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ON NON-z(mod k) DOMINATING SETS

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Abstract

For a graph G, a positive integer $k, k \geq 2$, and a non-negative integer with $z < k$ and $z \neq 1$, a subset D of the vertex set $V(G)$ is said to be a *non-z* (mod k) *dominating set* if D is a dominating set and for all $x \in V(G)$, $|N[x] \cap D| \not\equiv z \pmod{k}$.

For the case $k = 2$ and $z = 0$, it has been shown that these sets exist for all graphs. The problem for $k \geq 3$ is unknown (the existence for even values of k and $z = 0$ follows from the $k = 2$ case.) It is the purpose of this paper to show that for $k \geq 3$ and with $z < k$ and $z \neq 1$, that a non-z(mod k) dominating set exist for all trees. Also, it will be shown that for $k \geq 4$, $z \geq 1$, 2 or 3 that any unicyclic graph contains a non-z(mod k) dominating set. We also give a few special cases of other families of graphs for which these dominating sets must exist.

Keywords: dominating set, tree, unicyclic graph.

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1. Introduction

In 1989, Sutner [7] studied the following problem. Suppose each vertex of a graph is equipped with an indicator light and a switch. If the switch of a vertex is switched, the light of that vertex and all its neighbors will change from off to on or from on to off. Sutner asked whether there always was a set of switches which when switched resulted in the lights being on at all the vertices. He referred to this problem as the All-Ones Problem. He in fact showed that there such a set of switches exist for all graphs.

Transforming this problem into standard graph theory notation; does there exist a subset of vertices, D, so that $|N[x] \cap D| \equiv 1 (\text{mod } 2)$ (alternatively $|N[x] \cap D| \not\equiv 0 (\text{mod } 2)$) for every vertex x in G. Here $N[x]$ is the closed neighborhood of x, which consists of x and all its neighbors in G. Note, another way to define this is that $|N[x] \cap D|$ odd for all x in G. Sutner's proof used cellular automata and extends an earlier proof of Galvin who supplied an algorithmic proof to the case of trees. In 1996, Caro [3] gave a graph theoretic proof. These results will be further extended and generalized here.

Recently, the problem of generalizing this to values k other than 2 was raise and published in GRAPHNET by Caro. He suggested the question of whether for any graph G, there exists a set D so that $|N[x] \cap D| \not\equiv 0$ (mod k) for every vertex x in G. We explore a more general problem. For a graph G, a positive integer k, $k \geq 2$, and a non-negative integer with $z < k$, a subset D of the vertex set $V(G)$ is said to be a *non-z* (mod k) *dominating set* if D is a dominating set and for all $x \in V(G)$, $|N[x] \cap D| \not\equiv z$ (mod k).

As noted above, for the case $k = 2$ and $z = 0$, these sets exist for all graphs. In fact, this result implies that for any k even and $z = 0$, these dominating sets exist. The problem for $k \geq 3$ is unknown for odd values of k. It is the purpose of this paper to show that for $k \geq 3$ and with $z < k$ and $z \neq 1$, that a non-z(mod k) dominating set exists for all trees. Also, it will be shown that for $k \geq 4$, $z \neq 1$, 2 or 3 that any unicyclic graph contains a non-z(mod k) dominating set. We also give a few special cases of other families of graphs for which these dominating sets must exist.

2. Non-z(mod k) Dominating Sets for Trees

Observe that if $z = 1$, then by considering the $G = K_{1,m}$ where m is a multiple of k, then it is obvious that G does not contain a non-z(mod k)

dominating set. Consequently, the condition on z in the next result is necessary.

Theorem 1. *Let T be a tree, $k \geq 2$ a positive integer, and $z \neq 1$, be a member of the cyclic group Z_k. Then T contains non-z(mod k) dominating set.*

Proof. Let T be any tree, we will present an algorithm which will produce a non-z(mod k) dominating set, which in fact is independent. The process will consist of two steps, first a labeling of the vertices of T and then the construction of the non-z(mod k) dominating set. Choose any vertex x in T and consider T as being rooted at x.

Bottom − Up Labeling of $V(G)$ — Starting with the endvertices and labeling up the tree as follows:

1. Label all endvertices positive.
2. A vertex is labeled neutral if at least one of its children is negative.
3. A vertex, with no negative children, is labeled negative if the number of its positive children is positive and non-z(mod k), and is labeled positive otherwise.

This labeling is clearly well defined, and each vertex is labeled positive, neutral or negative. Now using this labeling we give a process for finding a set D, which will be shown to be a non-z(mod k) dominating set.

Top − Down construction of D — Starting at the root and adding vertices, one at a time to D as follows:

1. The root is in D if and only if it is a positive vertex. Now considering the children of a vertex for which it has been decided whether it is in D.
2. A neutral vertex is never in D.
3. A positive vertex is added to D if and only if its parent is not in D.
4. A negative vertex is added to D if the number of vertices in the intersection of the closed neighborhood of its parent and D is equivalent to a (mod k).

Continue this process until a decision about the inclusion of every vertex has been completed. It remains to show that the set D is a non-z(mod k) dominating set. Let x be any vertex in T.

First suppose that the vertex x was a neutral vertex. Of course x is not in D and since x is neutral it follows from the labeling scheme, that x had at least one negative child. Depending on how many of x's positive neighbors

were included in D, either 0 or 1 of x's negative children would have been included in D, thus assuring that x is adjacent to some element of D and having $|N[x] \cap D| \not\equiv z \pmod{k}$.

Now suppose that x was a positive vertex. If x is in D then its parent and none of its children are included in D, so $|N[x] \cap D| = 1 \pmod{k}$, and since $z \neq 1$, $|N[x] \cap D| \not\equiv z \pmod{k}$. If x is not in D, then its parent is in D as well as all of its positive children, and again this implies that x is adjacent to some element of D and has $|N[x] \cap D| \neq z \pmod{k}$, since to have x labeled positive it had to have $z(\text{mod } k)$ positive children.

Now suppose that x was a negative vertex, note its parent is neutral. If x is in D then its parent and none of its children are included in D, so $|N[x] \cap D| \equiv 1 \pmod{k}$, and since $z \neq 1$, $|N[x] \cap D| \not\equiv z \pmod{k}$. If x is not in D, all of its positive children are in D and by the labeling process the number of its positive children is positive and non-z(modk), so it follows that x is adjacent to some element of D and has $|N[x] \cap D| \not\equiv z \pmod{k}$.

Hence it follows that D is a non-z dominating set. It is easy to see that D is independent. Suppose a parent and child were both in D. Neither is neutral, since neutrals are never in D. The child can not be negative, since by the labeling of the parent would be neutral. But, if the child is positive, it would only be included if the parent were not, hence D is independent. ■

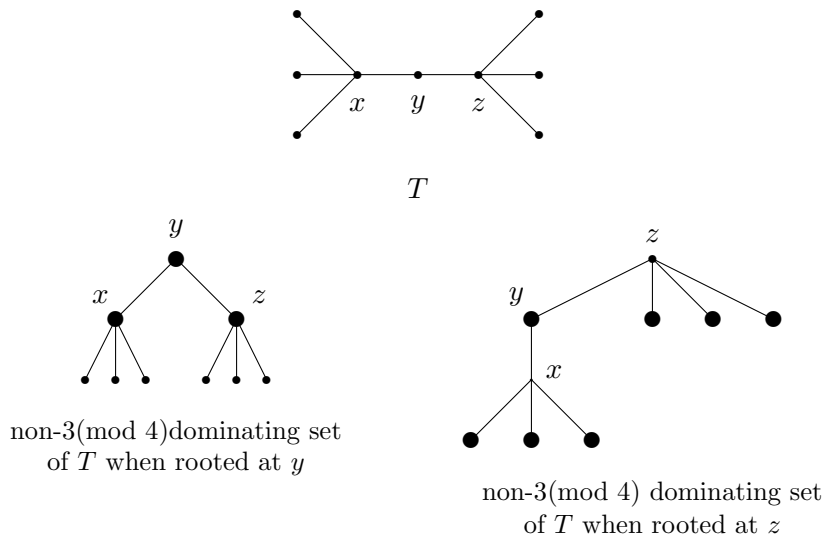

Figure 1. An example of a tree where the algorithm of Theorem 1 yields different sized sets when rooted at different vertices.

Figure 1 shows a tree for which the algorithm in Theorem 1 yields non-3(mod 4) dominating sets of vastly different sizes when the tree is rooted at different vertices. This tree can be generalized by taking larger stars with 3(mod 4) endvertices on each end or by picking alternate z and k and stars of appropriate order on the ends.

The algorithm in Theorem 1 is clearly linear in the number of vertices in T since the labeling only requires consideration of the children of any vertex, while the choice for inclusion in D only requires consideration of the unique parent of any vertex.

As is often the case for dominating sets, it would be nice to find the smallest non-z(mod k) dominating set. Unfortunately, the example above indicates that the algorithm gives non-z(mod k) dominating sets of vastly differing sizes. In addition, since the algorithm always yields and independent set and the smallest non-z(mod k) dominating set is not necessarily independent, for example the double star, where the middle edge would be a smallest non-z(mod k) dominating set when $z \neq 2$, and smallest independent sets can be arbitrarily large, the algorithm is not able to yield the smallest non-z(mod k) dominating set in all cases. But the algorithm does yield a bound; for convenience let $\gamma_{z,k}(G)$ denote the order of the smallest non-z(mod k) dominating set in G. Also let $\alpha(G)$ denote the independence number of G, the order of the largest set of independent vertices in G.

Corollary 2. *If T is a tree, k a positive integer $k \geq 2$, $z < k$ and $z \neq 1$, a non-negative integer, then it follows that $\gamma_{z,k}(T) \leq \alpha(T)$.*

Proof. This follows since the non-z(mod k) dominating set constructed in Theorem 1 is a maximal independent set. ■

3. Non-z(mod k) Dominating Sets for Other Graphs

In the next result we present an alternative proof for establishing the existence of non-z(mod k) dominating sets in trees as long as $z \neq 1$ or 2. In addition, the process yields the following criteria; that any vertex can be distinguished and have a further restricted intersection with the dominating set.

Before proceeding we make a useful observation to help to construct these dominating sets.

Lemma 3. *For any graph G, with cut edge $e = uv$, let $u \in G_1$ and $v \in G_2$ be the graphs of $G - e$. If there exists a non-z(mod k) dominating subset D_1 of G_1 such that $|N[u] \cap G_1| \not\equiv (z-1) \pmod k$, and a is not in D_1, then for any non-z(mod k) dominating subset D_2 of G_2 it follows that $D_1 \cup D_2$ is a non-z(mod k) dominating subset of G.*

Proof. This follows since the vertex u, which has $|N[u] \cap D_1| \equiv i \pmod k$ with $i \neq z-1$ or z can be dominated by at most one more vertex, possibly v, when rejoining the two subgraphs. Of course the number of vertices dominating v, and all of the other vertices of G, remains the same, since u is not in D_1. ■

This lemma will be useful in the next two results.

Theorem 4. *For any tree T, x any vertex in T, positive integer k, $k \geq 3$, and $z \neq 1$ or 2, a member of the cyclic group Z_k, T contains a non-z(mod k) dominating set D of T such that $|N[x] \cap D| \not\equiv (z-1) \pmod k$.*

Proof. Proceed by induction on the order of T. It is easy to check that this is true for all trees of order less than six. Suppose the result is true for all trees of order at most n and let T be a tree of order $n+1$ and choose x to be any vertex in T.

Case 4.1. Suppose x is an endvertex of T.
Let y be the vertex of T adjacent to x and let D' be a non-z(mod k) dominating set of $T - x$ such that $|N[y] \cap D'| \not\equiv (z-1) \pmod k$.

Subcase 4.1.1. Suppose y is an element of D'.
The set D' is a non-z(mod k) dominating set of T such that $|N[x] \cap D'| \neq (z-1) \pmod k$, since $z \neq 2$.

Subcase 4.1.2. Suppose a is not an element of D'.
The set $D = D' \cup \{x\}$ is a non-z(mod k) dominating set of T such that $|N[x] \cap D| \not\equiv (z-1) \pmod k$.

Case 4.2. The vertex x is not an endvertex of T.
Let $T_1, T_2, \ldots, T_r$ be the trees in the forest $T - \{x\}$, with y_i in T_i the vertex of T_i adjacent to x. For convenience let $S = \{y1, y2, \ldots, y_r\}$. Further, let D'_i be a non-z(mod k) dominating set of T_i such that $|N[y_i] \cap D'_i| \not\equiv (z-1) \pmod k$. Note, by Lemma 3 above, for each i, the vertex y_i must be contained in D'_i, for otherwise, by induction, choosing any non-z(mod k) dominating subsets D_1 of $T - T_i$ with $|N[x] \cap D_1| \not\equiv (z-1) \pmod k$, it would follow

that the set $D = D_1 \cup D'_i$ would be a non-z(mod k) dominating subset of T such that $|N[x] \cap D| \not\equiv (z-1) \pmod{k}$ and the result would follow.

Subcase 4.2.1. $|S| \equiv z \pmod{k}$.
The set $D' = D'_1 \ldots D'_2 \cup \ldots \cup D'_r \cup \{x\}$ is a non-z(mod k) dominating set of T such that $|N[x] \cap D'| \not\equiv (z-1) \pmod{k}$.

Subcase 4.2.2. $|S| \equiv i \pmod{k}$ for some non-negative integer $i < k$ with $i \neq z-1$ or z.
It this case it follows that the set $D' = D'_1 \cup D'_2 \cup \ldots \cup D'_r$ is a non-z(mod k) dominating set of T such that $|N[x] \cap D'| \not\equiv (z-1) \pmod{k}$.

Subcase 4.2.3. $|S| \equiv z-1 \pmod{k}$ and $|S| > k-1$.
If $|S| > k-1$ then it must follow that $T' = T - (T_1 \cup T_2 \cup \ldots \cup T_{z-1} \cup x)$ is a non-empty subforest of T. Furthermore, note that $|T' \cap S| \equiv 0 \pmod{k}$. By induction, there is a non-z(mod k) dominating set of $T_1 \cup T_2 \cup \ldots \cup T_{z-1} \cup x$, say D^*, such that $|N[x] \cap D^*| \not\equiv (z-1) \pmod{k}$. Subsequently, it follows that the set $D' = D^* \cup D'_{j+1} \cup D'_{j+2} \cup \ldots \cup D'_r$ is a non-z(mod k) dominating set of T such that $|N[x] \cap D'| \not\equiv (z-1) \pmod{k}$.

It only remains to consider that case when $|S| = z-1$.

Subcase 4.2.4. $|S| = z-1$.

Claim. For $j = 1, 2, 3, \ldots, z-1$ the degree of y_j in T_j is $z-1$.

Suppose this is not the case. Without loss of generality, suppose $\deg(y_1) \not\equiv z-1 \bmod k$. Furthermore, assume that $\deg(y_1) \not\equiv z \pmod{k}$. Let u_1, $u_2, \ldots, u_m$ be the neighbors of y_1 in T_1. Consider the trees of $T_1 - y_1$. By induction we can find a non-z(mod k) dominating set of each of these trees, say D_i*, such that $|N[u_i] \cap D^*_i| \neq z-1 \pmod{k}$. By Lemma 3, u_i must be contained in D^*_i, but then it follows that $D^*_1 \cup D^*_2 \cup \ldots \cup D^*_m$ would form a non-z(mod k) dominating set of T_1, say D^*, such that $|N[y_1] \cap D*| \neq z-1 \pmod{k}$, with $y_1 \notin D^*$. Consequently yielding the desired set by the Lemma. Thus, we may assume for each $j = 1, 2, \ldots, z-1$ that $\deg_{T_j}(y_j) \equiv z$ or $z-1$.

Suppose $\deg_{T_j} y_1 \equiv z \pmod{k}$. Let D_1 be a non-z(mod k) dominating set of $T_1 - y_1$ that, by Lemma 3 above, contains all the neighbors of y_1 and let D_2 be a non-z(mod k) dominating sets of the tree $T - (T_1 - y_1)$ such that $|N[x] \cap D_2| \not\equiv z-1 \pmod{k}$. Also observe that $|N[x] \cap D_2| = 1$ or 2, since the degree of y_1 in $T - (T_1 - y_1)$ is 1. Subsequently, it follows that the set $D^* = D_1 \cup D_2$ is a non-z dominating set of T such that $|N[x] \cap D| \not\equiv z-1 \pmod{k}$.

Hence it must be the case that the degree of y_j in T_j is $= z-1 \pmod k$ for $j = 1, 2, 3, \ldots, z-1$. Suppose, for some j, the degree of y_j in T_j is at least $k+z-1$. Again without loss of generality, we may assume that $j = 1$. Let the neighbors of y_1 in T_1 be $z_1, z_2, z_3, \ldots, z_m$, with $m \equiv z-1 \pmod k$ and m at least $k+z-1$. In $T_1 - y_1$, for $j = z, z+1, \ldots, m$, let T_j^* be the tree containing z_j, and $T^* = T - (T_z^* \cup T_{z+1}^* \cup \ldots \cup T_m^*)$. By Lemma 3, for $j = z, z+1, \ldots, m$, we can find non-$z \pmod k$ dominating sets D_j^* of T_j^* containing z_j having $|N[z_j] \cap D_j^*| \not\equiv z-1 \pmod k$. Also, by induction a non-$z \pmod k$ dominating set of T^*, say D^*, such that $|N[x] \cap D*| \not\equiv z-1 \pmod k$. Now since $m-z+1 = 0 \pmod k$ it follows that the set $D_1 = D^* \cup D_k^* \cup D_{k+1}^* \cup \ldots \cup D_m^*$ is a non-$z \pmod k$ dominating set of T such that $|N[x] \cap D_1| \not\equiv z-1 \pmod k$. Hence the claim follows.

Thus, we have x, with neighbors $y_1, y_2, y_3, \ldots, y_{z-1}$ and subtrees $T_1, T_2, T_3, \ldots, T_{z-1}$ of $T - x$ and let $u_1, u_2, u_3, \ldots, u_{z-1}$ and x be all of the neighbors of y_1. In $T_1 - y_1$, let $T_{u_1}, T_{u_2}, T_{u_3}, \ldots$ and $T_{u_{k-1}}$ be the trees containing $u_1, u_2, u_3, \ldots$ and u_{k-1} respectively. By induction, for $t = 1, 2, \ldots, z-1$ we can find D_{u_t} such that D_{u_t} is a non-$z \pmod k$ dominating set of T_{u_t} with $|N[u_t] \cap D_{u_t}| \not\equiv z-1 \pmod k$ and further having $u_t \in D_{u_t}$. Also, for $j = 2, 3, \ldots$ and $z-1$ let D_j be a non-$z \pmod k$ dominating set of T_j with $|N[y_j] \cap D_j| \neq z-1 \pmod k$. Again, by Lemma 3, we may assume that y_j is in D_j. As a consequence, the set

$$D = D_{u_1} \cup D_{u_2} \cup \ldots \cup D_{u_{k-1}} \cup D_2 \cup D_3 \cup \ldots \cup D_{k-1}$$

is a non-$z \pmod k$ dominating sets of T with $|N[x] \cap D| \neq z-1 \pmod k$ and the theorem follows. ■

We now use this result to allow us to show that if G is a unicyclic graphs, then for most k and z, a non-$z \pmod k$ dominating set of G exists.

Theorem 5. *For any unicyclic graph G, positive integer k, $k \geq 4$, and non-negative integer with $z < k$ and $z \neq 1, 2$ or 3, G contains a non-$z \pmod k$ dominating set.*

Proof. Let G be a unicyclic graph of order n, we will assume it is connected, for otherwise we simply consider the components separately. Also, we may assume that for all unicyclic graphs of order less than n, they contain non-$z \pmod k$ dominating sets.

Case 5.1. Suppose G is a cycle.
Let D be any maximal independent set. This is certainly a non-z(mod k) dominating set since $|N[x] \cap D| = 1$ or 2 for all vertices of G and $z \neq 1$ or 2.

Case 5.2. Suppose G is not a cycle.
Let $V = \{v_1, v_2, \ldots, v_t\}$ be the vertices adjacent to vertices of the unique cycle a in G, and let $T_1, T_2, \ldots, T_t$ be the trees in $G - C$ such that v_i is in T_i (some may be isolated vertices.) By Theorem 4, there is a non-z dominating set D_i of T_i such that $|N[v_i] \cap D_i| \not\equiv z - 1 \pmod k$ and by Lemma 3 we may assume that for each i, v_i is in D_i. Suppose, for some vertex x on C, there are T neighbors of x in V, for $T \neq 0$ or $T \neq z-2, z-1$ or z(mod k). Let H_1 be the subtree of G with x and all the subtrees of its neighbors in $G - C$. Let H_2 be the subtree $G - H_1$. A non-z(mod k) dominating set of G results, by taking the union of a non-z(mod k) dominating set of H_1 which contains all the neighbors of x and does not contain x and any non-z mod k dominating set of H_2. Consequently, we may assume that every vertex u on C is adjacent to t_u vertices, where $t_u = 0$ or $t_u \equiv z-2, z-1$ or z(mod k).

Now let $D = D_1 \cup D_2 \cup \ldots \cup D_t \cup V(C)$. It follows that $|N[x] \cap D| \not\equiv z \pmod k$, for each x on C. Furthermore, since $|N[v_i] \cap D_i| \not\equiv z - 1$ or z(mod k), it follows that $|N[v_i] \cap D| \not\equiv z \pmod k$, hence D is a non-z(mod k) dominating set of G completing the proof. ■

Now we give a few special cases of other families of graphs for which these dominating sets exist. Before proceeding we give two definitions. For graphs G and H, we say that G is H-free if G does not contain an induced copy of H. A graph G is m-separable if the set of vertices of degree at least m is independent. For a subset of vertices A we will employ the notation $N(A)$ to denote the set of neighbors of the vertices of A.

Theorem 6. *Let k and m be a positive integers with $k \geq 3$ and $1 < m < k$. If a is a member of the cyclic group Z_k, with $z = 0$ or $z \geq m$, and if G is a $K_{1,m}$-free graph, then G contains a non-z(mod k) dominating set.*

Proof. Let G, k, m and z be as above. Choose D to be any maximal independent set of G. Since D is a maximal independent set, D is a dominating set, and since G is $K_{1,m}$-free, it follows that $|N[x] \cap D| \not\equiv z \pmod k$, for each x in G, and the result follows. ■

Theorem 7. *Let k and m be a positive integers with $k \neq 3$ and $1 < m < k$. If z is a member of the cyclic group Z_k, with $z = 0$ or $z \geq m+1$, and if G is an m-separable graph, then G contains a non-z(mod k) dominating set.*

Proof. Let G, k, m and z be as above. Note if $\delta(G) \leq m-1$ then $V(G)$ is a non-z(mod k) dominating set. If $\delta(G) \geq m$, then let A be the set of vertices having degree at least m. Since G is m-separable, A is an independent set. Let $D = A \cup (V(G) - N(A))$. We now show that D is a non-z(mod k) dominating set. Suppose $x \in D$ and $\deg(x) \geq m$. It follows that $|N[x] \cap D| = 1 \pmod{k}$. If $x \in D$ with $\deg(x) < m$ then $|N[x] \cap D| \not\equiv z \pmod{k}$. Finally, for any other vertex x of G, since the $\deg(x) < m$, it follows that $|N[x] \cap D| \not\equiv z \pmod{k}$, thus D is a non-z(mod k) dominating set. ■

4. Problems and Conclusion

Although these results yield non-z(mod k) dominating sets for trees and specific other families of graphs, the general problem of determining whether these sets exist for all graphs remains open. Of course, when these sets do exist, the problem of determining the order of the smallest such set also would be a worthwhile question to resolve. In addition, the exploration of relationships between other types of dominating sets and non-z(mod k) dominating sets would be of interest, as would establishing relationships between the minimum sizes of these sets.

Another possible direction of study is to relax the definition slightly. As is the case with "standard" domination, one could define an "open" non-z(mod k) dominating set to be a dominating set D so that for every vertex x in $V - D$, $|N[x] \cap D| \not\equiv z \pmod{k}$. Note that $V(G)$ vacuously becomes such a dominating set. So finding the minimum order is the problem. There are other related problems that arise from this definition, for example, note that these sets exist, even when $z = 1$. For m a multiple of k, and $G = K_{1,m}$ it is obvious that the only "open" non-1(mod k) dominating set, is $V(G)$. Are unions of these stars the only such graphs? Note, when $z \neq 1$ and G is not the empty graph, it is easy to see that an "open" non-z(mod k) dominating set that is properly contained in $V(G)$ exists. This follows since, if it were not the case then for each x in $V(G)$, $\deg(x) = z \pmod{k}$, otherwise $V - \{x\}$ would be the required set. But then for any edge xy in G, it follows that $V - \{x, y\}$ would be an "open" non-z(mod k) dominating set.

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