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WEAK k-RECONSTRUCTION OF CARTESIAN PRODUCTS

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Abstract

By Ulam's conjecture every finite graph G can be reconstructed from its deck of vertex deleted subgraphs. The conjecture is still open, but many special cases have been settled. In particular, one can reconstruct Cartesian products.

We consider the case of k-vertex deleted subgraphs of Cartesian products, and prove that one can decide whether a graph H is a k-vertex deleted subgraph of a Cartesian product G with at least $k + 1$ prime factors on at least $k + 1$ vertices each, and that H uniquely determines G.

This extends previous work of the authors and Sims. The paper also contains a counterexample to a conjecture of MacAvaney.

Keywords: reconstruction problem, Cartesian product, composite graphs.

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1. Introduction

In [15] Ulam asked whether a graph G is uniquely determined up to isomorphism by its maximal subgraphs, that is, the graphs $G_x = G \setminus x$ obtained from G by deleting a vertex x and all edges incident with it. The answer is negative for infinite graphs [4]. For finite graphs the question is still open and has become known as Ulam's conjecture. Many partial results have been found. For example, Dörfler [1] proved the validity of this conjecture for finite nontrivial Cartesian products, that is, graphs which are the Cartesian product of at least two nontrivial factors. Actually a Cartesian product of at least two nontrivial factors is already uniquely determined by any one of its vertex deleted subgraps. This has first been shown by Sims [13] and has been presented in terms of the semistability of Cartesian products in [14]. For a different approach see [9].

Recently products of graphs have become popular objects of investigation from the algorithmic point of view [7]. In this vein Feigenbaum and Haddad have studied the problems of minimal Cartesian product extensions and maximal Cartesian product subgraphs of arbitrary graphs. Such problems arise in the design of computer networks and multiprocessing machines. Both problems were shown to be NP-complete [3]. We will consider the following problems in this paper:

1. Given a graph G' that is the result of the deletion of k vertices from a Cartesian product G, reconstruct G. (*Weak k-reconstruction*).
2. Given a graph G', decide whether it is possible to extend the graph to a Cartesian product by addition of k vertices and edges that are incident with at least one of the added vertices.

For the case $k = 1$, both problems were solved by Imrich and Žerovnik [9]. They showed that arbitrary nontrivial Cartesian products (finite or infinite) can be uniquely reconstructed, up to isomorphism, from an arbitrary vertex deleted subgraph. An $O(mn(\Delta^2 + m \log n))$ algorithm that reconstructs nontrivial Cartesian products from single vertex deleted subgraphs is presented in [5]. (As usual, n denotes the number of vertices, m the number of edges, and Δ the maximal degree of the vertices of a graph.)

In this paper we prove that a graph G is (up to isomorphism) uniquely determined by any one of its k-vertex deleted subgraphs if it has at least $k+1$ prime factors on at least $k+1$ vertices each (Theorem 1). We believe that the reconstruction can be effected in polynomial time. This does not contradict the NP-completeness results of [3], because in our case the given graph must be an induced graph of the resulting graph, whereas in the case of minimal Cartesian product extensions the addition of arbitrary edges is permitted.

MacAvaney conjectured [11] that a connected composite graph $G_1 \Box G_2$, where G_1 and G_2 have more than two vertices, is uniquely determined by any one of its two-vertex deleted subgraphs. This conjecture is stronger than our result, but unfortunately not true, as the counterexample of Figure 1 shows. It is due to Klavžar and can be extended to arbitrarily large counterexamples (see Figure 2).

2. Preliminaries

We assume familiarity with general graph theoretic concepts, but will introduce some basic notation and concepts pertaining to the Cartesian product. For a more detailed introduction we refer to [7].

We will consider finite undirected graphs without loops or multiple edges and write $V(G)$ for the vertex set of a graph G and $E(G)$ for its edge set. $E(G)$ will be considered as a set of unordered pairs $xy = \{x, y\}$ of distinct vertices of G.

$N(v)$ denotes the neighborhood of the vertex v, that is, the set of all vertices adjacent to v.

The *Cartesian product* of two graphs G_1 and G_2 is the graph $G_1 \Box G_2$ with vertex set $V(G_1) \times V(G_2)$, where $(x_1, x_2)(y_1, y_2) \in E(G_1 \Box G_2)$ whenever $x_1y_1 \in E(G_1)$ and $x_2 = y_2$, or $x_2y_2 \in E(G_2)$ and $x_1 = y_1$.

The Cartesian product of a K_2, i.e. the complete graph on two vertices, by itself is a square, $K_2 \Box K_2 \Box K_2$ is the cube and $K_2 \Box K_2 \Box K_2 \Box \cdots \Box K_2$ a hypercube. Other examples of Cartesian products are prisms (products of cycles by K_2) and grid graphs (products of paths).

The Cartesian product is commutative, associative, and has the one-vertex graph K_1 as a unit. A product of several factors will be denoted by $G = \Box_{i \in I} G_i$. It is connected if and only if every factor is.

We can consider the vertices of $G = \Box_{i \in I} G_i$ as vectors $x = (x_1, x_2, \ldots, x_{|I|})$ of length $|I|$. Moreover, two vertices x, y of G are adjacent if there exists an index $k \in I$ such that $x_i = y_i$ for all $i \neq k$ and $\{x_k, y_k\} \in E(G_k)$. Such an edge e is called a *G_k-edge.* For simplicity we will also say that e has *color* k with respect to the product decomposition $G = \Box_{i \in I} G_i$ of G.

For a vertex x of $G = \Box_{i\in I} G_i$, we call x_i the *projection* of x into the i-th factor G_i. In symbols, $x_i = p_i(x)$. Analogously one defines the projection of a subset of $V(G)$ into $V(G_i)$ or the projection of a subgraph of G into G_i.

The distance between two vertices x, y of a graph G will be denoted by $d_G(x, y)$ or simply $d(x, y)$. It is well known that

$$d_G(x, y) = \sum_{i\in I} d_{G_i}(x_i, y_i)$$

if $G = \Box_{i\in I} G_i$. It is not hard to see that any two shortest paths between two vertices x and y of a connected product G have the same number of edges of each color (see, for example [6]). If x and y differ in ℓ coordinates, than there exist at least ℓ vertex-disjoint shortest paths between x and y. In this case the coordinates in which x and y do not differ are identical for every vertex on any of the shortest paths between x and y.

A subgraph H of a graph G is called *convex* in G if all shortest paths of G between any two vertices of H are already in H. If this condition is satisfied only for paths of length 2, we speak of 2-*convexity*. It is easy to see that X is convex in Z if X is convex in Y and Y convex in Z.

A subgraph H of G is called *isometric* in G if $d_H = d_G$ on H. Convex subgraphs are isometric.

Now we define the *product relation* σ on the edge set $E(G)$ of the product $G = \Box_{i\in I} G_i$. We say that two edges e, f are in the relation $\sigma(\Box_{i\in I} G_i)$ if they have the same color with respect to the product representation $G = \Box_{i\in I} G_i$ of G. Clearly $\sigma(\Box_{i\in I} G_i)$ is an equivalence relation and depends on the product decomposition of G. For example, a cube can be represented in three ways as a product of a square by a K_2. Every one of these representations induces a different edge-coloring.

It is known that among all product relations of G there exists a finest one [12], which we will denote by $\sigma(G)$. All factors of this representation are *indecomposable*, or *prime*, as we will also call them. This decomposition is the so-called prime factorization of G. It is unique up to isomorphisms and up to the order of factors. We say that the Cartesian product has the *unique factorization property.*

Note that a graph P is prime if it is nontrivial, that is, different from K_1, and if $P = G \Box H$ implies that either G or H is K_1.

Decomposable graphs will be called *composite.* In this paper, a *Cartesian product graph* will always denote a composite graph.

Let $G = \Box G_i$. Then $G_k^a = \{v \mid v_i = a_i, i \neq k\}$ is called a G_k–*layer* through the vertex $a \in G$. If G_k is connected, then the G_k*-layers* are the connected components of the subgraph of G that consists of all edges of color k. Such layers are convex in G.

A subgraph H in $G = \Box G_i$ is a d-box in G if it is representable in the form $H = \Box p_i(H)$, where d of the factors $p_i(H)$ are nontrivial and convex in G_i and the others are one-vertex graphs. Note that for a d-box H the number of σ-equivalence classes $\sigma(H)$ is at least d and $\sigma(H) \subseteq \sigma(G)|_H$ because of the unique factorization property.

Now we define three relations Θ, τ and δ on $E(G)$ and describe their role in the prime factorization of Cartesian products. Let $e = xy \in E(G)$ and $f = x'y' \in E(G)$ be two edges of G. We say that e and f are in relation Θ, in symbols $e\Theta f$, if $d(x,x') + d(y,y') \neq d(x,y') + d(x',y)$. Two edges e and f are in relation τ if they are incident and if there is no chordless square spanned by e and f. We also set $e\tau e$. Thus τ is reflexive, but not necessarily transitive. Finally, the edges e and f are in relation δ if either $e\tau f$ or if they are opposite edges of a chordless square.

These relations are symmetric and reflexive, but in general not transitive. We denote their transitive closures by Θ^*, τ^*, and δ^*. From the definition it easily follows that any pair of incident edges which belong to distinct δ^* classes span a unique chordless square. We say that the relation δ^* has the *square property.*

Feder [2] showed that $\sigma = (\tau \cup \Theta)^*$. Imrich and Žerovnik extended this result to infinite graphs [8] and showed that σ is the convex hull of δ^*.

We will also need the restriction of relations to subgraphs. Let S be a subgraph of G. Then $\sigma(G)|_S$ denotes the restriction of the relation $\sigma(G)$ to S, or, more precisely, to the edge-set $E(S)$ of S.

Finally, for $X \subseteq V(G)$, G_X denotes the subgraph of G induced by the vertex set $V(G) \setminus X$. If $X = \{x\}$ we simply write G_x instead of $G_{\{x\}}$. For $|X| = k$, G_X is a k-vertex deleted subgraph.

3. Primality and Unique Reconstruction

Let $G = \Box_{i=1}^{k+1} G_i$ be a Cartesian product of $k+1$ factors with at least three vertices each, and X a set of k vertices of G.

Lemma 1. *Let H be a d-box in G with $d \geq k+1$. Then $S = H \setminus X$ is isometric in G_X, that is,*

$$d_S = d_{G_X}|_S,$$

and 2*-convex in* G_X*.*

Proof. Let x and y be arbitrary vertices of S.

Assume first that they differ in $k+1$ coordinates. Then there are at least $k+1$ disjoint shortest paths between x and y in G. The deleted vertices cannot be on all shortest paths, hence

$$d_S(x,y) = d_{G_X}(x,y) = d_G(x,y).$$

Now assume that x and y differ in $\ell < k+1$ coordinates. If there is a shortest path of length $d_G(x,y)$ in S then there is nothing to prove. Therefore we can assume that all disjoint shortest paths are "broken" by the vertices of X. Because H is a d-box in G with $d \geq k+1$, there are at least $k+1-\ell$ pairs of vertices x_i, y_i in H adjacent to x and y, respectively, which differ from x and y in exactly one of the $k+1-\ell$ coordinates common to x and y in H. The shortest paths between these pairs of vertices x_i and y_i are disjoint and cannot all be broken by $k-\ell$ vertices, therefore

$$d_{G_X}(x,y) = d_G(x,y) + 2$$

and

$$d_S(x,y) = d_{G_X}(x,y).$$

Furthermore, S is 2-convex in G_X, since H is convex in G and since there is no 2-path in $G \setminus H$ between vertices of S. ■

Lemma 2. *Let* S *be a* 2*-convex isometric subgraph of an arbitrary graph* G*, then*

$$\sigma(S) \subseteq \sigma(G)|_S.$$

Proof. From the fact that the distances in S are the same as in G and the definition of Θ we see that $\Theta(S) = \Theta(G)|_S$. Any pair of edges in S which are in relation τ are clearly in relation τ in G (otherwise S would not be 2-convex in G). Therefore $\tau(S) \subseteq \tau(G)|_S$ and because of $\sigma = (\tau \cup \Theta)^*$ the assertion follows. ■

From now on we will assume that $G = \Box G_i$ is a Cartesian product of at least $k+1$ prime factors on at least $k+1$ vertices each. Clearly, for any j,

$V(G_j) \setminus p_j(X) \neq \emptyset$, because $|V(G_j)| > |X|$. Furthermore, since $k > 1$, for any j and any $x \in V(G_j) \setminus p_j(X)$ the inverse image $p_j^{-1}(x)$ of x is composite; in fact, it is isomorphic to the product of the other factors.

The lemmas will be used in the proof of our main theorem. Another consequence of the lemmas is the following interesting result on the primality of G_X. Since it will not be used in the proof of Theorem 1, the proof of Proposition 1 will be given in the last section.

Proposition 1. *Let G be a Cartesian product graph of at least $k+1$ prime factors on at least $k+1$ vertices each and $X \subseteq V(G)$, $|X| = k$. Then G_X is prime.*

Our assumptions imply that for each i there is at least one box of the form

$$S_i = H_i \Box (\Box_{j \neq i} G_j).$$

There is at least one set of such boxes S_i ($i \in I$) in G_X, such that $\cap_{i \in I} S_i \neq \emptyset$. To see this, take a vertex v such that $p_i(v) \notin p_i(X)$ for every $i \in I$ and construct S_i as a convex maximal Cartesian product subgraph containing $p_i^{-1}(p_i(v))$. We call such a set of boxes a *box skeleton* of G in G_X.

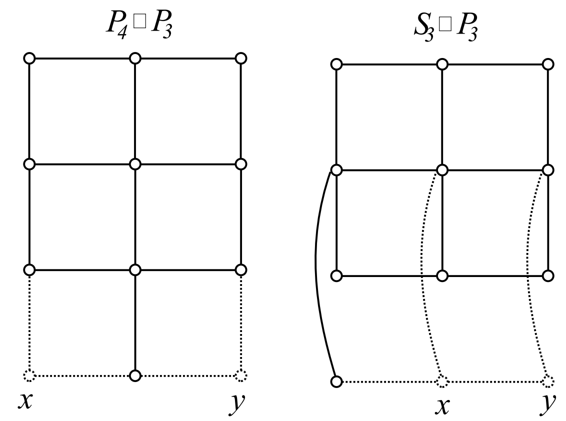

Figure 1: A counterexample to MacAvaney's conjecture

We are now ready to prove the main result of this paper.

Theorem 1. *Let G be a Cartesian product graph with at least $k+1$ prime factors on at least $k+1$ vertices each and $G' = G_X$ the graph induced by*

$V(G) \setminus X$, $|X| = k$. If G'' is a Cartesian product with at least $k+1$ factors on at least $k+1$ vertices each such that G' is an induced subgraph of G'' and $|V(G'')| = |V(G')| + k$, then $G'' \simeq G$.

Proof. The case $k = 1$ was proved in [9]. Thus, let $k > 1$. The proof is effected by the following construction.

1. Find a maximal box skeleton $\{S_i\}$.
2. For all $i \in I$ compute $\sigma(S_i)$.
   By Lemma 2,
   $$\sigma(S_i) \subseteq \sigma(G)|_{S_i}.$$
   This takes account of the fact that the $H_i \subseteq G_i$ may have more than one equivalence class.
3. Compute the transitive closure, say R, of the union of the $\sigma(S_i)$. In other words, R is the equivalence relation $(\cup_{i \in I}\sigma(S_i))^\star$ on $S = \cup_{i \in I} S_i$. For each factor G_i of G, there is a G_i-layer in S and all edges of this layer are in the same equivalence class of R, therefore $\sigma(G)|_S = R$.

Because of the unique factorization property, any extension of R to G'' satisfying the square property yields a product relation on graph isomorphic to G. ■

For $k = 2$ this partially solves the conjecture of MacAvaney, that a connected Cartesian product $G_1 \Box G_2$, where G_1 and G_2 have more than two vertices, is uniquely determined by any of its two vertex deleted subgraphs. It should be noted that MacAvaney does not require the factors to be prime, so there was hope that the conjecture held despite Theorem 1. However, this is not the case, as the counterexample of Figure 1 due to Klavžar [10] shows. In fact, there exists an infinite family of counterexamples (see Figure 2). We pose the following problem.

Problem 1. Is it true that any connected product graph G with $k \geq 2$ prime factors on more than $\max\{3, k\}$ vertices each is uniquely determined by each of its k-vertex deleted subgraphs?

The properties required are perhaps too weak. The reason for our choice is that we hoped to design an algorithm which would reconstruct graphs enjoying the properties given in Problem 1.

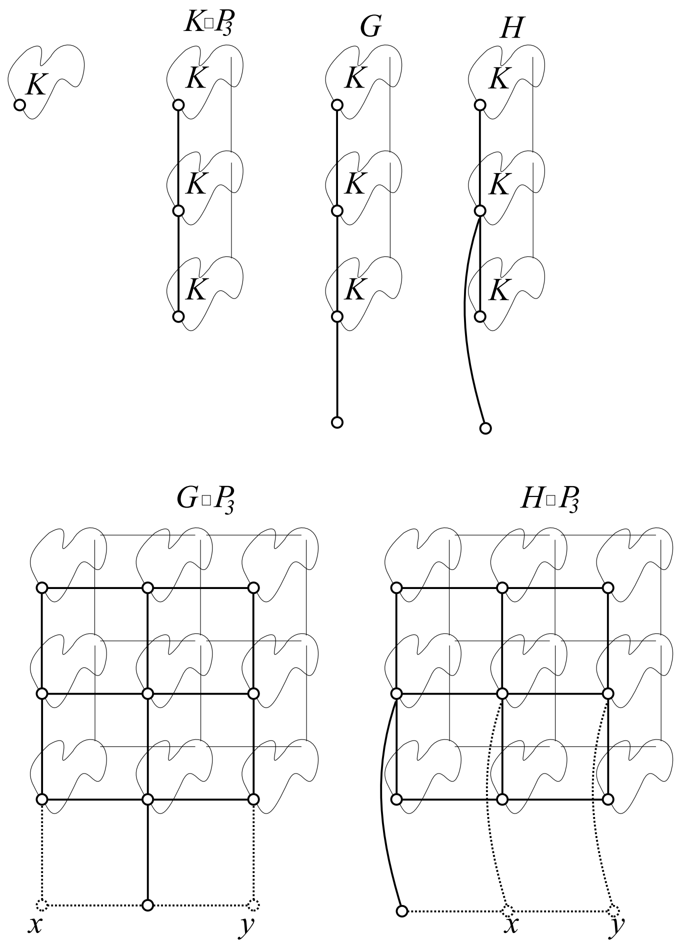

Figure 2. $G\Box P_3 \setminus \{x, y\} \simeq H\Box P_3 \setminus \{x, y\}$

As one of the referees suggested, it is likely that it is possible to reconstruct graphs under weaker conditions. For example, it might be true that a graph with k factors on more than $\max\{3, k\}$ vertices each is uniquely determined by each of its $((k+1)^{k-1} - 1)$-vertex deleted subgraphs! We have no counterexample.

Another possibility to strengthen the conjecture is to weaken either the condition on the number of factors or the condition on the size of factors.

4. Proof of Proposition 1

First a lemma:

Lemma 3. *Prime factors of G are subgraphs of prime factors of G_X. In symbols,*

$$\sigma(G)|_{G_X} \subseteq \sigma(G_X)$$

Proof. Take an arbitrary edge $e = uv$ from G_X. Without loss of generality, we can assume that e lies in a G_1-layer of G. In G there is at least one G_1-layer G_1^x that does not meet X. Let $e' = u'v'$ be the edge of G_1^x where u' has the same first coordinate as u and v' has the same first coordinate as v. Clearly, $p_1(e) = p_1(e')$.

Since G is the Cartesian product of $k+1$ factors, there exist at least k shortest paths $P \subset G$ from u to u'. Hence, there exist at least k minimal subgraphs $P \Box K_2$ (say $B_1, B_2, \ldots, B_k$) in G which connect e and e'.

We consider two cases:

1. Suppose there exists a subgraph B_i which does not intersect the vertex set X. Because any pair of incident δ^*-nonequivalent edges in G can span only one chordless square (square property of δ^*), $e\delta^* e'$ on G_X and therefore e and e' are $\sigma(G_X)$-equivalent.

2. All subgraphs B_i meet the vertex set X. Then each such subgraph contains exactly one vertex from X. Take any subgraph B_i and denote it by B. On B there exists a vertex $z \in X$ and two edges $g = pq$ and $g' = rs$ in B with $p_1(g) = p_1(g') = p_1(e)$ and qz, $sz \in E(G)$ (see Figure 3). Now we have three subcases:

(a) There is no vertex $x \notin B$ adjacent to p and r (see Figure 3a). Then $g\tau f \tau f' \tau g'$ and therefore $g\delta^* g'$ on G_X.

(b) There is a vertex $x \notin B$ adjacent to p and r and no vertex $y \notin B$ adjacent to x, q and s (see Figure 3b). Then $g\tau f \delta f' \tau g'$ and therefore $g\delta^* g'$ on G_X.

(c) There is a vertex $x \notin B$ adjacent to p and r and a vertex $y \notin B$ adjacent to x, q and s (see Figure 3c). Let $w \in B$ be the common neighbor of p, r and z in G. Then replacement of w and z by x and y in B gives rise to a subgraph of G_X isomorphic to $P \Box K_2$, in contradiction to the assumption of Case 2.

Because any pair of incident δ^*-nonequivalent edges in G can span only one chordless square (square property of δ^*), $e\delta^* g\delta^* g'\delta^* e'$ on G_X and therefore e and e' are $\sigma(G_X)$-equivalent.

Since the edges of G_1^x are $\sigma(G_X)$-equivalent by Lemma 2 this means that any edge in an arbitrary G_1-layer is $\sigma(G_X)$-equivalent to an edge in G_1^x. This proves the lemma. ∎

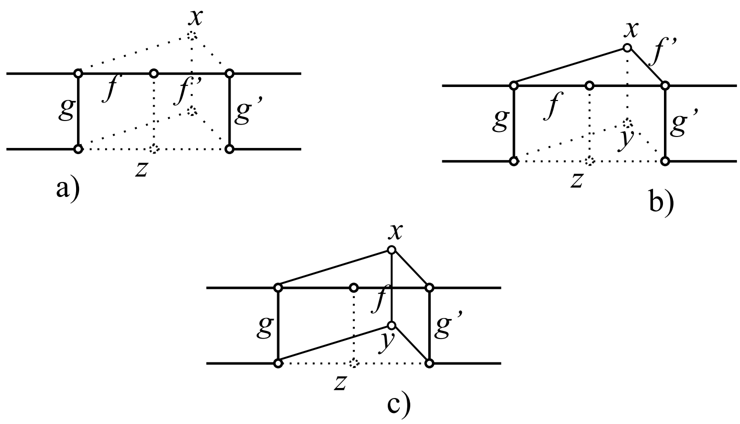

Figure 3. Three possible subcases

Now we can prove Proposition 1:

Proof. We proceed by induction on the number of missing vertices. The case $k = 1$ was proved in [9]. We therefore assume that any Cartesian product of $l+1$ prime factors on $l+1$ vertices each and with at most l (but at least one) missing vertex is prime for $l < k$. Let

$$G = (G_1 \Box G_2) \Box (\Box_{i=3}^{k+1} G_i) = G_I \Box G_{II}$$

be a factorization of G. We consider two cases:

1. If $|p_I(X)| = 1$, let $p_I(X) = v$. From [9] we infer that $G_I \setminus v$ is prime. By Lemma 1 all subgraphs $G_I^x \setminus X$ are 2-convex and isometric in G_X, and by Lemma 2

$$\sigma(G_I^x \setminus X) \subseteq \sigma(G_X)|_{G_I^x \setminus X}.$$

By Lemma 3 the edges in the subgraphs $G_I^x \setminus X$ are $\sigma(G_X)$-equivalent. Let σ_I denote the σ-class of edges in G_I-layers.

Select an arbitrary $\sigma(G)$-class $\beta \neq \sigma_I$.

Each (prime) factor of G has at least $k+1$ vertices. Since $|X| = k$, there exists a G_{II}-layer G_{II}^u in G disjoint to the vertex set X and adjacent to at least one vertex in X. Because $|G_{II}^u| > k$ and since each vertex is incident with all $\sigma(G_{II})$-classes there exists at least one edge $e \in G_{II}^u$ in β with one endpoint x adjacent to $x' \in p_I^{-1}(v) \setminus X$ and the other endpoint y adjacent to $y' \in p_I^{-1}(v) \cap X$.

There is a unique chordless square in G spanned by the incident edges e and xx'. In G_X they do not span such a square. Therefore e and xx' are $\sigma(G_X)$-equivalent, whence β and σ_I are in the same $\sigma(G_X)$-class. Therefore all edges of G_X are $\sigma(G_X)$-equivalent, i.e., G_X is prime.

2. If $|p_I(X)| > 1$ we consider the following two subcases.

(a) If $k = 2$, we can choose the factorization $G = (G_1 \square G_2) \square G_3 = G_I \square G_{II}$ such that two vertices from X differ in the third coordinate. As there are two G_I-layers with a missing vertex, the edges of all subgraphs $G_I^x \setminus X$ are $\sigma(G_X)$-equivalent by Lemmas 1, 2 and 3. Because there exists at least one G_{II}^u-layer in G_X, where the vertex u is adjacent to X in G, we infer the primality of G_X as in the first case.

(b) If $k > 2$, there is at least one G_{II}-layer (say G_{II}^x) with at least one and at most $k-2$ vertices from X.
(As there are at least two G_{II}-layers intersecting X, at least one of them, say G_{II}^x, contains at most $\lfloor \frac{k}{2} \rfloor \leq k-2$ vertices from X.) By the induction hypothesis, $G_{II}^x \setminus X$ is prime and therefore the edges of all subgraphs $G_{II}^y \setminus X$ are $\sigma(G_X)$-equivalent by the same arguments as before.
Let σ_{II} denote the $\sigma(G_X)$-class of edges in the subgraphs $G_{II}^y \cap X$.
Let $\beta \neq \sigma_{II}$ be an arbitrary $\sigma(G)$-class.
Each prime factor of G has at least $k+1$ vertices. Since $|X| = k$, there exists a G_I-layer G_I^u in G disjoint to X and adjacent to at least one vertex in X. Because $|G_I^u| > k$ and since each vertex is incident to all $\sigma(G_I)$-classes there exists at least one edge $xy = e \in G_I^u$ that is in β and where x is adjacent to a vertex $x' \in G_{II}^v \setminus X$ and y is adjacent to a vertex $y' \in G_{II}^v \cap X$.
There is a unique chordless square in G spanned by the edges $xy = e$ and xx' because they are incident. In G_X they do not span such a

square. Therefore they are $\sigma(G_X)$-equivalent, whence β and σ_{II} are the same $\sigma(G_X)$-class. Thus all edges of G_X are $\sigma(G_X)$-equivalent, i.e., G_X is prime. ■

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