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ARBOREAL STRUCTURE AND REGULAR GRAPHS OF MEDIAN-LIKE CLASSES

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Abstract

We consider classes of graphs that enjoy the following properties: they are closed for gated subgraphs, gated amalgamation and Cartesian products, and for any gated subgraph the inverse of the gate function maps vertices to gated subsets. We prove that any graph of such a class contains a peripheral subgraph which is a Cartesian product of two graphs: a gated subgraph of the graph and a prime graph minus a vertex. Therefore, these graphs admit a peripheral elimination procedure which is a generalization of analogous procedure in median graphs. We characterize regular graphs of these classes whenever they enjoy an additional property. As a corollary we derive that regular weakly median graphs are precisely Cartesian products in which each factor is a complete graph or a hyperoctahedron.

Keywords: median graph, tree, gatedness, amalgam, periphery, regular graph.

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1. Introduction

Classes of graphs that are regarded as median-like usually admit an elimination procedure. The first result of this type and a model for several others was an expansion procedure for median graphs due to Mulder [11, 12], cf. [9, 10]. Later a similar concept of gated amalgamation procedure was

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generalized to several median-like classes [2, 4, 5, 8]. Gatedness is a strong condition that can be used in a quite general setting [15], and amalgamation concepts appear to be useful also in other branches of graph theory, cf. [6].

All graphs considered in this paper are undirected, simple and finite. A subset U of the vertex-set $V(G)$ of a graph G is called *gated* if for every $x \in V(G)$ there exists a vertex u in U such that for any $v \in U$, u lies on a shortest path from x to v. If, for some x, such a vertex u in U exists, it is unique, and is called the *gate* $\alpha_U(x)$ of x in U. Note that $\alpha_U(x)$ is always the closest to x among vertices of U. A subgraph of G induced by a gated subset is called a *gated subgraph* of G. Clearly for any graph G its singletones and G itself are gated subgraphs (which we call *trivial* gated subgraphs). For each gated subset U the mapping $\alpha_U : V(G) \to U$ which maps x to $\alpha_U(x)$ will be called the *gate function* with respect to U.

Now, a graph G is said to be the (*gated*) *amalgam* of two gated subgraphs G', G'' if $G' \cup G'' = G$, $G' \cap G'' \neq \emptyset$, and there are no edges between $G' - G''$ and $G'' - G'$. Note that $G' \cap G''$ is also a gated subgraph. In other words, we say that G is obtained by an *amalgamation* along the common gated subgraph $G' \cap G''$ of G' and G''.

The Cartesian product $G = G_1 \square G_2 \square \ldots \square G_k$ of graphs $G_1, G_2, \ldots, G_k$ has the set of vertices $V(G) = V(G_1) \times V(G_2) \times \ldots \times V(G_k)$, and two vertices $u = (u_1, u_2, \ldots, u_k)$, $v = (v_1, v_2, \ldots, v_k)$ of G are adjacent if there exists j $(1 \leq j \leq k)$ such that $u_j v_j \in E(G_j)$ and $u_i = v_i$ for all $i \in \{1, 2, \ldots, k\} \setminus \{j\}$. By π_{G_i} we denote the natural projection to a factor G_i, that is $\pi_{G_i}(u) = u_i$.

It seems reasonable to require that a median-like class enjoys the following properties:

(G) closed for gated subgraphs,

(A) closed for gated amalgamations,

and often we wish the class to be

(C) closed for Cartesian multiplication.

Another interesting property is shared by several median-like classes:

(I) inverses of gate functions are gated, that is $\alpha_H^{-1}(x)$ is a gated subset for any gated subgraph H and any $x \in V(H)$.

We say that a class of graphs $\mathcal{A}$ is a GACI *class* if it enjoys the above four properties. Property (I) is characteristic for *fiber-complemented graphs* as introduced by Chastand [8]. (Alternatively, a graph enjoys (I) if for any gated subgraph K of H a subgraph induced by $\alpha_H^{-1}(V(K))$ is gated.) Hence any GACI class is a subclass of fiber-complemented graphs. Moreover, fiber-complemented graphs are a GACI class in our sense, cf. [8]. Another two examples of GACI classes are quasi-median and weakly median graphs, cf. [2, 5].

A graph G of a GACI class $\mathcal{A}$ is called *prime* if it cannot be represented as a Cartesian product of two smaller graphs of $\mathcal{A}$ nor as a gated amalgam of two such graphs. It was proved in [8] that the following property of a class of graphs $\mathcal{A}$

(P) for any prime graph G of $\mathcal{A}$ the only gated subgraphs in G are the trivial ones,

is characteristic for fiber-complemented graphs. Hence, alternatively we could say that $\mathcal{A}$ is a GACI class of graphs if it enjoys properties (G), (A), (C) and (P).

In the following section we prove that a GACI class admits a special amalgamation procedure by which we obtain any graph of this class from Cartesian products of prime graphs, where this procedure is performed in an arboreal (alias tree-like) way. More precisely, for any gated subgraph W of such a graph, $G - W$ contains a so-called peripheral subgraph which is a Cartesian product of a prime graph minus a vertex with a gated subgraph of G. The corresponding procedure is a generalization of the peripheral elimination procedure in median graphs where peripheral sets (the so-called sets U_{ab}) can be contracted in each step [13], see also [9, 10]. In Section 3 we characterize regular graphs of a GACI class that has an additional (weak) property; that is, we prove that they are precisely Cartesian products of regular prime graphs of this class. As a corollary regular weakly median graphs are characterized as Cartesian products, of which each factor is a complete graph or a hyperoctahedron. Finally, we rediscover regular pseudo-median graphs [3].

2. Arboreal Structure

Unless stated otherwise a graph will always mean a connected graph. Let G be a graph of a GACI class $\mathcal{A}$. A subgraph U of G is called *peripheral* if there exist graphs G', G'', H and a prime graph P of $\mathcal{A}$ such that G is a gated amalgam of G' and G'' along H, where $G' \cong H \Box P$ and $U = G' - H$.

Note that a peripheral subgraph does not necessarily belong to $\mathcal{A}$, although by a removal of a peripheral subgraph we obviously get a smaller graph from $\mathcal{A}$, namely G''. Clearly, if G is a gated amalgam of two boxes (a *box* is a Cartesian product of prime graphs) then it contains at least two peripheral subgraphs.

On the left-hand side of Figure 1 a graph G that is a gated amalgam of G' and G'' is shown schematically. On the right-hand side of the figure G' is depicted as a Cartesian product of a prime graph P (isomorphic to $K_4 - e$) and a gated subgraph H (isomorphic to P_4). In this example the peripheral subgraph is isomorphic to $P_4 \Box K_3$.

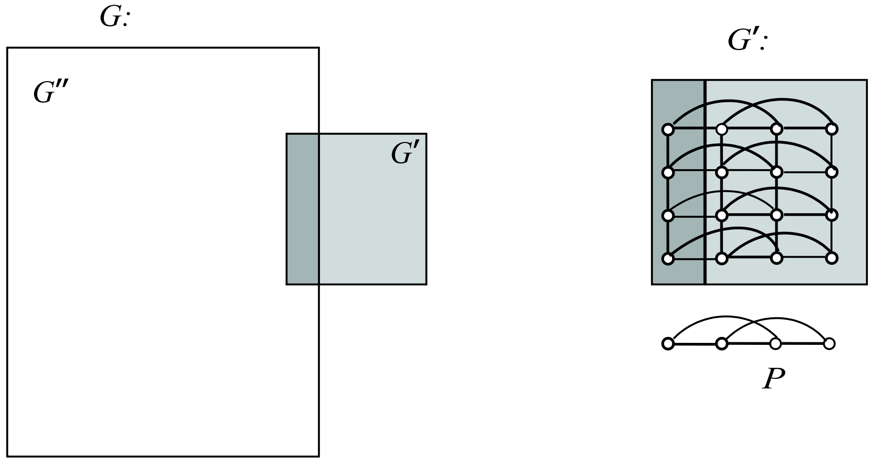

Figure 1. A periphery of a graph

One of the basic properties of gated subgraphs is that the intersection of two such subgraphs is also gated. Another result is also a straightforward consequence of the definition.

Lemma 2.1. *Let A, B, C be graphs such that B is a gated subgraph of A and C is a subgraph of B. Then C is gated in B if and only if C is gated in A.*

The following easy result can be found in [14] or [7].

Lemma 2.2. *Let $G = G_1 \Box \cdots \Box G_l$ be a Cartesian product of connected graphs. Then K is gated in G if and only if $K = K_1 \Box \ldots \Box K_l$, where each K_i is gated in G_i.*

From now on we shall use a simplified notation to avoid too many symbols. That is, a symbol for a set of vertices will occasionally mean a graph induced by these vertices, and a symbol for a graph will in some cases stand also for the set of its vertices.

Let G be a connected graph. A subset X of the vertex-set $V(G)$ is a *cutset* of G if $G - X$ is a disconnected graph.

Lemma 2.3. *Let X be a cutset of G. Then X induces a gated subgraph if and only if G is an amalgam of gated subgraphs along the subgraph induced by X.*

Proof. Indeed if G is a gated amalgam of two subgraphs, such that their intersection is induced by the vertices of X, then X is obviously a gated cutset.

Conversely, if X is a gated cutset, let $H_1, \ldots, H_k$ be the connected components of $G - X$. We first prove that each $H_i \cup X =: G_i$ is a gated subgraph. Let u be a vertex of $G - G_i$. As X is gated there exists a unique vertex $\alpha_X(u) \in X$ which lies on a shortest path from u to any vertex $v \in X$. Let $w \in H_i$. As X is a cutset there exists a vertex $x \in X$ which lies on a shortest path from u to w. Since $\alpha_X(u)$ lies on a shortest path from u to x, it also lies on a shortest path from u to w. In other words $\alpha_{G_i}(u) = \alpha_X(u)$. We easily conclude that G is an amalgam of two gated subgraphs, for instance $H_1 \cup X$ and $H_2 \cup \ldots \cup H_k \cup X$. ■

Lemma 2.4. *Let G be a gated amalgam of G_1, G_2 along G_0, and X_0 a cutset of G_1. If $X_0 \cap G_0 \neq \emptyset$, then $\alpha_{G_0}^{-1}(X_0 \cap G_0)$ is a cutset of G.*

Proof. As X_0 is a cutset of G_1, there exist distinct components X_1, X_2 of $G_1 - X_0$. Suppose that $\alpha_{G_0}^{-1}(X_0 \cap G_0)$ is not a cutset of G. Then there must be path P in G between vertices of X_1 and vertices of X_2. Clearly, there is no such path lying entirely in G_1, thus P goes through $G_2 - G_0$. Hence $X_1 \cap G_0$, $X_2 \cap G_0$ are both nonempty, thus $X_0 \cap G_0$ is a cutset of G_0 Thus there exist consecutive vertices x, y on P that belong to $G_2 - G_0$ such that $\alpha_{G_0}(x) \in X_1$ and $\alpha_{G_0}(y) \in X_2$. Moreover, $d(\alpha_{G_0}(x), x) = d(\alpha_{G_0}(y), y) =: k$ because x and y are adjacent. As $\alpha_{G_0}(x)$ is a gate for x in G_0, it lies on a shortest path from x to $\alpha_{G_0}(y)$. Thus $k+1 = d(x, \alpha_{G_0}(y)) = k + d(\alpha_{G_0}(x), \alpha_{G_0}(y))$. This implies the adjacency of $\alpha_{G_0}(x)$ and $\alpha_{G_0}(y)$ which contradicts the assumption that they are in different components of $G_1 - X_0$. ■

Lemma 2.5. *Let G be a gated amalgam of G_1, G_2 along G_0. Then a subgraph H of G is gated if and only if either*

- *H is a gated subgraph of $G_1 - G_0$, or*
- *H is a gated subgraph of $G_2 - G_0$, or*
- *$H \cap G_1$ is a gated subgraph of G_1 and $H \cap G_2$ is a gated subgraph of G_2.*

The latter gated subgraph H is contained in a subgraph of G induced by $\alpha_{G_0}^{-1}(H \cap G_0)$.

Proof. Let G be a gated amalgam of G_1, G_2 along G_0, and suppose that H is gated in G. If H has no vertices in G_0 then obviously one of the first two possibilities occurs. If H has a vertex in G_0 then without loss of generality it is enough to prove that $H \cap G_1$ is a gated subgraph of G_1.

Suppose $H \cap G_1$ is not gated in G_1. Then there exists a vertex $x \in V(G_1)$ for which there is no gate in $H \cap G_1$. Since G_1 is gated, it is convex (that is, every shortest path between vertices of G_1 lies in G_1), hence x does not have a gate in H also with respect to G, a contradiction with H being gated.

Now the proof of the converse. For the first two cases note that by Lemma 2.1 a gated subgraph of $G_i - G_0$ is also gated in G. For the last case it is enough to prove that a vertex $x \in V(G_1)$ has a gate in H. We claim that $\alpha_H(x)$ is the same as the gate for x in $H \cap G_1$ which exists by the gatedness of $H \cap G_1$. Let u be a vertex in $(G_2 - G_0) \cap H$. As G_2 is gated, $\alpha_{G_2}(x)$ lies on a shortest path from x to u. If $\alpha_{G_2}(x) \in H$ then $\alpha_{H \cap G_1}(x)$ lies on a shortest path from x to $\alpha_{G_2}(x)$, and we are done. If $\alpha_{G_2}(x) \notin H$ then $\alpha_H(\alpha_{G_2}(x)) \in G_0$ and clearly it is equal to $\alpha_{H \cap G_1}(x)$. Thus H is a gated subgraph of G.

The last sentence of the theorem follows from the fact that gated subgraphs are convex. Indeed, suppose that $x \in H \cap (G_1 - G_0)$ and $x \notin \alpha_{G_0}^{-1}(H \cap G_0)$. Then $\alpha_{G_0}(x) \notin H$ hence every shortest path from x to vertices of $H \cap G_0$ would contain a vertex outside H, a contradiction to convexity of H. ∎

The following lemma implies the main result.

Lemma 2.6. *Let $\mathcal{A}$ be a GACI class, G a graph of $\mathcal{A}$ which is not a box, and W a proper gated subgraph of G. Then $G - W$ contains a peripheral subgraph of G.*

Proof. We prove this by induction on the number of vertices of G. Let G be a gated amalgam of G_1, G_2 along G_0, and W a gated subgraph of G. By Lemma 2.5, W can be one of the three types of subgraphs.

Case 1. W is in G_1 or G_2.
We may assume without loss of generality that W is in G_1. Hence W is disjoint with $G_2 - G_0$. Suppose first that G_2 is not a box. As G_2 is smaller than G, the claim holds in G_2. Hence $G_2 - G_0$ contains a peripheral subgraph which is clearly in $G - W$, and which is obviously peripheral also in G. If G_2 is a box then G_0 is also a box, a subproduct of G_2 (using Lemma 2.2). Obviously then $G_2 - G_0$ contains a peripheral subgraph and we are done in this case.

Case 2. W contains vertices of $G_1 - G_0$ and $G_2 - G_0$.
By Lemma 2.5, W is a gated amalgam of gated subgraphs W_1 and W_2 along W_0 where $W_i = W \cap G_i$ for $i = 0, 1, 2$. By the last statement of this lemma W is a subgraph of $\alpha_{G_0}^{-1}(W_0)$. Moreover, as we need to prove the existence of a peripheral subgraph in $G - W$, it is enough to prove that there is a peripheral subgraph in a complement of the whole $\alpha_{G_0}^{-1}(W_0)$ (which is gated by (I)). So we may assume that W is equal to $\alpha_{G_0}^{-1}(W_0)$. As G_1 is smaller than G, there exists a peripheral subgraph U_1 in G_1 which is disjoint from W_1. By definition of peripheral subgraphs there exist gated subgraphs H_1, H_1', H_1'' of G_1, and a prime graph P, such that G_1 is a gated amalgam of H_1' and H_1'' along H_1, where $H_1' \simeq H_1 \Box P$ and $U_1 = H_1' - H_1$. If $H_1 \cap G_0 = \emptyset$ then U_1 is peripheral also in G and the proof is done. So let us assume that $H_1 \cap G_0 \neq \emptyset$. As $H_1 \cap G_0$ is gated in G, the set $\alpha_{G_0}^{-1}(H_1 \cap G_0)$ induces a gated subgraph of G. By Lemma 2.4, the set $\alpha_{G_0}^{-1}(H_1 \cap G_0)$ is a cutset of G (because H_1 induces a cutset of G_1 and $H_1 \cap G_0 \neq \emptyset$). Hence using Lemma 2.3 we derive that G is a gated amalgam of two gated subgraphs along their intersection $\alpha_{G_0}^{-1}(H_1 \cap G_0)$. As $U_1 \cap W_1 = \emptyset$, one of these two gated subgraphs obviously contains $W = \alpha_{G_0}^{-1}(W_0)$, so this case is reduced to Case 1, and the proof is complete. ■

Theorem 2.7. *Every graph of a* GACI *class can be reduced to a box* (*Cartesian product of prime graphs*) *by a succesive removal of peripheral subgraphs.*

Note that this operation is a generalization of the removal of pendant vertices in trees. Moreover by Lemma 2.6 we can start the procedure of removals with any peripheral subgraph in any part of G with a gated complement, by which the name arboreal structure of these graphs is justified.

The inverse operation of the removal of peripheral subgraphs is closely related to the peripheral expansion for median graphs as defined by Mulder [13]. In fact it is its generalization, so let us formulate it explicitly (we prefer a term peripheral amalgamation here). Let G be a graph of $\mathcal{A}$, and H a gated subgraph of G. Then the *peripheral amalgamation* of G with respect to H and a prime graph P of $\mathcal{A}$ is the graph obtained as amalgam of G and $H \Box P$ along their common gated subgraph H.

Corollary 2.8. *A graph G belongs to a* GACI *class $\mathcal{A}$ if and only if G can be obtained from K_1 by successive peripheral amalgamations from prime graphs of $\mathcal{A}$.*

3. Regular Graphs of Median-Like Classes

Lemma 3.1. *Let G and H be graphs and $G \Box H$ the Cartesian product of G and H. Then $G \Box H$ is regular if and only if G and H are regular.*

Proof. Obviously, the degree of a vertex of $G \Box H$ is the sum of degrees of its coordinate vertices. Let $G \Box H$ be regular and $x = (x_1, x_2) \in G \Box H$ with $\deg(x_1) = k, \deg(x_2) = l$. Hence for any $a \in H$, the vertex (x_1, a) must have degree $k + l$, hence $\deg(a) = l$. The converse is obvious. ■

By assuming an additional (weak) condition for a GACI class we can nicely characterize regular graphs of such a class.

Theorem 3.2. *Let $\mathcal{A}$ be a* GACI *class of graphs such that for any prime graph P*

(R) *if $|V(P)| - 1$ vertices have the largest degree in P, then P is regular. Then the subclass of regular graphs of $\mathcal{A}$ consists precisely of the Cartesian products of regular prime graphs.*

Proof. Clearly, the Cartesian products of regular prime graphs of a GACI class $\mathcal{A}$ are regular graphs of $\mathcal{A}$.

For the proof of the converse let G be a regular graph of the GACI class $\mathcal{A}$ enjoying (R). If G is a box then by Lemma 3.1 it is regular only if its factors (prime graphs) are regular. Thus we may assume that G is not a box. Then by Lemma 2.6, G contains a peripheral subgraph U. Thus there exist gated subgraphs H, H', H'' of G and a prime graph P such that G is a gated amalgam of H' and H'' along H, where $H' \simeq H \Box P$ and $U = H' - H$. By looking at H' as the Cartesian product of H and P, set $\lambda = \pi_P(x)$, where x is a vertex of $H' \cap H''$. Suppose H is not regular. Then there exist vertices $a, b \in U$, such that $\pi_P(a) = \pi_P(b) \neq \lambda$, which have different degrees in G, contrary to the assumption that G is regular. Hence H must be regular, and from the same reason all vertices in $P - \lambda$ must have the same degrees. Thus using (R) we derive that P is either regular or λ is the unique vertex with the largest degree in P. But then observe that the vertices of H are adjacent to vertices of $G - H'$, thus G is not regular, a contradiction. ■

Bandelt and Chepoi introduced weakly median graphs as a common generalization of quasi-median and pseudo-median graphs. Unlike pseudo-median graphs, weakly median graphs are closed for the Cartesian product operation, and as both classes they are also closed for gated amalgamation. Moreover, they are a GACI class in our sense, as it is shown in the following characterization from [2].

Theorem 3.3. *A nontrivial graph G is a weakly median graph if and only if it can be obtained by successive gated amalgamations from Cartesian products of the following prime graphs: complete graphs with* 2 *vertices,* 5*-wheels, induced subgraphs (which contain a K_4 or an induced* 4*-wheel) of hyperoctahedra, and* 2*-connected K_4- and $K_{1,1,3}$-free bridged graphs. The latter bridged graphs are exactly the graphs which can be realized as plane graphs such that all inner faces are triangles and all inner vertices have degrees larger than* 5*. A weakly median graph is prime if and only if it does not have any proper gated subgraphs other than singletones.*

Recall that a *hyperoctahedron* (alias a cocktail party graph) is a graph on $2n$ vertices, obtained from a complete graph K_{2n} by deletion of edges forming a perfect matching. A graph is *bridged* [1, 6] if it does not contain any isometric (distance preserving) cycle of length greater than 3, that is each cycle of length greater than 3 has a shortcut.

In the aim of applying Theorem 3.2 to weakly median graphs the only remaining question is whether prime weakly median graphs enjoy also the property (R). This is trivial in the case of regular prime weakly median graphs (these are precisely all complete graphs on at least two vertices and all hyperoctahedra on at least 6 vertices). The property is also obvious for wheels. On the other hand, if P is an induced subgraph of a hyperoctahedron and is not regular then note that it has at least two vertices with not maximum degree (these are any two vertices of this graph which are not adjacent). Finally, it is straighforward to check that the bridged graphs introduced in Theorem 3.3 also enjoy (R) (by [2, Lemma 7] there are at least two vertices of degree 2 or 3, and except for K_3 and $K_4 - e$ all graphs of this type have at least one vertex of degree at least 4). Combining these observations with Theorem 3.2 we derive

Corollary 3.4. *A graph G is a regular weakly median graph if and only if $G = G_1 \Box \cdots \Box G_k$, $k \geq 1$, where each G_i is a hyperoctahedron or a complete graph.*

From Corollary 3.4 we easily rediscover a characterization of regular pseudo-median graphs due to Bandelt and Mulder [3]. Recall that a pseudo-median graph is indecomposable (with respect to gated amalgamation) if and only if it is a Cartesian product $Q_n \Box H$, where H is either a wheel, or a snake, or an induced subgraph of a hyperoctahedron, cf. [4] (note that snakes are bridged graphs introduced in Theorem 3.3 with no inner vertices).

Corollary 3.5. *A graph G is a regular pseudo-median graph if and only if G is $Q_n \Box H$ for $n \geq 0$ where H is either a hyperoctahedron or a complete graph.*

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