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MODULAR AND MEDIAN SIGNPOST SYSTEMS AND THEIR UNDERLYING GRAPHS

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Abstract

The concept of a signpost system on a set is introduced. It is a ternary relation on the set satisfying three fairly natural axioms. Its underlying graph is introduced. When the underlying graph is disconnected some unexpected things may happen. The main focus are signpost systems satisfying some extra axioms. Their underlying graphs have lots of structure: the components are modular graphs or median graphs. Yet another axiom guarantees that the underlying graph is also connected. The main results of this paper concern if-and-only-if characterizations involving signpost systems satisfying additional axioms on the one hand and modular, respectively median graphs on the other hand.

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1. Introduction

Usually, when we travel and want to find our way from u to w, we look for signposts towards w and follow them until we reach w. That is, at u we follow the signpost to w, by which we first arrive at v. Then at v we look for the next signpost to w. We can denote this step in our journey by the triple (u, v, w): here u is the point where we currently are, w is the final destination, and v is the point where we first arrive after following the direction of the signpost at u.

In this paper we introduce the concept of a signpost system on a set V as a ternary relation satisfying three basic axioms. We introduce the underlying graph of a signpost system. As a first step in the study of signposts systems, we consider signpost systems satisfying additional axioms. Thus we get the modular and median signpost systems. There is a close relation between these signpost systems and modular and median graphs. Especially median graphs constitute an important class in the universe of all graphs. Loosely speaking, there are as many median graphs as there are connected triangle-free graphs in the universe of all graphs, see [3]. Median graphs are well studied, they have quite diverse applications in e.g. location theory, dynamic search problems, consensus theory, conflict models, the measurement of dissimilarities, and the transcription history of mediaeval manuscripts. There are relations with discrete structures from several other mathematical areas. For a recent survey see [4]. The importance of these classes of graphs gives rise to the study of modular and median signpost systems.

We present some basic results on signpost systems, and we present some important examples that we need to clarify certain subtleties in the remaining of the paper. Our main results are of the following type. If Q is a modular (median) signpost system on a set V, then each component of the underlying graph of Q is modular (respectively median). Note that we do not have an 'if and only if' here. It turns out that, when the underlying graph is disconnected, then modularity of the components does not guarantee modularity of the signpost system. The same can be said in the median case. In the final section of the paper, we introduce an additional axiom. Combined with the modular or median axioms, this guarantees connectivity of the underlying graph. In this case we are able to obtain 'if and only if' results: now Q is a modular (median) signpost system with the additional axiom if and only if its underlying graph is a modular (median) graph.

2. Preliminaries

Throughout this paper all graphs and sets are supposed to be *finite*. Let $G = (V, E)$ be a (finite) graph. Let u and v be two vertices of G. Then $d(u, v)$ denotes the distance between u and v, that is, the length of a shortest path between u and v. The *interval* between u and v is the set $I(u, v)$ defined by

$$I(u, v) = \{w \mid w \text{ lies on a shortest } u, v\text{-path}\}.$$

We call I the *interval function* of G. See [5] for an extensive study of the interval function of a graph. For $u, v, w \in V$ we write

$$I(u, v, w) = I(u, v) \cap I(v, w) \cap I(w, u).$$

A connected graph G is a *modular graph* if $I(u, v, w) \neq \emptyset$, for all u, v, w in V, cf. [2]. It is an easy exercise to prove that a modular graph is bipartite. Any vertex in the intersection $I(u, v, w)$ is called a *median* of u, v, and w. A connected graph G is a *median graph* if $|I(u, v, w)| = 1$, for all u, v, w in V. Thus, a median graph is a modular graph with unique medians. One characterization of median graphs that is quite useful for our purposes is given by Theorem 3.1.8 in [5]: *a connected graph G is a median graph if and only if it is triangle-free and $|I(u, v, w)| = 1$, for all u, v, w in V with $d(u, w) = 2$.* In [5] this theorem was proved in the search for minimal sufficient conditions for a graph to be a median graph. As an easy byproduct of the proof of this theorem we get the following characterization of modular graphs: *a connected graph G is a modular graph if and only if it is triangle-free and $I(u, v, w) \neq \emptyset$, for all u, v, w in V with $d(u, w) = 2$.*

Note that median graphs are connected triangle-free graphs of a very special type. Therefore it may be surprising at first sight that there is an easy one-to-one correspondence between a quite restricted subclass of the class of median graphs on the one hand and the class of all connected triangle-free graphs on the other hand, see [3]. In other words this means that the "density" of the median graphs in the universe of all graphs is the same as that of the connected triangle-free graphs.

Examples of modular graphs are the covering graphs of modular lattices. Actually, this is the example from which they derive their name. Examples of median graphs are the covering graphs of distributive lattices, the trees, the hypercubes, and the grid graphs. Note that a hypercube is the covering graph of a finite Boolean lattice. The survey paper [4] may serve as a first introduction to median graphs, their applications and the various guises in which they occur in quite diverse mathematical areas. There exist various algebraic structures related to median graphs, see [4]. Amongst these are ternary relations on the vertex set of the graph. These ternary relations are then characterized completely in terms of axioms on the ternary relation. As an example may serve the so-called *median algebras*, a ternary relation $m : V \times V \times V \to V$ on a set V that satisfies the axioms

$(m1)$ $m(u, u, u) = u$,

$(m2)$ $m(m(u, v, w), m(u, v, x), y) = m(m(w, x, y), u, v)$,

see [11, 1]. For another set of axioms for such median algebras see [6, 7].

The ternary relation discussed in this paper is new and is quite different from those listed in [4]. It is closely related to the notion of a step system of a graph, see below or [8]. Yet another way to deal with the algebraic properties of median-type structures can be found in [9], where trees are characterized using a binary operation. This result can be interpreted as a special case of our Theorem 10 below.

The paper is organized as follows. In Section 3 we introduce the notion of a signpost system and its underlying graph. Some basic results and examples are given. It turns out that connectivity of the underlying graph is an issue: if the underlying graph is disconnected, then some peculiar things can happen. In Section 4 we introduce the modular and median signpost systems, and we prove that the components of their underlying graphs are modular, respectively median. We present examples to show what may happen when the underlying graph is disconnected. In the last section we introduce an additional axiom. With the help of this axiom we are able to prove our main results: a signpost system Q satisfying this additional axiom is modular (median) if and only if its underlying graph is a modular (median) graph.

Recall that we restrict ourselves to the finite case. Some of the results below also hold (trivially) for the infinite case, others may or may not hold in the infinite case, but we will not pursue that here.

3. Signpost Systems

Throughout this paper $Q \subseteq V \times V \times V$ denotes a *ternary relation* on a (finite) set V. The *underlying graph* G_Q of Q has V as its vertex set, and u

and v are joined by an edge in G_Q if and only if $(u, v, v) \in Q$.
A ternary relation Q on V is a *signpost system* or a *signposting* on V if Q satisfies the following axioms:

($s1$) if $(u, v, w) \in Q$, then $(v, u, u) \in Q$, for u, v, w in V,

($s2$) if $(u, v, w) \in Q$, then $(v, u, w) \notin Q$, for u, v, w in V,

($s3$) if $u \neq v$, then there exists a $t \in V$ such that $(u, t, v) \in Q$, for u, v in V.

Axiom ($s1$) guarantees us that we can find our way back. Axiom ($s2$) prevents us from getting stuck in a loop. Axiom ($s3$) guarantees that at any point there are signposts to all other points. In practice this may be realized only by combining the existing signpost system with a map.

An axiom for a signpost system Q giving conditions for triples in Q will be called a *signpost axiom*. Note that, by axiom ($s1$), we have $(u, v, v) \in Q$ if and only if $(v, u, u) \in Q$. Moreover, it follows from axiom ($s1$) that uv is an edge in G_Q if and only if $(u, v, w) \in Q$ for some w in V. Axiom ($s2$) implies that G_Q does not contain loops, whence is a simple graph.

The following basic lemma was presented in [8] in the context of step systems. For the sake of completeness we include a proof here.

Lemma 1. *Let Q be a signpost system on a set V, and let $u, v, w \in V$. If $(u, v, w) \in Q$, then $v \neq u \neq w$.*

Proof. Let $(u, v, w) \in Q$. By ($s1$), we have $(v, u, u) \in Q$, whence also $(u, v, v) \in Q$. Then it follows from ($s2$) that $u \neq v$. Now assume that $u = w$, so that $(w, v, w) \in Q$. Then ($s1$) implies that $(v, w, w) \in Q$. On the other hand, by ($s2$) we have $(v, w, w) \notin Q$. This contradiction implies that $u \neq w$. ∎

An immediate consequence of Lemma 1 is that Q is empty whenever $|V| \leq 1$. Therefore, in the sequel we will always assume that $|V| \geq 2$, although this is not essential. Note that, in this case, it follows from ($s3$) that G_Q has no isolated vertices.

Let us have a closer look at the relation between a signpost system Q on the set V and its underlying graph $G = (V, E)$. Fix a vertex x in G. Assume that (u, v, x) is in Q. Then, by axiom ($s2$), (v, u, x) is *not* in Q. Thus, seen from x, we may orient edges in E as follows: $u \to v$ if and only if $(u, v, x) \in Q$. By E_x we denote the set of oriented edges (or *arcs*), as seen from x. Let G_x be the subgraph of G induced by E_x. Since all edges in G_x are oriented, G_x is a directed graph. Note that between any pair of vertices there is at most one arc. We call G_x the *x-orientation* of G. By axiom $(s3)$, G_x is a spanning subgraph of G in which all vertices distinct from x have positive outdegree. By definition of the underlying graph, (u, x, x) is in Q for each neighbor u of x. So all edges incident with x are oriented towards x. Thus x is the unique sink of G_x, where a *sink* is a vertex with positive indegree and zero outdegree. Note that not all edges of G need to be oriented.

These observations give us an easy way to construct signpost systems from graphs. Take any graph $G = (V, E)$. For each vertex x of G, we choose a spanning subgraph G_x containing all edges incident with x. Now we orient the edges of G_x such that x is the unique sink in the resulting digraph. Then we construct the signpost system Q as follows: $(u, v, x) \in Q$ if and only if $u \to v$ is an arc in G_x. It is easy to see that Q satisfies the three signpost axioms $(s1), (s2), (s3)$. Obviously, G is the underlying graph of Q.

First we consider the case that G is connected. Then we can always construct the necessary digraphs G_x: for instance, take a spanning subtree of G that contains all edges incident with x and orient the edges in this spanning subtree towards x. Another prime example is the following: if uv is an edge in G with $d(v, x) = d(u, x) - 1$, then we call (u, v, x) a *step* in G, that is, by going from u to v along the edge uv, we take a step closer to x. The *step system* Q_G of G is the ternary relation on V consisting of all the steps in G as defined in this way. It is easily seen that the step system of a graph G is a signpost system if and only if G is connected.

The step system of a graph G was introduced in [8] to study the system of all geodesics in G, see also [10]. The main result of these papers is a characterization of step systems in terms of axioms on the ternary relation. More precisely, let Q be a ternary relation with underlying graph G. Then Q is the step system of G if and only if G is connected and Q satisfies a prescribed set of axioms. Note that connectivity of G is essential: it is an open problem whether connectivity of G may be replaced by axioms on Q.

If G is disconnected, then it is not always possible to construct the digraphs G_x. For, if H is a component of G, and x is in another component, then the orientation of edges in H must result in a digraph H_x without sinks. This is only possible if H contains a cycle, that is, if H is *not* a tree. For the component containing x we can proceed as above in the connected case.

Let G_{Q_G} be the underlying graph of the step system Q_G of a graph G.

Then we have

$$uv \in E(G_{Q_G}) \Leftrightarrow (u, v, v) \in Q_G \Leftrightarrow uv \in E(G).$$

So we have $G_{Q_G} = G$.

Let G be a graph, and let Q be its step system. Then we can easily verify that Q satisfies the following axiom:

($s4$) if $(u, v, w), (w, x, v) \in Q$, then $(u, v, x) \in Q$, for u, v, w, x in V.

Let Q be a signpost system on V with underlying graph G_Q, and let Q_{G_Q} be the step system of G_Q. Then one might ask whether we have $Q_{G_Q} = Q$. It turns out that for this question connectivity of G_Q is essential. The result of Nebeský [8, 10] mentioned above now reads as follows: let Q be a ternary relation on V; then $Q_{G_Q} = Q$ if and only if Q satisfies a set of five axioms (one of which is our axiom ($s4$) above) *and* G_Q is connected. In the present paper we study a special case, see Section 5. Here we do not need to require that G_Q is connected because now connectivity *follows* from the axioms. But let us first consider the disconnected case a little more closely.

Let H be a connected graph with at least one cycle. A *sinkless orientation* of H is an orientation of *all* the edges of H such that a digraph results without any sinks. Note that the digraph necessarily contains a directed cycle. Now we describe a construction for signpost systems that will provide us with convenient examples. Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two non-trivial connected graphs, each with a sinkless orientation (so that each contains a cycle). Let $G = (V, E)$ be the disjoint union of G_1 and G_2, that is $V_1 \cap V_2 = \emptyset$ and $E_1 \cap E_2 = \emptyset$, and $V = V_1 \cup V_2$ and $E = E_1 \cup E_2$. Note that G is a disconnected graph with a sinkless orientation. Now we construct the ternary relation Q on V as follows. Let Q_{G_i} be the step system of G_i, for $i = 1, 2$. Then we set

$$Q = Q_{G_1} \cup Q_{G_2} \cup \{(u, v, x) \mid uv \text{ is an arc in } G_1 \text{ and } x \text{ is in } G_2\}$$
$$\cup \{(u, v, x) \mid uv \text{ is an arc in } G_2 \text{ and } x \text{ is in } G_1\}.$$

It is easily checked that Q is a signpost system that also satisfies axiom ($s4$). We call Q the signpost system of G with its sinkless orientation. Clearly, the underlying graph of Q is G. Moreover, we have $Q_G = Q_{G_1} \cup Q_{G_2}$. Thus we have constructed an example of a signpost system for which we have $Q_{G_Q} \neq Q$.

Hence, in general, we do not have $Q_{G_Q} = Q$ for signpost systems, even if they satisfy the additional axiom ($s4$). We call a signpost system Q a *graphic signpost system* if the equation $Q_{G_Q} = Q$ holds for Q. By the above observations, a graphic signpost system necessarily satisfies axiom ($s4$). As observed above, in [8] a characterization of graphic signpost systems (in the guise of step systems) is given in the case that connectivity of the underlying graph is presupposed. It is an open problem to find a characterization solely in terms of signpost axioms in this case.

4. Modular and Median Signpost Systems

Recall that V be a finite set with $|V| \geq 2$. A *modular signpost system* on V is a signpost system Q that satisfies axiom ($s4$) and the following axiom:

($s5$) if $(u, v, v), (v, w, w) \in Q$, $(u, v, x), (w, v, x) \notin Q$ and $u \neq w$, then there exists $t \in V$ such that $(u, t, x), (w, t, x) \in Q$, for u, v, w, x in V.

In terms of the underlying graph, axiom ($s5$) reads as follows: if we have three distinct vertices u, v, w with edges uv, vw, and from u or w we cannot get 'closer' to x via v, then there exists a common neighbor t of u and w such that we can get 'closer' to x via t.

First we present two basic facts on modular signpost systems.

Lemma 2. *Let Q be a modular signpost system on a finite set V. If (u, v, v), $(v, w, w) \in Q$, and $u \neq w$, then $(u, v, w) \in Q$.*

Proof. Suppose, to the contrary, that $(u, v, w) \notin Q$. Since $(v, w, w) \in Q$, it follows from ($s2$) that $(w, v, w) \notin Q$. Hence, by ($s5$), there exists a vertex t in V such that (u, t, w), $(w, t, w) \in Q$. Thus we have a conflict with Lemma 1. ■

Lemma 3. *Let Q be a modular signpost system on a finite set V with underlying graph G. Then G is triangle-free.*

Proof. Assume that G has a triangle on u, v, w, so that $u \neq v \neq w \neq u$. By the definition of G, we have (u, v, v), (v, w, w), (v, u, u), $(u, w, w) \in Q$. So, by Lemma 2, we have (u, v, w), $(v, u, w) \in Q$. This is impossible by ($s2$). ■

The following lemma tells us that the restriction of a modular signpost system to a component of the underlying graph is just the step system of that component.

Lemma 4. *Let Q be a modular signpost system on a finite set V with underlying graph G. Then, for any component H of G, the ternary relation*

$$P = \{(u, v, w) \in Q \mid u, v, w \in V(H)\}$$

defined on $V(H)$ is the step system of H.

Proof. Note that H is the underlying graph of P. Let R denote the step system of H. Suppose, to the contrary, that $P \neq R$. Then there exist vertices u, v, w in H with $v \neq u \neq w$ such that

$$(u, v, w) \in P \triangle R \tag{1}$$

and

$$\begin{aligned} &(r, s, t) \in P \text{ if and only if } (r, s, t) \in R \\ &\text{for all } r, s, t \in V(H) \text{ such that } d(r, t) < d(u, w), \end{aligned} \tag{2}$$

where d is the distance in H. Write $n = d(u, w)$. Since $u \neq w$, we have $n \geq 1$. We consider two cases.

Case 1. $(u, v, w) \in R$.
Note that, by (1), we have $(u, v, w) \notin P$. Since R is the step system of the graph H, we have $uv \in E(H)$ and $d(v, w) = n - 1$. Since H is the underlying graph of P, it follows that $(u, v, v) \in P$. Hence we have $v \neq w$, that is, $n - 1 > 0$, so that $n \geq 2$. If we would have $n = 2$, then vw would be an edge in $E(H)$, so that $(v, w, w) \in P$. But now, u and w being distinct, Lemma 2 would imply that $(u, v, w) \in P$, which is not so. Therefore, we have $n \geq 3$.

Now there exists a neighbor x of v in H such that $d(x, w) = n-2$, so that $u \neq x$. Then (v, x, w) is a step in R, whence (v, x, w) is in P, by (2). By $(s2)$ we have $(x, v, w) \notin P$. Since v is a neighbor of u and x, it follows from the definition of the underlying graph of Q that $(u, v, v), (v, x, x) \in P$. Recall that $(u, v, w) \notin P$. Hence we deduce from $(s5)$ the existence of a vertex y in V such that $(u, y, w), (x, y, w) \in Q$. Obviously, y is a vertex in H. Since $d(x, w) = n - 2$, we have $(x, y, w) \in R$, by (2). Hence $d(y, w) = n - 3$. But $(u, y, w) \in Q$ implies that u and y are adjacent, and we have a conflict with $d(u, w) = n$. This settles Case 1.

Case 2. $(u,v,w) \notin R$.
Note that now we have $(u,v,w) \in P$. So, by $(s1)$, we have $(v,u,u) \in P$.

Assume that $n = 1$, that is, uw is an edge in H. Then $(u,w,w) \in P \cap R$. Since $(u,v,w) \notin R$, we must have $v \neq w$. So, by Lemma 2, we have $(v,u,w) \in P$. But this is impossible by $(s2)$. This implies that $n \geq 2$.

Now let z be a neighbor of u in H with $d(z,w) = n-1$. Then we have $(u,z,w) \in R$. Note that we must have $v \neq z$. Moreover, we have $(z,u,w) \notin R$. Since $d(z,w) < d(u,w)$, it follows that also $(z,u,w) \notin P$. Since $(u,v,w) \in P$, we have, by $(s2)$, $(v,u,w) \notin P$. Now, uv and uz being edges in H, we have $(v,u,u),(u,z,z) \in P$. Therefore we can apply axiom $(s5)$, which gives us a vertex t of G with $(v,t,w),(z,t,w) \in Q$. Then vt and zt are edges in G, whence in H, so that t is a vertex of H, and $(z,t,w) \in P$. Since $d(z,w) < d(u,w)$, it follows that $(z,t,w) \in R$, so that $d(t,w) = d(z,w) - 1 = n-2$. But now we deduce that $d(v,w) \leq n-1$. Since $d(u,w) = n$, we have $d(v,w) = n-1$. This implies that $(u,v,w) \in R$, which produces a contradiction. This settles Case 2.

Thus we have shown that $R = P$. ∎

In view of Theorem 5 below, axiom $(s5)$ is much too strong in the sense that it is *not* a characterization of the signpost systems with the property in Lemma 4 (the restriction to any component of the underlying graph is precisely the step system of that component). So one may ask for an axiom or a set of axioms that characterizes such signpost systems. This is a variation of the problem of characterizing the graphic signpost systems in terms of signpost axioms only.

Theorem 5. *Let Q be a modular signpost system with underlying graph G. Then each component of G is a modular graph.*

Proof. Note that, by Lemma 4, the restriction of Q to any component H of G is precisely the step system of H. This means that, for any three vertices u,v,w of a component H, we have

$$(u,v,w) \in Q \text{ if and only if } d(u,v) = 1 \text{ and } d(v,w) = d(u,v) - 1.$$

From Lemma 3 it follows that G is triangle-free. So it suffices to prove that $I(u,x,w) \neq \emptyset$, for any three vertices u,x,w in the same component H of G with $d(u,w) = 2$. If $x \in I(u,w)$, then we are done. So we may assume that $x \notin I(u,w)$.

Take three vertices u, x, w of H with $d(u,w) = 2$ and $x \notin I(u,w)$. Note that we have $d(u,x) + d(x,w) \geq 3$. Without loss of generality we assume that $d(u,x) \geq d(w,x)$. Since $d(u,w) = 2$, it follows that $d(u,x) \geq d(w,x) \geq d(u,x) - 2 \geq 0$. If we have $d(w,x) = d(u,x) - 2$, then we have $d(u,w) + d(w,x) = d(u,x)$, so that $w \in I(u,x)$. This implies that $I(u,x,w) \neq \emptyset$, and we are done. So we may assume that $d(u,x) \geq d(w,x) \geq d(u,x) - 1 \geq 1$. Let v be a common neighbor of u and w. Note that we have $(u,v,v), (v,w,w) \in Q$. We distinguish two cases.

Case 1. $d(u,x) = d(w,x)$.
Then we have $d(u,x) - 1 \leq d(v,x) \leq d(u,x) + 1$. If $d(v,x) = d(u,x) - 1$, then clearly $v \in I(u,x,w)$, and we are done. So suppose that $d(v,x) \geq d(u,x)$. Since $d(u,x) \geq 2$, it follows from the fact that Q is the step system of G that (u,v,x), $(w,v,x) \notin Q$. Hence, by axiom $(s5)$, there exists a vertex t in G such that $(u,t,x), (w,t,x) \in Q$. Then t is a common neighbor of u and w with $d(t,x) = d(u,x) - 1 = d(w,x) - 1$, so that t lies in $I(u,x,w)$, and we are done.

Case 2. $d(u,x) = d(w,x) + 1$.
Since v is a common neighbor of u and w, we have $d(w,x) \leq d(v,x) \leq d(u,x)$. Assume that $d(v,x) = d(w,x) = d(u,x) - 1$. If $d(w,x) = 1$, then w, v, x would induce a triangle. By Lemma 3, this is impossible. So we necessarily have $d(w,x) \geq 2$ and $d(u,x) \geq 3$. Now (u,v,x) is a step in Q. Let z be a neighbor of v with $d(z,x) = d(v,x) - 1 = d(u,x) - 2$. Then (v,z,x) is a step in Q as well, so that (z,v,x) is not a step in Q. Since $d(v,x) = d(w,x)$, it follows that (w,v,x) is not a step in G, hence also not a step in Q. On the other hand, v being a common neighbor of z and w, we know that (z,v,v) and (v,w,w) are steps in Q. Hence it follows from axiom $(s5)$ that there exists a vertex t such that (z,t,x) and (w,t,x) are steps in Q, and thus in G. This implies that t is a common neighbor of z and w in G with $d(t,x) = d(z,x) - 1 = d(u,x) - 3$ as well as $d(t,x) = d(w,x) - 1 = d(u,x) - 2$. This impossibility shows that $d(v,x) > d(w,x)$, so that $d(v,x) = d(u,x)$.

Now it follows that (u,v,v) and (v,w,w) are steps in Q, but (u,v,x) and (w,v,x) are not. So, by axiom $(s5)$, we can find a vertex v' with (u,v',x), $(w,v',x) \in Q$. Then (u,v',x) is a step in the step system of G, whence $d(v',x) = d(u,x) - 1$. Moreover v' is a common neighbor of u and w. Now we replace v by v' and apply the previous arguments on u, v', w, x instead of on u, v, w, x. This settles Case 2 and concludes the proof of the Theorem. ■

Let H be the graph consisting of two disjoint 4-cycles and an additional vertex v adjacent to one vertex on each of the 4-cycles, say to u on the one cycle and to w on the other cycle. We obtain a sinkless orientation of H in the following way: orient the two 4-cycles so that they become directed cycles, and orient the two edges incident with v towards the 4-cycles. Let G be the graph consisting of the disjoint union of H and yet another 4-cycle C. Orient C to make it a directed cycle as well. Thus we obtain a sinkless orientation of G from the sinkless orientations of H and C. Let Q be the signpost system constructed from G with its sinkless orientation (as constructed at the end of Section 3). Then each of the two components of G is a median graph, whence also a modular graph. On the other hand, let x be any vertex in C. Then (u, v, v) and (v, w, w) are in Q, but (u, v, x) and (w, v, x) are not in Q. Since v is the only common neighbor of u and w, we do not find a vertex t such that (u, t, x) and (w, t, x) are both in Q. That is, Q does not satisfy axiom $(s5)$, so that it is *not* a modular signpost system.

In the next section we will see that, for an 'if and only if' situation, we need connectivity of the underlying graph of the signpost system.

As soon as one has modular signpost systems the question arises whether there is a "median analogue". Consider the following axiom:

$(s6)$ if $(u, v, v), (v, w, w) \in Q$ with $u \neq w$, then there exists at most one $x \in V$ such that $x \neq v$ and $(u, x, x), (x, w, w) \in Q$, for u, v, w, x in V.

In terms of the underlying graph axiom $(s6)$ 'forbids' $K_{2,3}$ in G.

A *median signpost system* is a modular signpost system satisfying axiom $(s6)$. The difference between modular and median graphs is the unicity of medians. For median signpost systems we could replace axioms $(s5)$ and $(s6)$ by a single axiom. But we prefer the situation where median signpost systems are modular signpost systems satisfying an additional axiom.

Theorem 6. *Let Q be a median signpost system with underlying graph G. Then each component of G is a median graph.*

Proof. By Theorem 5 it follows that each component of G is a modular graph. Axiom $(s6)$ guarantees that medians of triples of vertices are unique, which completes the proof. ∎

The same signpost system above serves as an example that the condition that all components of the underlying graph are median graphs is not enough to guarantee that the signpost system itself is median.

5. Modular and Median Graphs

Let Q be a signpost system on the finite set V with $|V| \geq 2$. Let G be the underlying graph of Q. In this section we will study the connected case. It turns out that in the present case connectivity follows from the axioms $(s1)$ up to $(s5)$ together with the following additional axiom:

$(s7)$ if $(u,v,x),(v,w,x),(u,y,y),(y,w,w) \in Q$, then (u,y,x), $(y,w,x) \in Q$, for u, v, w, x, $y \in V$.

Lemma 7. *Let Q be a modular signpost system on V satisfying axiom $(s7)$ with G as its underlying graph. Let $x \in V$, and let H be a component of G. If $v_0, \ldots, v_m$ are vertices in H, $m \geq 1$, with*

$$(v_i, v_{i+1}, x) \in Q \quad \textit{for each } i, 0 \leq i < m,$$

then $d(v_0, v_m) = m$, where d denotes the distance in H.

Proof. Let $P \subseteq Q$ be defined as in Lemma 4. Then P is the step system of H. We proceed by induction on m. The case $m = 1$ follows from the definition of underlying graph. The case $m = 2$ follows from Lemma 3. Let $m \geq 3$. Let $v_0, \ldots, v_m$ be vertices in H with (v_0, v_1, x), (v_1, v_2, x), $\ldots$, $(v_{m-1}, v_m, x) \in Q$. We want to prove that $d(v_0, v_m) = m$. Assume, to the contrary, that $d(v_0, v_m) \neq m$. By the induction hypothesis we have $d(v_0, v_{m-2}) = m - 2$ and $d(v_0, v_{m-1}) = m - 1$. So $d(v_0, v_m) \leq m - 1$. Since H is a modular graph, it is bipartite. Since $d(v_{m-1}, v_m) = 1$, we get $d(v_0, v_m) = m - 2$. Now (v_{m-1}, v_{m-2}, v_0) and (v_{m-1}, v_m, v_0) are steps in H, so that (v_{m-2}, v_{m-1}, v_0), $(v_m, v_{m-1}, v_0) \notin Q$. Since $(v_{m-2}, v_{m-1}, v_{m-1})$, $(v_{m-1}, v_{m-1}, v_m) \in P$, axiom $(s5)$ implies that there exists a vertex t in V such that (v_{m-2}, t, v_0), $(v_m, t, v_0) \in Q$. Then, by definition of G, $v_{m-2}t$ is an edge in G, whence in H. Hence $(v_{m-2}, t, v_0), (v_m, t, v_0)$ are in P and therefore,

$$d(v_0, t) = m - 3. \tag{3}$$

Since (v_{m-2}, v_{m-1}, x), (v_{m-1}, v_m, x), (v_{m-2}, t, t), $(t, v_m, v_m) \in Q$, axiom $(s7)$ implies that $(v_{m-2}, t, x) \in Q$. Since we have (v_0, v_1, x), (v_1, v_2, x), $\ldots$, $(v_{m-3}, v_{m-2}, x) \in Q$, it follows from the induction hypothesis that $d(v_0, t) = m - 1$, which contradicts (3). ∎

Lemma 8. *Let Q be a modular signpost system on V satisfying axiom $(s7)$ with G as its underlying graph. Then G is connected.*

Proof. Assume, to the contrary, that G is disconnected. Let H be an arbitrary component of G. Then there exists a vertex x of G outside H. Choose a vertex u_0 in H. Then, obviously, $x \neq u_0$. So, by axiom $(s3)$, we can find a vertex u_1 with (u_0, u_1, x) in Q. Then u_0 and u_1 are adjacent in G, whence they both lie in H. In particular we have $u_1 \neq x$. Similarly, we can find an infinite sequence $u_1, u_2, u_3, \ldots$ in H such that

$$(u_n, u_{n+1}, x) \in Q \text{ for all } n = 0, 1, 2, \ldots.$$

Since H is finite, there exist i and j with $0 \leq i < j$ such that $u_i = u_j$ and $u_i, \ldots, u_{j-1}$ are mutually distinct. From Lemma 1 and $(s2)$ it follows that $j > i + 2$, so that $\{u_i, \ldots, u_{j-1}\}$ induces a cycle in H. Since H is bipartite this cycle is even. Hence $j = i + 2m$, for some positive integer m. Clearly, $d(u_i, u_{i+m+1}) < m$. But by Lemma 7 we have $d(u_i, u_{i+m+1}) = m + 1$. This contradiction completes the proof. ∎

Let V be a finite nonempty set. We now present the main results of our paper.

Theorem 9. *Let Q be a signpost system on V, and let G be its underlying graph.*

- (a) *If Q is a modular signpost system satisfying axiom $(s7)$, then G is a modular graph and Q is its step system.*
- (b) *If G is a modular graph, then Q is a modular signpost system satisfying axiom $(s7)$ and Q is the step system of G.*

Proof. (a) Let Q be a modular signpost system satisfying axiom $(s7)$. From Lemma 8 it follows that G is a connected graph, so that G is a modular graph, by Theorem 5. By Lemma 4, Q is the step system of G.

(b) Let G be a modular graph, so that G is connected. Let u, v, w, x be vertices in V with $(u, v, v), (v, w, w) \in Q$, $(u, v, x), (w, v, x) \notin Q$ and $u \neq w$. From the fact that $(u, v, v), (v, w, w) \in Q$ it follows that v is a common neighbor of u and w. Since G is bipartite, $u \neq w$ implies that $d(u, w) = 2$. From $(u, v, x), (w, v, x) \notin Q$ it follows that $d(v, x) \geq d(u, x), d(w, x)$, so that $v \notin I(u, w, x)$. Moreover, we have $d(u, x) - 1 \leq d(w, x) \leq d(u, x) + 1$. In particular, this means that u is not between w and x and w is not between u and x.

Hence, by modularity, there exists a vertex t in $I(u, w, x)$. This implies that t is a common neighbor of u and w with $d(t, x) = d(u, x) - 1 = d(w, x) - 1$. So, G being the underlying graph of Q, we have $(u, t, x), (w, t, x) \in Q$. That is, Q satisfies axiom ($s5$) and is a modular signpost system. By Lemma 4, Q is the step system of G. It is easily verified that Q also satisfies axiom ($s7$). ■

Theorem 10. *Let Q be a signpost system on V, and let G be the underlying graph of Q.*

(a) *If Q is a median signpost system satisfying axiom ($s7$), then G is a median graph and Q is its step system.*

(b) *If G is a median graph, then Q is a median signpost system satisfying axiom ($s7$) and Q is the step system of G.*

Proof. Let Q be a median signpost system satisfying axiom ($s7$). By the previous theorem, we only need to check unicity of medians. But this is a simple consequence of axiom ($s6$).

On the other hand, if G is a median graph, then by the previous theorem Q is modular and it is the step system of G. It is straightforward to check that Q also satisfies axiom ($s6$). ■

To conclude the paper we rephrase these two theorems from the perspective of graphs. This provides us with a new characterization of modular graphs as well as a new characterization of median graphs.

Corollary 11. *Let $G = (V, E)$ be a graph. Then G is a modular graph if and only if there exists a modular signpost system Q satisfying axiom ($s7$) such that $G = G_Q$.*

Corollary 12. *Let $G = (V, E)$ be a graph. Then G is a median graph if and only if there exists a median signpost system Q satisfying axiom ($s7$) such that $G = G_Q$.*

By now, median-type graphs are abundant in the literature: pseudo-modular graphs, pseudo-median graphs, quasi-median graphs, weakly modular graphs, etcetera, etcetera. It may be interesting to continue the study of signpost systems by searching for weaker axioms than ($s5$) up to ($s7$) to obtain "pseudo-modular signpost systems", and so on. This could be a first step in finding the axiom(s) characterizing the graphic signpost systems.

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