_1 and B_{2k}, we obtain an inflation of $C^{k-2}_{3(k-1)-1}$. But by induction hypothesis, this graph contains a connected dominating matching M' which contracts to a complete graph on half its number of vertices. So then $M \cup M'$ is a connected dominating matching in the inflation of C^{k-1}_{3k-1} and hence upon contraction this matching yields a complete graph on half the number of vertices of the inflation and the theorem is proved. ∎

5. Induced Subgraphs

Theorem 5.1. *Let G be a graph with n vertices and with $\alpha(G) \leq 2$. If G does not contain an induced C_5, then either G contains $K_{\lceil n/2 \rceil}$ or else G contains a dominating edge.*

Proof. Assume that G does not contain a dominating edge. Then G is not complete, and there are two vertices $x, y \in V(G)$ such that $x \not\sim y$. Denote by A the set of neighbors of x which are not neighbors of y, by C the set of neighbors of y which are not neighbors of x and by B the set of vertices adjacent to both x and y.

Since $\alpha(G) \leq 2$, both induced subgraphs $G[A \cup \{x\}]$ and $G[C \cup \{y\}]$ are complete. Therefore, if $B = \emptyset$, then G contains a $K_{\lceil n/2 \rceil}$ and we are done. Otherwise, consider an arbitrary vertex $b \in B$. Since G does not contain a dominating edge, there is a common non-neighbor c of b and x as well as a common non-neighbor a of b and y. Clearly, $a \in A$ and $c \in C$ and, because $\alpha(G) \leq 2$, ac is an edge of G. But then $axbyc$ is an induced C_5 in G, contrary to hypothesis. ∎

Corollary 5.2. *Let G be a graph with n vertices and with $\alpha(G) \leq 2$. If G does not contain an induced C_5, then $G \succeq K_{\lceil n/2 \rceil}$.* ∎

Figure 5.1 displays the six graphs $G_4^1, G_4^2, G_4^3, G_4^4, G_4^5, G_4^6$ as well as graph H_7, which are used in the next theorem and the next corollary.

Theorem 5.3. *Let G be a graph with n vertices and with $\alpha(G) \leq 2$. If G does not contain an induced H_7, then $G \succeq K_{\lceil n/2 \rceil}$.*

Proof. We use induction on n. For $n \leq 4$, the result is clear since $\alpha(G) \leq 2$.

So suppose $n \geq 5$ and suppose the result is true for graphs with fewer than n vertices and let G be a graph with n vertices. If G does not contain an induced C_5, we are done by Corollary 5.2. So suppose G contains an induced $C_5 = abcdea$. If $M = \{ab, cd\}$ is a connected dominating matching in G, then since $G-a-b-c-d \succeq K_{\lceil (n-4)/2 \rceil}$ by the induction hypothesis, we are done. Otherwise, since M is a connected matching, one of the edges, say ab, is not dominating and, therefore, there is a common non-neighbor z of a and b in G. But then z is adjacent to each of c, d, and e. If $M' = \{ae, bc\}$ is a connected dominating matching, then again we are done. Otherwise, since M' is a connected matching, one of the two edges, say ae, is not dominating and therefore there is a common non-neighbor z' of a and e. But then z' is adjacent to all of b, c, d and z, and $G[\{a, b, c, d, e, z, z'\}] = H_7$ is an induced subgraph of G, contrary to the hypothesis. ■

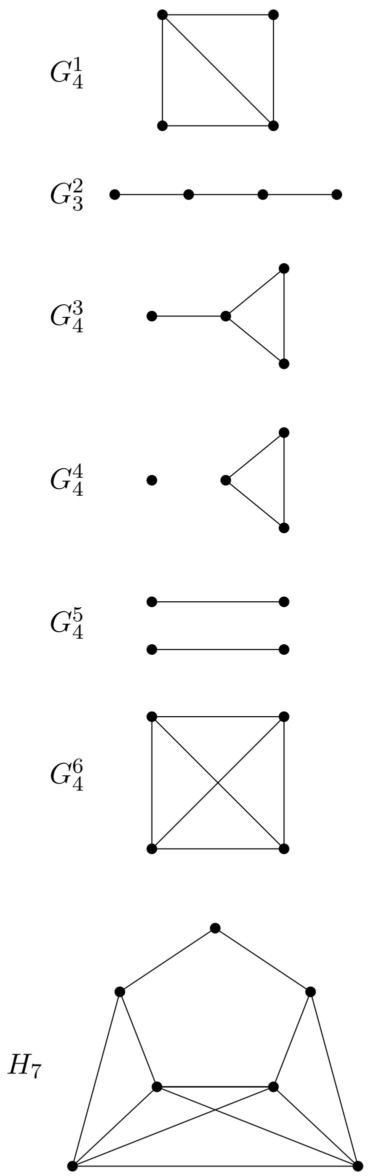


Figure 5.1

Corollary 5.4. *Let G be a graph with n vertices and with $\alpha(G) \leq 2$. Moreover, suppose graph $H \in \{G_4^1, G_4^2, G_4^3, G_4^4, G_4^5, G_4^6\}$. If G does not contain an induced H, then $G \succeq K_{\lceil n/2 \rceil}$.* ■

The list of forbidden induced subgraphs in Corollary 5.4 contains all 4-vertex graphs with $\alpha \leq 2$, except C_4. However C_4 may be added to this list as we shall now show.

Theorem 5.5. *Let G be a graph with n vertices and with $\alpha(G) \leq 2$. If G does not contain an induced C_4, then $G \succeq K_{\lceil n/2 \rceil}$.*

Proof. We use induction on n. For $n \leq 4$, the statement is evident since $\alpha(G) \leq 2$.

So suppose $n \geq 5$, suppose the result is true for graphs with fewer than n vertices and let G be a graph with n vertices. If G does not contain an induced C_5, we are done by Corollary 5.2. So suppose G contains an induced C_5. Let H denote a largest induced inflated C_5 in G with non-empty atoms. Let the five atoms of the inflated C_5 be denoted by $B_1, \ldots, B_5$ in clockwise order.

First, we claim that every vertex outside H is adjacent to all vertices of H. For the proof, suppose on the contrary that there is a vertex $z \in V(G) - V(H)$ such that $z \not\sim b$ for some vertex b of H, say $b \in B_1$. Then $\alpha(G) \leq 2$ implies that z is adjacent to all vertices of $B_3 \cup B_4$. If z has a neighbor $b_2 \in B_2$ as well as a neighbor $b_5 \in B_5$, then (b, b_2, z, b_5) is an induced C_4 in G, contrary to hypothesis. Therefore, by symmetry, we may assume that z has no neighbor in B_2. Then $\alpha(G) \leq 2$ implies that z is adjacent to all vertices of B_5 and, hence, to all vertices of $B_3 \cup B_4 \cup B_5$. But then z must be adjacent to some vertex $b_1 \in B_1$, since otherwise H would not be a largest induced inflated C_5 in G. Thus, for every vertex $b_2 \in B_2$ and every vertex $b_3 \in B_3$, we obtain an induced $C_4 = (b_2, b_3, z, b_1)$, contrary to hypothesis. This proves the claim.

Now we construct a matching M in H as follows. Let B_1 be a smallest atom of H and let B_3 and B_4 be the two opposite atoms. By symmetry we may assume that $|V(B_3)| \leq |V(B_4)|$. Let M_1 be a matching of all of B_3 into B_4 and let M_2 be a matching of all of B_1 into B_5. Then $M = M_1 \cup M_2$ is a non-empty connected dominating matching in H. But then, since every z not in H is adjacent to all vertices in H, matching M dominates all such z's. Now let $H' = G - V(M)$. Now $\alpha(H') \leq 2$, so by induction hypothesis, $H' \succeq K_{\lceil n'/2 \rceil}$, where $n' = |V(H')|$. Now contract all edges of matching M

and we have a complete graph with $\geq |V(H)|/2+|M|/2 = |V(G)|/2$ vertices as desired. ■

6. α-Criticality

To prove Hadwiger's Conjecture for $\alpha \leq 2$, clearly it is sufficient to do so for those graphs with $\alpha \leq 2$ which are α-critical.

Theorem 6.1. *Suppose G is a connected graph with $\alpha(G) \leq 2$ and suppose that G is α-critical. Then if x and y are any two vertices in G, $d(x,y) \leq 2$. Moreover, every pair of adjacent edges in G lie either in a K_3 or a chordless C_5.*

Proof. Suppose x and y are any two non-adjacent vertices in G. Clearly, since $\alpha(G) \leq 2$, we have $d(x,y) \leq 3$ and moreover, $N(x) \cup N(y) = V(G)$. So suppose $d(x,y) = 3$ and let P be a path of length 3 joining x and y. Then $N(x) \cap N(y) = \emptyset$ and each of $N(x)$ and $N(y)$ is a complete graph. Let e be the edge of P joining $N(x)$ and $N(y)$. Then $\alpha(G-e) = 2$, a contradiction.

Let e and f be adjacent edges in G. In any α-critical graph, every pair of adjacent edges share a chordless odd cycle. (See [1, 13].) If this cycle were a C_7 or larger, we would have $\alpha(G) \geq 3$, a contradiction. ■

For an α-critical graph G and two non-adjacent vertices x and y of G, we cannot produce any restriction on the structure of the subgraph induced by the common neighborhood of x and y. The next theorem explains why. More particularly, we show that *any* graph with $\alpha \leq 2$ can be embedded in a graph having $\alpha = 2$ which is α-critical.

Theorem 6.2. *Let H be any graph with $\alpha(H) \leq 2$. Then there exists an α-critical graph G with $\alpha(G) = 2$ which contains H as an induced subgraph. Furthermore, H can be embedded in G so that there exist two non-adjacent vertices x and y in G such that $N_G(x) \cap N_G(y) = H$.*

Proof. Let $V(H) = \{h_1, \ldots, h_r\}$ and define two new sets of vertices $A' = \{a_1, \ldots, a_r\}$ and $C' = \{c_1, \ldots, c_r\}$. Join vertices a_i and h_j if and only if $i \neq j$ and join vertices c_i and h_j if and only if $i \neq j$.

Now for each edge $e_{ij} = h_i h_j$ in H, insert a new vertex v_{ij} into either A' or C' thus obtaining vertex sets $A \supseteq A'$ and $C \supseteq C'$. Then join v_{ij} to all h_k, $k \neq i$ and $k \neq j$. (Note that since each v_{ij} may be put into either A or C, the graph G under construction is by no means unique.)

Let x and y be two new non-adjacent vertices. Join both x and y to all vertices of H, join x to all vertices of A and join y to all vertices of C.

Now join all vertices of A to each other and all vertices of C to each other. Finally, suppose $a \in A$ and $c \in C$. Join a to c if and only if there exists an h_i such that $a \not\sim h_i \not\sim c$.

It is then routine to check that $\alpha(G) = 2$ and that all edges of G are critical. ■

The following theorem is stated in [2, 3] in complementary form. The equivalence of (i) and (ii) is due to Brandt and the equivalence of (ii) and (iii) is due to Pach [15].

Theorem 6.3. *The following three statements are equivalent for any graph G:*

(i) *Graph G is an inflation of the graph C_{3k-1}^{k-1} (where empty atoms are not allowed), for $k \geq 1$.*

(ii) *$\alpha(G) = 2$, G is α-critical and G does not contain the triangular prism as an induced subgraph.*

(iii) *$\alpha(G) = 2$, G is α-critical and any complete subgraph of G is in the non-neighborhood of some vertex of G.* ■

We have shown in Theorem 4.2 that Hadwiger's Conjecture is true for all graphs which satisfy Theorem 6.3. So now let us assume that $\alpha(G) = 2$, that G *does* contain the triangular prism as an induced subgraph and that G is α-critical. Let P_6 denote the triangular prism subgraph of G and let e_1, e_2 and e_3 be the three edges of P_6 which do not lie in the two triangles of P_6. Since edge e_1 is critical, there must be a seventh vertex v_1 such that v_1 is not adjacent to either endvertex of e_1. Similarly there must be vertices v_2 and v_3 relative to e_2 and e_3 respectively. But then since $\alpha(G) = 2$, for each $i = 1, 2, 3$, v_i must be adjacent to the other four vertices of P_6 different from the two endvertices of edge e_i. In particular, all three v_i must be distinct and hence $|V(G)| \geq 9$. In particular, this implies that Hadwiger's Conjecture is true for all graphs G having $\alpha(G) \leq 2$ and $|V(G)| \leq 8$, as well as their inflations, since any such graph contains a spanning α-critical subgraph with $\alpha \leq 2$ satisfying the conditions of Theorem 6.3. Moreover, we have shown that the $K_{\lceil n/2 \rceil}$ minor can be obtained by contracting the edges of a connected dominating matching.

But now the set $\{v_1, v_2, v_3\}$ cannot be independent, so there exists at least one edge among the three v_i's. Thus we may suppose we have as an induced subgraph of G one of the three graphs designated as Γ_1, Γ_2 and Γ_3

shown in Figure 6.1 where the vertices v_1, v_2 and v_3 have been relabeled v_5, v_7 and v_9 respectively.

We label by Cases 1, 2 and 3 the situations when G contains graphs Γ_1, Γ_2 and Γ_3, respectively, as induced subgraphs. Note that these three

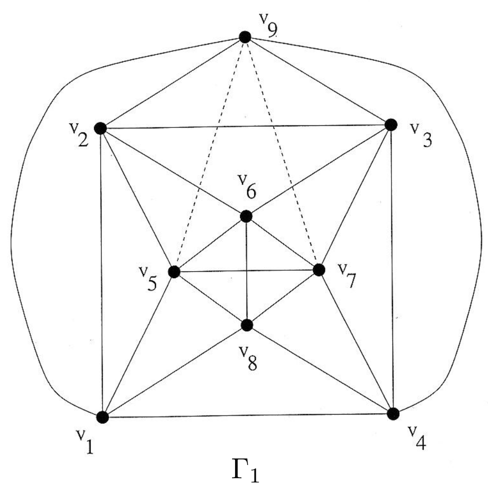


Figure 6.1(a)

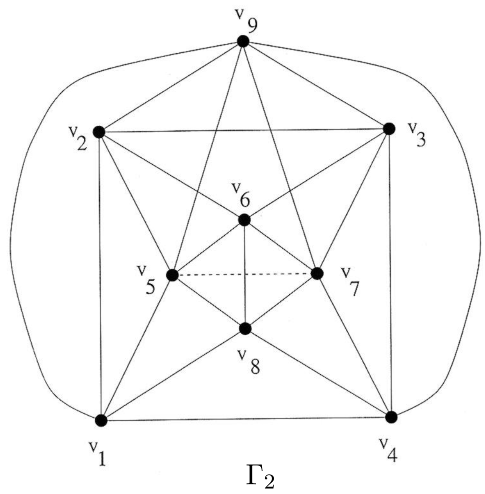
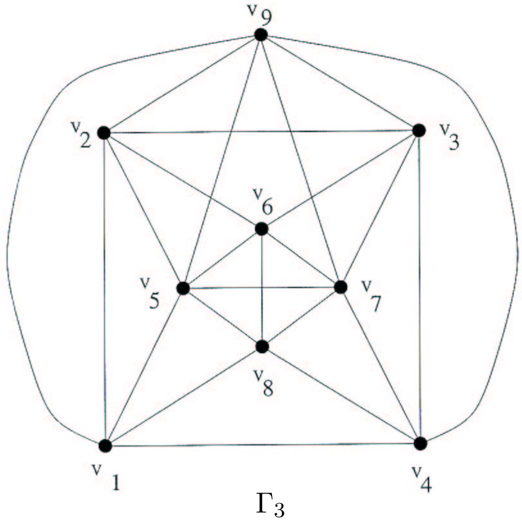


Figure 6.1(b)



Γ_3

Figure 6.1(c)

cases are *not* mutually exclusive! We shall use these three cases to prove Theorem 4.1; in particular, we demonstrate that Hadwiger's Conjecture is true for all graphs G with $\alpha(G) \le 2$ and $|V(G)| \le 11$ and all inflations of such graphs.

Theorem 6.4. *For each $i = 1, 2, 3$, a smallest α-critical graph with $\alpha = 2$ and containing Γ_i as an induced subgraph is unique. If G_i denotes this graph, then $|V(G_1)| = 9, |V(G_2)| = 11$ and $|V(G_3)| = 10$. The graphs G_1, G_2 and G_3 are shown in Figure 6.2. Moreover, G_i and any inflation of G_i satisfies Hadwiger's Conjecture for $i = 1, 2, 3$.*

Proof. In Case 1, the graph Γ_1 is already α-critical; hence $G_1 = \Gamma_1$.

In Case 3, edges v_5v_7, v_5v_9 and v_7v_9 are not critical in Γ_3. Hence G_3 contains at least one new vertex joined neither to v_5 nor v_7, one new vertex adjacent to neither v_5 nor v_9 and one new vertex joined neither to v_7 nor v_9. Suppose that these three new vertices are equal; call it v_{10}. Then v_{10} is joined to none of v_5, v_7 and v_9. Since $\alpha(\Gamma_3) = 2$, v_{10} is joined to all other vertices of Γ_3. It's easy now to check that the resulting graph is α-critical. Hence the smallest possible G_3 is unique and is, in fact, equal to the complement of the Petersen graph, $\overline{P_{10}}$. This completes Case 3.

In Case 2, the edges v_5v_9 and v_7v_9 are not critical in Γ_2. Hence G_2 contains a new vertex v_{10} joined neither to v_7 nor to v_9 and a new vertex v_{11} joined to neither v_5 nor to v_9. The vertices v_{10} and v_{11} are not equal, for if they were, the vertices v_5, v_7 and v_{10} would be independent and this is impossible.

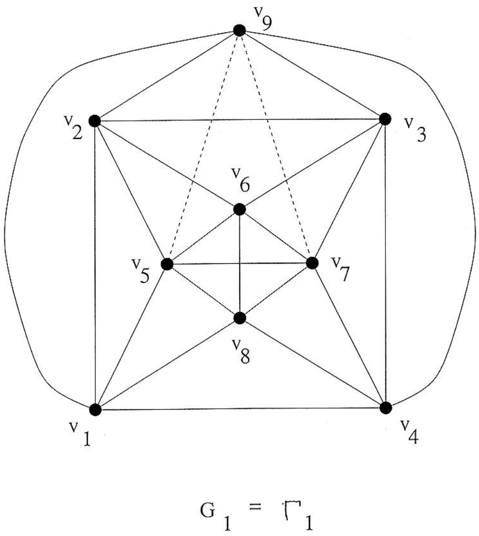


Figure 6.2(a)

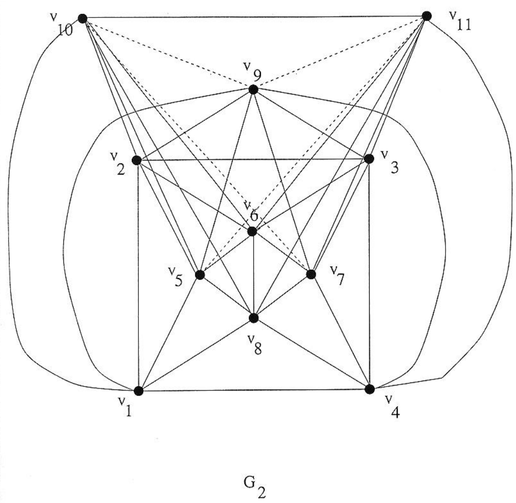


Figure 6.2(b)

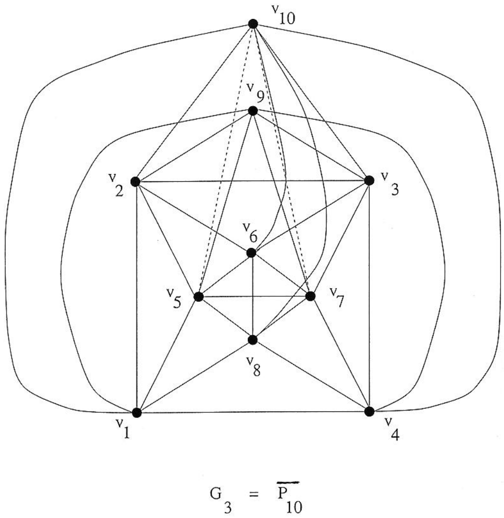


Figure 6.2(c)

Suppose now that G_2 has only these eleven vertices. Then, since $\alpha = 2$, vertex v_{10} is adjacent to v_1, v_2, v_5, v_6, v_8 and v_{11} and v_{11} is adjacent to v_3, v_4, v_6, v_7, v_8 and v_{10}. Moreover, $v_{10} \not\sim v_4$, for if $v_{10} \sim v_4$, edge v_4v_{10} is not critical. Similarly, $v_{11} \not\sim v_1$. So G_2 is an α-critical graph on eleven vertices and is hence unique.

Now we turn to inflations of the graphs G_1, G_2 and G_3. Let us begin by considering an inflation H of G_1. Let the atoms of H be labeled $A_i, i = 1, \ldots, 9$ so as to correspond to vertices $v_1, \ldots, v_9$ as shown in Figure 6.2(a). We henceforth adopt the notation $A_i \leq A_j$ to mean $|A_i| \leq |A_j|$.

Case 1(a). First suppose that $A_1 \leq A_2$ and $A_7 \leq A_3$. Let M_1 be a complete matching of A_1 into A_2 and M_2, a complete matching of A_7 into A_3. Then $M_1 \cup M_2$ is a connected dominating matching. Let A_1^c and A_7^c denote the atoms resulting from the contraction of each edge of the matching $M_1 \cup M_2$. Then the atoms not involved in this contraction may be designated by $A_2', A_3', A_4, A_5, A_6, A_8$, and A_9, where (A_2' and A_3' denote the "left-over" vertices of the original atoms A_2 and A_3 which were not involved in the contraction.) But these seven atoms are the vertex set of an inflation of a seven-vertex graph having $\alpha \leq 2$ and we have already shown that such a graph can be contracted to a complete graph $K_{\lceil n'/2 \rceil}$, where n' denotes the number of vertices in the graph. Thus H satisfies Hadwiger's Conjecture.

Case 1(b). Suppose now that $A_1 \leq A_5$ and $A_3 \leq A_7$. In this case, let M_1 be a complete matching of A_1 into A_5 and M_2, a complete matching of A_3 into A_7. Then $M_1 \cup M_2$ is a connected dominating matching and again the "left-overs" form an inflation of a graph on seven vertices and having $\alpha \leq 2$. So again we are done.

We now claim that Cases 1(a) and 1(b) cover all possibilities. Suppose not. Thus suppose we are not in one of these cases, nor are we in any case symmetric to one of these cases. By symmetry, we may assume, without loss of generality, that $A_1 \leq A_2$. Then since we are not in Case 1(a), it follows that $A_7 > A_3$. Moreover, since we are not in Case 1(b), $A_1 > A_5$. Now if $A_3 \leq A_4$, we get a case which is symmetric to Case 1(a). So we assume $A_4 < A_3$. If, then, $A_2 \geq A_5$, we get a case symmetric to Case 1(a) again. Thus we may assume $A_2 < A_5$. But then the inequalities involving the sizes of atoms A_1, A_2 and A_5 violate transitivity.

Let us next consider Case 3. Let H be an inflation of $G_3 = \overline{P_{10}}$. Due to the symmetry of the Petersen graph, we may assume that atom A_{10} is a smallest atom. Again by symmetry, without loss of generality, we may suppose that atom A_9 is smallest among A_5, A_7 and A_9. Let M_1 be a complete matching of A_{10} into A_6 and M_2, a matching of A_9 into A_5. Then again $M_1 \cup M_2$ is a connected dominating matching and the remaining "left-overs" induce an inflation of an eight-vertex graph and hence we are done.

Finally, consider an inflation H of G_2.

Case 2.1. Suppose $A_2 \leq A_3$ and $A_8 \leq A_4$. As usual, let M_1 and M_2 be complete matchings of A_2 into A_3 and A_8 into A_4 respectively. The graph made up of the "left-overs" has nine atoms and we know it contracts to a graph on half its total number of vertices by Case 1.

Case 2.2. Suppose $A_4 \leq A_1$ and $A_6 \leq A_2$. This is symmetric with Case 2.1.

Case 2.3. Suppose $A_1 \leq A_4$ and $A_6 \leq A_3$. This is also symmetric with Case 2.1.

Case 2.4. Suppose $A_2 \leq A_6$ and $A_4 \leq A_8$. Then we let M_1 and M_2 be complete matchings of A_2 into A_6 and A_4 into A_8 respectively and proceed as before.

Now suppose none of the above four subcases occurs. Then without loss of generality we may assume that $A_2 \leq A_3$. So since we are not in Case 2.1, we may assume $A_4 < A_8$. Since we are not in Case 2.4, we may assume $A_6 < A_2$. Then by transitivity, $A_6 < A_3$. Then since we are not in Case 2.3, we may assume that $A_4 < A_1$. Then since we are not in Case 2.2, we may assume that $A_2 < A_6$. But this is a contradiction. So Case 2 is complete and with it the proof of Theorem 6.4. ■

The preceding Theorem shows that all graphs G with $\alpha(G) \leq 2$ and having $|V(G)| \leq 9$ satisfy Hadwiger's Conjecture. In fact a $K_{\lceil n/2 \rceil}$ minor can be obtained by contracting the edges of a connected dominating matching.

Suppose now that G is α-critical, $\alpha(G) = 2$, and $|V(G)| = 10$. Then if $G = C_{3k-1}^{k-1}, G_1, G_2$ or G_3, or any inflation thereof, we have shown that G satisfies Hadwiger's Conjecture. So suppose G is not one of these. Then G must contain $\Gamma_1 = G_1$ as an induced subgraph. This 9-vertex subgraph G_1 is α-critical.

We proceed to investigate how vertex v_{10} is adjacent to the vertices $v_1, \ldots, v_4$. Since $\{v_1, v_3\}$ is independent, v_{10} is joined to vertex v_1 and/or vertex v_3. Similarly, since $\{v_2, v_4\}$ is independent, v_{10} is joined to vertex v_2 and/or vertex v_4. So vertex v_{10} is joined to two, three or all four of $\{v_1, \ldots, v_4\}$.

Case 1. Suppose v_{10} is adjacent to all four of $\{v_1, \ldots, v_4\}$. Edge v_1v_{10} must be critical, so at least one of v_6 and v_7 is not adjacent to v_{10}. But then $v_{10} \sim v_9$. If $v_{10} \sim v_5, v_6, v_7$ or v_8, then the corresponding edge would not be critical, so v_{10} is adjacent to none of $v_5, \ldots, v_8$. So G is an inflation of G_1 where $\{v_9, v_{10}\}$ lie in the same atom and so we are done.

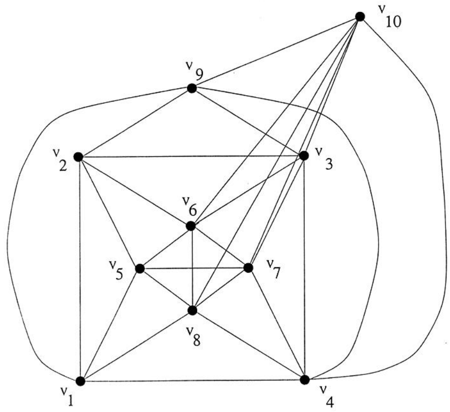
So let us assume that vertex v_{10} is not adjacent to at least one of $v_1, \ldots, v_4$.

Case 2. Suppose now that $v_{10} \sim v_2, v_3$ and v_4, but $v_{10} \not\sim v_1$ (without loss of generality). Then $v_{10} \sim v_6$ and $v_{10} \sim v_7$. Then $v_{10} \not\sim v_8$ (since edge v_2v_{10} is critical), $v_{10} \not\sim v_5$ (since edge v_4v_{10} is critical), and $v_{10} \sim v_9$ (since $\alpha = 2$). But then G must be an inflation of G_1 where vertices v_3 and v_{10} belong to the same atom and we are done.

Case 3. So suppose v_{10} is adjacent to *exactly* two of the vertices v_1, v_2, v_3, v_4. Then these two must in turn be adjacent. So suppose $v_{10} \sim v_3, v_4$ and $v_{10} \not\sim v_1, v_2$. Then $v_{10} \sim v_8, v_6$ and v_7. If $v_{10} \sim v_5$, then $v_{10} \not\sim v_9$. But then G is an inflation of G_1 (where vertices v_{10} and v_7 belong to the same atom) and once again we are done.

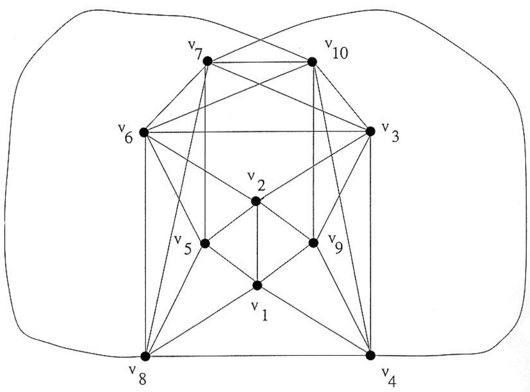
So suppose $v_{10} \not\sim v_5$. Thus $v_{10} \sim v_9$. So G is isomorphic to the graph G_4 pictured in Figure 6.3(a) and redrawn in Figure 6.3(b) to better exhibit its symmetries.

This is a new α-critical graph which we haven't encountered before. Let us now consider any inflation H of graph G_4.



G_4

Figure 6.3(a)



G_4 (redrawn)

Figure 6.3(b)

Case 3.1. Suppose $A_6 \leq A_2$ and $A_4 \leq A_1$. Let M_1 and M_2 be complete matchings of A_6 into A_2 and A_4 into A_1, respectively. Clearly $M_1 \cup M_2$ is a connected dominating matching. The vertices not spanned by $M_1 \cup M_2$ induce an 8-atom graph H' of "left-over vertices" which is an inflation of an 8-vertex graph with $\alpha \leq 2$. Then H' satisfies HC and thus has $K_{\lceil n'/2 \rceil}$ as a minor. Contracting the edges of $M_1 \cup M_2$ into single vertices results in a graph containing $K_{\lceil n/2 \rceil}$. Hence H satisfies HC.

Case 3.2. Suppose $A_6 \leq A_3$ and $A_1 \leq A_4$. Let M_1 be a complete matching of A_6 into A_3 and M_2, a complete matching of A_1 into A_4 and continue as before.

Case 3.3. Suppose $A_8 \leq A_4$ and $A_2 \leq A_3$. Let M_1 be a complete matching of A_8 into A_4 and M_2, a complete matching of A_2 into A_3 and continue as before.

Case 3.4. Suppose $A_4 \leq A_8$ and $A_2 \leq A_6$. This case is symmetric to Case 3.2.

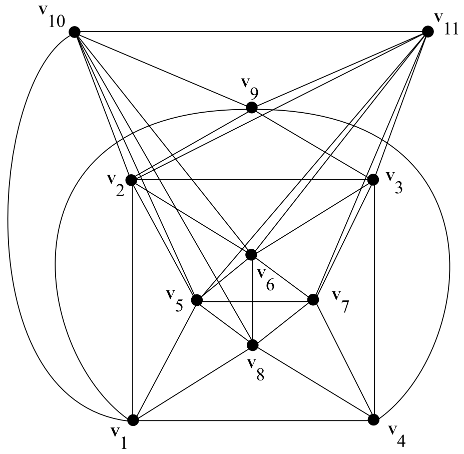
Now also suppose that H does not satisfy HC. Without loss of generality, we may assume that $A_6 \leq A_3$. By Case 3.2 we may assume that $A_4 < A_1$. Then by Case 3.1 it follows that $A_2 < A_6$. But then by Case 3.4, $A_8 < A_4$ and by Case 3.3, $A_3 < A_2$. But then $A_3 < A_2 < A_6 \leq A_3$, a contradiction.

So we may conclude that any inflation of a graph with $\alpha \leq 2$ and having no more than ten vertices (and any inflation of such a graph) satisfies Hadwiger's Conjecture and moreover we can contract such a graph to a graph containing a clique on at least half the number of its vertices by contracting the edges of a connected dominating matching.

We have carried out a similar investigation when $|V(G)| = 11$. As α-critical graphs with $\alpha = 2$ and having eleven vertices we obtain two new graphs, G_5 and G_6, (shown below in Figures 6.4 and 6.5 respectively) not covered before. More specifically, G_5 arises in Case 1 and G_6 arises in Case 3.

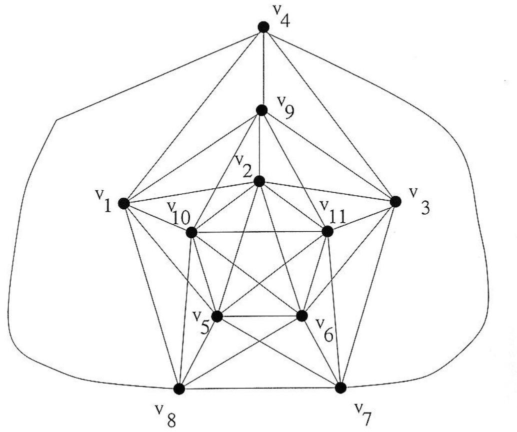
We assert that any inflation H of graph G_6 satisfies Hadwiger's Conjecture. This follows by an argument similar to those above. More particularly, let Case 1 denote the situation when $A_2 \leq A_3$ and $A_8 \leq A_4$; Case 2, the situation when $A_4 \leq A_1$ and $A_6 \leq A_2$; Case 3, the situation when $A_1 \leq A_4$ and $A_6 \leq A_3$; and Case 4, the situation when $A_2 \leq A_6$ and $A_4 \leq A_8$. Assume then that H does not satisfy Hadwiger's Conjecture. Without loss of generality, we may assume that $A_2 \leq A_3$. But then by Case 1, $A_4 < A_8$; by Case 2, $A_1 < A_4$; by Case 3, $A_3 < A_6$ and by Case 4, $A_6 < A_2$. But then we have $A_6 < A_2 \leq A_3 < A_6$, a contradiction.

We now turn our attention to graph G_5. The reader is directed to the second drawing of G_5 shown in Figure 6.4 to more clearly see the symmetries which we shall appeal to below. We prove that any inflation H of G_5 satisfies HC.



G_5

Figure 6.4(a)



G_5 (redrawn)

Figure 6.4(b)

Let us denote by Case A, the situation when $A_1 \leq A_5$ and $A_3 \leq A_6$. The associated complete matchings of A_1 into A_5 and A_3 into A_6 give the desired result. The reader sees that by (rotational) symmetry in Figure 6.4 (b), four other cases follow.

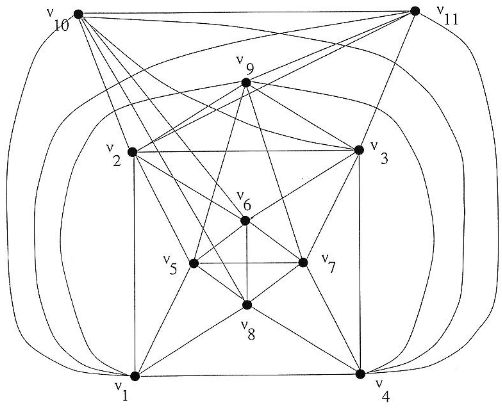


Figure 6.5

Let Case B denote the situation in which $A_4 \le A_9$, $A_5 \le A_2$ and $A_6 \le A_{10}$. The three associated complete matchings then give the desired result. This time there are four additional cases which follow by rotating G_5 clockwise succesively through angles of $2\pi/5$ radians and another five cases which follow by reflecting the graph about a vertical axis of symmetry such as the axis through atom A_2 and the midpoint of the family of edges joining A_5 and A_6.

Case C denotes the situation in which $A_9 \le A_4, A_5 \le A_7$ and $A_6 \le A_8$. Again the associated complete matchings serve to give the desired result. Also here there are an additional four cases which are settled by rotating the graph by multiples of $2\pi/5$ radians clockwise.

Finally, let Case D denote the situation in which $A_4 \le A_9 \le A_{10}$, $A_9 \le A_{11}$, $A_2 \le A_1$, $A_2 \le A_3$, $A_5 \le A_7$ and $A_6 \le A_8$. First, let M_1 and M_2 be complete matchings of A_5 into A_7 and A_6 into A_8, respectively. Then let A_7' and A_8' be the vertices of A_7 and A_8, respectively, that are not covered by the matching $M_1 \cup M_2$. By symmetry, we may assume that $A_8' \le A_7'$. This implies that there is a complete matching M_3 of A_8' into A_7'. Furthermore, there are complete matchings M_4 and M_5 of A_4 into A_9 and A_2 into A_3, respectively. Eventually, let A_9' be the vertices of A_9 not covered by M_4. Then there is a complete matching M_6 of A_9' into A_{11}. Clearly, $M = \bigcup_{i=1}^{6} M_i$ is a matching that covers all vertices from $A_2 \cup A_4 \cup A_5 \cup A_6 \cup A_8 \cup A_9$ and the reader can easily check that the matching M is connected and dominating. Then it gives the desired result. Also here there are four additional cases.

To finish the proof, we claim that these four cases are sufficient. Suppose that this is not true, i.e., none of the situations described in Case A,B,C or D occur in the inflation H of G_5. To arrive at a contradiction, we first choose the smallest atom among the five atoms A_2, A_5, A_6, A_{10} and A_{11} that belong to the K_5 of G_5. By symmetry, we may assume that this atom is A_2. Then $A_2 \le A_i$ for $i \in \{5, 6, 10, 11\}$. Furthermore, by symmetry, we may assume that $A_{11} \le A_{10}$. Then we conclude that $A_8 < A_4$, since otherwise we have Case B, because of $A_{11} \le A_{10}$ and $A_2 \le A_5$. Now, we distinguish two cases.

Case 1. Suppose $A_3 \le A_2$. Then $A_{10} < A_8$, since otherwise we have Case A. Because of $A_3 \le A_2 \le A_{10} < A_8 \le A_4$, it follows that $A_3 < A_4$. Also $A_3 \le A_6$ follows by transitivity and, therefore, $A_5 < A_1$, since otherwise we have Case A. But then, because of $A_3 < A_4$, $A_5 < A_1$ and $A_{10} < A_8$, we have Case C, a contradiction.

Case 2. Suppose $A_2 < A_3$. Then $A_9 < A_{11}$, since otherwise, because of $A_8 < A_4$, we have Case C. Then, using Case A, we conclude that $A_6 < A_8$ and, using transitivity, we conclude that $A_9 < A_{10}$. This, by Case A, implies $A_5 < A_7$. Then, using Case C, we infer that $A_4 < A_9$. Since $A_8 < A_4 < A_9 < A_{10}$, it follows that $A_8 < A_{10}$ and, therefore, $A_6 < A_{10}$, by transitivity. Furthermore, we infer that $A_1 < A_2$, since otherwise we have Case D with $A_5 < A_7$, $A_6 < A_8$, $A_4 < A_9 < A_{10}$, $A_9 < A_{11}$ and $A_2 \le A_3$. Because of $A_1 < A_2$, it follows from Case A, that $A_{11} < A_7$. Since $A_4 < A_9 < A_{11} < A_7$, we have $A_4 < A_7$. Furthermore, since $A_1 < A_2 \le A_5$, we have $A_1 < A_5$. Then, using Case A, we conclude that $A_6 < A_3$. Since $A_{11} < A_7$, it then follows by Case C that $A_4 < A_1$. Now, we have $A_8 < A_4 < A_1$ and $A_1 < A_2 \le A_6 < A_8$, a contradiction.

Thus, in both cases we arrived at a contradiction. This proves our claim and completes the proof of Theorem 4.1. ∎

7. Connected Matchings

Let us denote by the name CONNECTED MATCHING the following problem (already posed in our Introduction):

Given a graph G and a positive integer k, is there a connected matching M in G such that $|M| \ge k$?

A well-known NP-complete problem called CLIQUE is stated as follows [9]:

Given a graph G and a positive integer k, is there a clique K in G with $|K| \geq k$?

Theorem 7.1. CONNECTED MATCHING *is* NP-*complete.*

Proof. Clearly the problem is in NP and we shall reduce CLIQUE to CONNECTED MATCHING.

Let G be any graph and construct a new graph H as follows. Given two disjoint copies G_1 and G_2 of G, join all vertices of G_1 to all vertices of G_2. Now attach to each vertex v of $V(G_1) \cup V(G_2)$ a new edge joining v to a new vertex v' and let H be the resulting graph on $4|V(G)|$ vertices.

Suppose first that H has a connected matching M. Then M is composed of edge set M_{11} consisting of edges of the form vv' where endvertex v lies in G_1, edge set M_1 having both endvertices in G_1, edge set M_{12} consisting of edges having one endvertex in G_1 and the other in G_2, edge set M_2 consisting of edges having both endvertices in G_2 and M_{22} made up of edges of the form vv' where v lies in G_2. Then for $i = 1, 2$, the endvertices of M_{ii} in G_i a an a complete subgraph Q_i of G_i. Thus

$$\begin{aligned} |M| &= |M_{11}| + |M_{22}| + |M_1| + |M_{12}| + |M_2| \\ &\leq |V(Q_1)| + |V(Q_2)| + (2|V(G)| - |V(Q_1)| - |V(Q_2)|)/2 \\ &\leq |V(G)| + |V(Q)|, \end{aligned}$$

where Q is a maximum complete subgraph of G.

Conversely, it is easy to see that H has a connected matching of size $|V(G)| + |V(Q)|$, where Q is a largest complete subgraph in G. Thus G has a complete subgraph of size at least k if and only if H has a connected matching of size at least $|V(G)| + k$. Therefore CLIQUE has been reduced to CONNECTED MATCHING and thus the latter is NP-complete. ■

8. Concluding Remarks

When commencing this investigation, our feeling was that Hadwiger's Conjecture might fail for some $\alpha = 2$ graphs and that a counterexample might possibly be obtained as an inflation of some small graph G having $\alpha = 2$ (in the same way that a counterexample to the related conjecture of Hajós turned out to be simply an inflation of the 5-cycle, as noted by Catlin [4], see also [12]).

The main outcome of our investigations is that this seems not to be so; at least G will have to have at least twelve vertices.

It is unfortunate that we have not been able to carry our investigation through to a final conclusion for Hadwiger's Conjecture with respect to $\alpha = 2$ graphs. It has been likewise disappointing not even to be able to improve the trivial constant 1/3. (See the Introduction.) A possible improvement would be to the value 2/5 perhaps by being able to repeatedly contract three edges of a 5-cycle into two vertices.

So Hadwiger's Conjecture seems to remain one of the great challenges of discrete mathematics, even for graphs with $\alpha = 2$.

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