Discussiones Mathematicae
Graph Theory 23 (2003) 177–187

THE SIZE OF MINIMUM 3-TREES: CASES 0 AND 1 MOD 12

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Abstract

A 3-uniform hypergraph is called a minimum 3-tree, if for any 3-coloring of its vertex set there is a heterochromatic triple and the hypergraph has the minimum possible number of triples. There is a conjecture that the number of triples in such 3-tree is $\lceil \frac{n(n-2)}{3} \rceil$ for any number of vertices n. Here we give a proof of this conjecture for any $n \equiv 0, 1 \bmod 12$.

Keywords: tight hypergraphs, triple systems.

2000 Mathematics Subject Classification: 05B07, 05C65, 05D10.

1. Introduction

A 3-*graph* is an ordered pair of sets $G = (V, \Delta)$. The elements of V are called *vertices*. The elements of Δ are subsets of vertices of cardinality 3 and are called *triples*. Given a 3-graph $G = (V, \Delta)$ and a vertex v the trace

$Tr_G(v)$ of v in G is the graph with vertex set $V\setminus\{v\}$, and a pair $\{x,y\}$ is an edge of $Tr_G(v)$ if and only if $\{v,x,y\}$ is a triple of G.

A 3-*coloring* of a 3-graph is a surjective map from the vertex set onto a set of three elements. A 3-graph is said to be *tight* (see [1]) if any 3-coloring has a heterochromatic triple i.e., a triple whose vertices are colored differently. A tight 3-graph is called a 3-*tree* if whenever we delete a triple from it we obtain an untight 3-graph. Different 3-trees on n vertices may have a different number of triples. From the results of [4], we know that the maximum number of triples in any 3-tree is $\binom{n-1}{2}$. It is not difficult to show that the minimum number of triples in such a 3-tree is not less than $\lceil\frac{n(n-2)}{3}\rceil$. In [1] it was proved that this bound is sharp for any n of the form $\frac{p-1}{2}$ where p is a prime number, and it was conjectured that the bound is sharp for any n. In [2] the case when $n\equiv 3,4 \bmod 6$ was solved and in [3] a full proof for the case $n\equiv 2 \bmod 3$ is given.

Here we give the proof of the cases $n\equiv 0,1 \bmod 12$. The case $1 \bmod 12$ is solved via a generalization of a construction from [2].

2. The Case $0 \bmod 12$

In order to prove the conjecture for any n it is sufficient to construct a 3-tree with $\lceil\frac{n(n-2)}{3}\rceil$ triples. In this section we deal only with the case $n\equiv 0 \bmod 12$.

Let us consider the cyclic group $\mathbb{Z}_n=\{0,1,\ldots,n-1\}$, its elements are the vertices of the 3-graph H_n defined below.

Of course, we know how to add vertices. If $e=\{x_1,x_2,x_3\}$ is a triple and y is a vertex, then $e+y=\{x_1+y,x_2+y,x_3+y\}$. If F is any set of triples and S any set of vertices then $F+S=\{f+s|f\in F, s\in S\}$. It is important to observe that all operations must be interpreted in the appropriate cyclic group.

Denote by $\mathbb{A}_n=\{1,\ldots,\frac{n}{6}\}\subset\mathbb{Z}_n$ and $\mathbb{B}_n=\{1,\ldots,\frac{n}{12}\}\subset\mathbb{A}_n$.
For $a\in\mathbb{A}_n$ and $b\in\mathbb{B}_n$, let us consider the following triples:

$$\begin{aligned}
\varepsilon_a &= \{0, 2\tfrac{n}{3}, 2a\},\\
\zeta_b &= \{0,2,3-4b\},\\
\eta_b &= \{0, 2\tfrac{n}{3}+2b, 4b-1\}.
\end{aligned}$$

Those triples generate the set of triples of the 3-graph H_n i.e., any triple will be of the form ε_a+y or ζ_b+y or η_b+y where $y\in\mathbb{Z}_n$. Formally, denote

$$H_n = (\mathbb{Z}_n, (\{\varepsilon_a \mid a \in \mathbb{A}_n\} \cup \{\zeta_b, \eta_b \mid b \in \mathbb{B}_n\}) + \mathbb{Z}_n)\,.$$

Our purpose is to show that H_n is a 3-tree with $\frac{n(n-2)}{3}$ triples.

Proposition 1. *H_n has $\frac{n(n-2)}{3}$ triples.*

Proof. There are $n(\frac{n}{6} - 1) + \frac{n}{3}$ triples generated by ε_a. The number of triples generated by ζ_b and η_b is $\frac{n^2}{6}$. Those triples are all different and a straightforward calculation gives the result. ∎

Let us construct an auxiliary hypergraph. For this, let $m \equiv 0 \bmod 3$ and denote $\alpha_a = \{0, 2\frac{m}{3}, a\}$.

The hypergraph G_m is by definition $(\mathbb{Z}_m, \{\alpha_a \mid a \in \{1, \ldots, \frac{m}{3}\}\} + \mathbb{Z}_m)$.

Observe that the hypergraph generated by the set of even vertices in H_n contains a copy of $G_{n/2}$ and also the hypergraph generated by the set of odd ones by the automorphism $x \mapsto x + 1$ of H_n.

Lemma 2. *Let f be a non heterochromatic 3-coloring of G_m. Then, all the cosets of $\mathbb{Z}_m$ by the subgroup $\langle \frac{m}{3} \rangle \cong \mathbb{Z}_3$ are monochromatic.*

Proof. Denote $t = \frac{m}{3}$. Let f be a red-blue-yellow 3-coloring for which the lemma is false. Let $y \in \mathbb{Z}_m$, observe that for the 3-coloring $f + y : a \mapsto f(a + y)$ the lemma is also false. So we can suppose that $|f(\alpha_t)| = 2$, and $f(0) = f(-t) = R$ and $f(t) = B$. So for any $a \in \{1, \ldots, t\}$ we have

$$\left.\begin{array}{c} \alpha_a + t = \{t, 0, a + t\} \in G_m \\ \text{and } f(0) = R,\ f(t) = B. \end{array}\right\} \Rightarrow f\,(a + t) \neq Y,$$

$$\left.\begin{array}{c} \alpha_a - t = \{-t, t, a - t\} \in G_m \\ \text{and } f(-t) = R,\ f(t) = B. \end{array}\right\} \Rightarrow f\,(a - t) \neq Y.$$

Therefore, since any 3-coloring is a surjective map there must be an $x \in \{1, \ldots, t - 1\}$ such that $f(x) = Y$. In this case we have

$$\left.\begin{array}{c} \alpha_{t-x} + x = \{x, x - t, t\}\,,\ \alpha_x - t = \{-t, t, x - t\} \in G_m \\ \text{and } f(-t) = R,\ f(t) = B,\ f(x) = Y. \end{array}\right\} \Rightarrow f(x - t) = B,$$

$$\left.\begin{array}{r}\alpha_{t-x}+x+t=\{x+t,x,-t\},\ \alpha_x+t=\{t,0,x+t\}\in G_m\\ \text{and } f(-t)=f(0)=R,\ f(t)=B,\ f(x)=Y.\end{array}\right\}\Rightarrow f(x+t)=R$$

and this is a contradiction because $\alpha_t + x = \{x, x-t, x+t\} \in G_m$. ∎

Of course, the lemma is equivalent to the fact that any non heterochromatic 3-coloring of G_m factorizes through a 3-coloring of the quotient hypergraph $G_m/\langle\frac{m}{3}\rangle$, i.e., the 3-graph whose vertices are the cosets modulo $\langle\frac{m}{3}\rangle$ and the triples are the images of the triples in G_m by the natural map (see [1] for a more formal definition).

Let us prove a key property of the hypergraph H_n.

Lemma 3. *If f is a non heterochromatic 3-coloring of H_n, then f is surjective in the set of odd vertices or is surjective in the set of even vertices.*

Proof. For two vertices $x, y \in \mathbb{Z}_n$ define the distance between them as the minimal natural number d such that $(d \bmod n)+x = y$ or $(d \bmod n)+y = x$.

Let f be a non heterochromatic 3-coloring of H_n. Both cosets, $\langle 2\rangle$ and $\langle 2\rangle + 1$ can not be monochromatic.

Suppose that $f(\langle 2\rangle+1) = Y$, then $f(\langle 2\rangle) = \{R, B\}$ and since $x \mapsto x+2$ is an automorphism of H_n we also may assume that $f(0) = R$ and $f(2) = B$. Therefore the triple $\zeta_1 = \{0, 2, -1\}$ contradicts the fact that f is non heterochromatic. So, both cosets are bichromatic.

Let Y be the common color to both cosets. Let x and y be vertices such that $f(\{x, y\}) = \{R, B\}$ and the distance between x and y is minimal. Since $x \mapsto x + 1$ is an automorphism of H_n we may assume that $y = 0$, $f(0) = R$ and $f(x) = B$. Therefore, $f(\langle 2\rangle) = \{R, Y\}$, $f(\langle 2\rangle + 1) = \{B, Y\}$ and $x \in \{\frac{n}{2}+1, \frac{n}{2}+3, \ldots, -1\}$. Of course, by the minimality of the distance between x and y, for all $z \in \{x+1, x+2, \ldots, -1\}$ we have $f(z) = Y$.

For $x \in \{\frac{n}{2}+1, \frac{n}{2}+3, \ldots, 2\frac{n}{3}-1\}$, let d be the solution in $\mathbb{B}_n$ of $2\frac{n}{3} - 2d + 1 = x$. In this case the triple $\eta_d + 1 - 4d = \{1 - 4d, 0, x\} \in H_n$ is heterochromatic and this is a contradiction.

On the other hand, let $x \in \{2\frac{n}{3}+1, 2\frac{n}{3}+3, \ldots, -1\}$.

If $x \equiv 1 \bmod 4$ then let us consider the solution d in $\mathbb{B}_n$ of $1 - 4d = x$. In this case, the triple $\zeta_d - 2 = \{-2, 0, x\} \in H_n$ gives a contradiction.

If $x \equiv 3 \bmod 4$ then let d be the solution in $\mathbb{B}_n$ of $3 - 4d = x$. In this case we have

$$\left.\begin{array}{r}\eta_d + x = \left\{x, x + 2\frac{n}{3} + 2d, 2\right\} \in H_n, \\ f(x) = B,\ f(x + 2\frac{n}{3} + 2d) = Y,\ f(2) \neq B.\end{array}\right\} \Rightarrow f(2) = Y$$

and the triple $\zeta_d = \{0, 2, x\} \in H_n$ is heterochromatic, which is impossible.

■

Lemma 4. *If f is a non heterochromatic 3-coloring of H_n, then f is surjective in the set of odd vertices and is also surjective in the set of even vertices.*

Proof. Let f be a non heterochromatic 3-coloring of H_n then by the preceding lemma we may suppose that $f(\langle 2\rangle) = \{R, B, Y\}$ and $R \notin f(\langle 2\rangle + 1)$.

Since the hypergraph generated by $\langle 2\rangle$ is isomorphic to $G_{n/2}$, hence by Lemma 2, for all $\alpha \in \langle 2\rangle$ the coset $\left\langle \frac{n}{3}\right\rangle + \alpha$ must be monochromatic. So, we can suppose that $f\left(\left\langle \frac{n}{3}\right\rangle\right) = R$ and $f\left(\left\langle \frac{n}{3}\right\rangle + 2\right) = B$.

For a better understanding, we urge the reader to remember (see the beginning of Section 2) that we can add a set of vertices to a triple thus obtaining in this way a set of triples.

For all $b \in \mathbb{B}_n$ we have that

$$\left.\begin{array}{r}\zeta_b + \left\langle \frac{n}{3}\right\rangle = \{0, 2, 3 - 4b\} + \left\langle \frac{n}{3}\right\rangle, \\ f\left(\left\langle \frac{n}{3}\right\rangle\right) = R,\ \ f\left(\left\langle \frac{n}{3}\right\rangle + 2\right) = B \\ \text{and } R \notin\ f\left(\left\langle \frac{n}{3}\right\rangle + 3 - 4b\right).\end{array}\right\} \Rightarrow f\left(\left\langle \frac{n}{3}\right\rangle + 3 - 4b\right) = B.$$

Observe that

$$\bigcup_{b \in \mathbb{B}_n} \left(\left\langle \frac{n}{3}\right\rangle + 3 - 4b\right) = \bigcup_{b \in \mathbb{B}_n} \left(\left\langle \frac{n}{3}\right\rangle + 4b - 1\right)$$

and therefore for any $b \in \mathbb{B}_n$, $f\left(\left\langle \frac{n}{3}\right\rangle + 4b - 1\right) = B$ holds.

On the other hand

$$\left.\begin{array}{r}\zeta_b + 4b - 3 + \left\langle \frac{n}{3}\right\rangle = \{4b - 3, 4b - 1, 0\} + \left\langle \frac{n}{3}\right\rangle, \\ f\left(\left\langle \frac{n}{3}\right\rangle\right) = R,\ f\left(\left\langle \frac{n}{3}\right\rangle + 4b - 1\right) = B \\ \text{and } R \notin\ f\left(\left\langle \frac{n}{3}\right\rangle + 4b - 3\right).\end{array}\right\} \Rightarrow f\left(\left\langle \frac{n}{3}\right\rangle + 4b - 3\right) = B.$$

Since every odd vertex is either in some coset of the form $\left\langle \frac{n}{3} \right\rangle + 4b - 1$ or in some coset of the form $\left\langle \frac{n}{3} \right\rangle + 4b - 3$, hence $f\left(\langle 2 \rangle + 1\right) = B$.

Let $x \in \langle 2 \rangle$ a vertex colored yellow. Recall that $f\left(\left\langle \frac{n}{3} \right\rangle + x\right) = Y$ so we can suppose that $x \in \{2, 4, \ldots, \frac{n}{3} - 2\} = 2\mathbb{B}_n \cup \left(\frac{n}{3} - 2\mathbb{B}_n\right)$. If $x = 2b$, $b \in \mathbb{B}_n$ we have the heterochromatic triple $\eta_b = \left\{0, x - \frac{n}{3}, 4b - 1\right\} \in H_n$. In any other case, $x = \frac{n}{3} - 2b$, $b \in \mathbb{B}_n$ and the triple $\eta_b + x = \left\{x, 0, \frac{n}{3} + 2b - 1\right\} \in H_n$ is heterochromatic and this is a contradiction. ■

Lemma 5. *If f is a non heterochromatic 3-coloring of H_n, then all the cosets of $\mathbb{Z}_n$ by the subgroup $\left\langle \frac{n}{3} \right\rangle \cong \mathbb{Z}_3$ are monochromatic.*

Proof. Let f be a non heterochromatic 3-coloring of H_n, then by Lemma 4 f is surjective in the set of odd vertices and in the set of even vertices. Both sets of vertices induce hypergraphs that are isomorphic to $G_{n/2}$. By Lemma 2 the cosets mod $(n/6)$ in $G_{n/2}$ are monochromatic but these cosets are precisely the cosets mod $(n/3)$ in $\mathbb{Z}_n$ (by the two isomorphisms). ■

Lemma 6. *H_n is tight if and only if $H_n / \left\langle \frac{n}{3} \right\rangle$ is tight.*

Proof. Any non heterochromatic 3-coloring f' of $H_n / \left\langle \frac{n}{3} \right\rangle$ lifts to a non heterochromatic 3-coloring f of H_n. On the other hand (by the preceding lemma) any non heterochromatic 3-coloring f of H_n factorizes (i.e., $f = f' \circ \mathrm{nat}$) through a non heterochromatic 3-coloring f' of $H_n / \left\langle \frac{n}{3} \right\rangle$. ■

Theorem 7. *H_n is tight.*

Proof. Denote $\widehat{H}_n = H_n / \left\langle \frac{n}{3} \right\rangle$. Let f' be a non heterochromatic 3-coloring of H_n. As in the preceding lemma the map f' factorizes through a non heterochromatic 3-coloring f of $\widehat{H}_n$, moreover by Lemma 4 f' (and so f) is surjective in the set of odd and in the set of even vertices. Denote by $t = \frac{n}{3}$ and recall that $f : \mathbb{Z}_n / \left\langle \frac{n}{3} \right\rangle \cong \mathbb{Z}_t \to \{R, B, Y\}$ is a non heterochromatic red-blue-yellow 3-coloring of $\widehat{H}_n$.

First we shall prove that there is an x such that $f(x) = f(x+1)$. Suppose not. If there is no y such that $f(y) = f(y+2)$ then, $t \equiv 0 \bmod 3$, the cosets $\langle 3 \rangle$, $\langle 3 \rangle + 1$ and $\langle 3 \rangle + 2$ are monochromatic and the triple $\zeta_{\frac{t}{4}-1} \bmod t = \{0, 2, 7\} \in \widehat{H}_n$ gives a contradiction. So, there exists $y \in \mathbb{Z}_t$ such that $f(y) = f(y+2) = R$. If $f(y+1) = R$ or $f(y+3) = R$ then we are done. Let $f(y+1) = B$. The triple $(\zeta_1 \bmod t) + y + 1 = \{y+1, y+3, y\} \in \widehat{H}_n$ shows that $f(y+3) = B$. Taking as a new y the vertex $y+1$ and repeating this argument the needed number of times we conclude that there is not a yellow vertex which is a contradiction.

Therefore we can suppose that $f(0) = R$, $f(1) = f(2) = B$.
For all $b \in \mathbb{B}_n = \{1, \ldots, \frac{n}{12}\} \subset \mathbb{Z}_n$ denote $b' = -4b \bmod t \in \mathbb{Z}_t$. We have that

$$\left.\begin{array}{c} \zeta_b \bmod t = \{0, 2, b' + 3\} \in \widehat{H}_n, \\ f(0) = R,\ f(2) = B. \end{array}\right\} \Rightarrow f(b' + 3) \neq Y.$$

Observe that $\{b' : b \in \mathbb{B}_n\} = \langle 4 \rangle \subset \mathbb{Z}_t$. Since f is surjective in the set of odd vertices there must be a vertex $c' \in \langle 4 \rangle$ such that $f(c' + 1) = Y$ and $c' \neq 0$. Let c be the element in $\mathbb{B}_n$ such that $c' = -4c \bmod t$. We have that

$$\left.\begin{array}{c} \left(\zeta_{\frac{n}{12}} \bmod t\right) - 2 = \{-2, 0, 1\} \in \widehat{H}_n, \\ (\zeta_c \bmod t) - 2 = \{-2, 0, c' + 1\} \in \widehat{H}_n \\ \text{and } f(0) = R,\ f(1) = B,\ f(c' + 1) = Y. \end{array}\right\} \Rightarrow f(-2) = R.$$

Now, let d be the element in $\mathbb{B}_n$ such that $c' + 4 = 4d \bmod t$. We have that

$$\left.\begin{array}{c} (\zeta_d \bmod t) + c' + 1 = \{c' + 1, c' + 3, 0\} \in \widehat{H}_n, \\ \zeta_c \bmod t = \{0, 2, c' + 3\} \in \widehat{H}_n \\ \text{and } f(0) = R,\ f(2) = B,\ f(c' + 1) = Y. \end{array}\right\} \Rightarrow f(c' + 3) = R.$$

Since f is surjective in the set of even vertices there must be a vertex $x \in \langle 2 \rangle$ such that $f(x) = B$. If $x \in \langle 4 \rangle$ then $b' = x - c' - 4 \in \langle 4 \rangle$. In this case the triple

$$(\zeta_b \bmod t) + c' + 1 = \left\{c' + 1, c' + 3, x\right\} \in \widehat{H}_n$$

gives a contradiction. If $x \notin \langle 4 \rangle$ then, $b' = x - c' - 2 \in \langle 4 \rangle$ and we have

$$\left.\begin{array}{c} (\zeta_b \bmod t) + c' - 1 = \{c' - 1, c' + 1, x\} \in \widehat{H}_n, \\ f(c' - 1) \neq Y,\ f(c' + 1) = Y,\ f(x) = B. \end{array}\right\} \Rightarrow f(c' - 1) = B.$$

Therefore, the triple $(\zeta_d \bmod t) + c' - 1 = \{c' - 1, c' + 1, -2\} \in \widehat{H}_n$ is heterochromatic, which is impossible. ■

3. THE CASE 1 mod 12

When $n \equiv 1 \bmod 3$ the bound for the number of triples in a tight 3-graph is $\frac{n(n-2)+1}{3}$. This bound can be reached in a 3-graph in which the trace of one vertex is a cycle and the trace of any other vertex is a tree. Such 3-graph will be called an *almost 3-tree.*

Let M be a 3-tree with n vertices with $n \equiv 0 \bmod 3$ and suppose that M has a set T of $\frac{n}{3}$ disjoint triples. Let C be a cycle passing through every vertex of M. Define the 3-graph $\widetilde{M}$ obtained from M by the following procedure:

- add a new vertex $*$,
- add the triples $\{*, v, w\}$ where $\{v, w\}$ is an edge of C,
- delete all the triples of T.

It is easy to see, that if all the traces of vertices in $\widetilde{M}$ are connected then $\widetilde{M}$ is an almost 3-tree. In particular, if we can prove that $\widetilde{M}$ is tight then we have a proof of the conjecture on the minimum size of tight 3-graph for the case $n + 1$.

In this section we construct a 3-graph $\widetilde{H_n}$ which is an almost 3-tree and prove that it is tight.

Recall our definition of H_n from Section 1

$$H_n = (\mathbb{Z}_n, (\{\varepsilon_a \mid a \in \mathbb{A}_n\} \cup \{\zeta_b, \eta_b \mid b \in \mathbb{B}_n\}) + \mathbb{Z}_n),$$

where

$$\begin{aligned} \varepsilon_a &= \{0, 2\tfrac{n}{3}, 2a\},\ \zeta_b = \{0, 2, 3 - 4b\}, \eta_b = \{0, 2\tfrac{n}{3} + 2b, 4b - 1\}, \\ \mathbb{A}_n &= \{1, \ldots, \tfrac{n}{6}\} \subset \mathbb{Z}_n,\ \mathbb{B}_n = \{1, \ldots, \tfrac{n}{12}\} \subset \mathbb{A}_n \end{aligned}$$

and $n \equiv 0 \bmod 12$.

Let T be the set of triples $\{\varepsilon_{n/6} + \mathbb{Z}_n\}$ and C be the cycle $\{\{x, x + 1\} \mid x \in \mathbb{Z}_n\}$. Let $\widetilde{H}_n$ be the 3-graph obtained as above, i.e.,

$$\widetilde{H}_n = \big(\mathbb{Z}_n \cup \{*\}, \big(\{\varepsilon_a \mid a \in \mathbb{A}_n \backslash \{\tfrac{n}{6}\}\} \cup \{\zeta_b, \eta_b \mid b \in \mathbb{B}_n\} \cup \{*, 0, 1\}\big) + \mathbb{Z}_n\big)$$

where, by definition, $* + x = *$ for all $x \in \mathbb{Z}_n$.

Theorem 8. *$\widetilde{H}_n$ is tight.*

Proof. The proof bellow is not valid for the case $n = 12$. However, for that case we can prove that $\widetilde{H}_{12}$ is tight checking all possible colorings (the number of colorings can be reduced using the symmetries of $\widetilde{H}_{12}$ and the fact that H_{12} is tight).

So, let $s = n/12$, $s \geq 2$ and let f be a non heterochromatic 3-coloring of $\widetilde{H}_n$.

There must be a vertex x in $\mathbb{Z}_n$ such that $f(x) = f(*)$ for if this is not the case, then there are two consecutive vertices y, $y+1$ such that $f(y) \neq f(y+1)$ and therefore the triple $\{*, y, y+1\}$ gives a contradiction.

Then f is surjective in $\mathbb{Z}_n$. By Theorem 7 there must be an heterochromatic triple $\varepsilon_{2s} + x \in H_n$. Since $x \mapsto x+1$ is an automorphism of H_n and $\widetilde{H}_n$, we can suppose that $x = 0$. Let $f(0) = R$, $f(4s) = B$ and $f(8s) = Y$.

We divide the proof in two cases when $f(0) = f(2)$ and otherwise.
If $f(0) = f(2) = R$ then

$$\left.\begin{array}{r} \varepsilon_1 + 8s = \{8s, 4s, 8s+2\}\,, \\ \varepsilon_{2s-1} + 2 = \{2, 8s+2, 4s\}\,, \\ f(2) = R,\ f(4s) = B,\ f(8s) = Y. \end{array}\right\} \Rightarrow f(8s+2) = B,$$

$$\left.\begin{array}{r} \varepsilon_s = \{0, 8s, 2s\}\,,\ \varepsilon_{s-1} + 2 = \{2, 8s+2, 2s\}\,, \\ f(0) = R,\ f(2) = R,\ f(8s) = Y,\ f(8s+2) = B. \end{array}\right\} \Rightarrow f(2s) = R,$$

$$\left.\begin{array}{r} \zeta_1 + 8s = \{8s, 8s+2, 8s-1\}\,, \\ \eta_s + 4s = \{4s, 8s-1, 2s\}\,, \\ f(2s) = R,\ f(4s) = B,\ f(8s) = Y,\ f(8s+2) = B. \end{array}\right\} \Rightarrow f(8s-1) = B,$$

$$\left.\begin{array}{r} \varepsilon_1 + 4s = \{4s, 0, 4s+2\}\,, \\ \varepsilon_{2s-1} + 4s + 2 = \{4s+2, 2, 8s\}\,, \\ f(0) = R,\ f(2) = R,\ f(4s) = B,\ f(8s) = Y. \end{array}\right\} \Rightarrow f(4s+2) = R,$$

$$\left.\begin{array}{r} \varepsilon_{2s-2} = \{0, 8s, 4s-4\}\,, \\ \varepsilon_{2s-3} + 2 = \{2, 8s+2, 4s-4\}\,, \\ f(0) = R,\ f(2) = R,\ f(8s) = Y,\ f(8s+2) = B. \end{array}\right\} \Rightarrow f(4s-4) = R,$$

$$\left.\begin{array}{r} \varepsilon_{2s-2} + 4s = \{4s, 0, 8s-4\}\,, \\ \varepsilon_2 + 8s - 4 = \{8s-4, 4s-4, 8s\}\,, \\ f(0) = R,\ f(4s-4) = R,\ f(4s) = B,\ f(8s) = Y. \end{array}\right\} \Rightarrow f(8s-4) = R$$

and

$$\left.\begin{array}{r}\eta_1 + 8s = \{8s, 8s+3, 4s+2\}\,,\\ \eta_2 + 8s - 4 = \{8s-4, 8s+3, 4s\}\,,\\ f(4s) = B,\ f(4s+2) = R,\ f(8s-4) = R,\ f(8s) = Y.\end{array}\right\} \Rightarrow f(8s+3) = R.$$

Moreover, if $f(8s+1) = R$ then no matter the color of $*$ is, some of the triples $\{*, 8s-1, 8s\}$, $\{*, 8s, 8s+1\}$ or $\{*, 8s+1, 8s+2\}$ gives a contradiction. Hence

$$\left.\begin{array}{r}\zeta_s + 8s - 1 = \{8s-1, 8s+1, 4s+2\} \in \widetilde{H_n},\\ f(4s+2) = R,\ f(8s-1) = B,\ f(8s+1) \neq R.\end{array}\right\} \Rightarrow f(8s+1) = B$$

and the triple $\zeta_1 + 8s + 1 = \{8s+1, 8s+3, 8s\}$ gives a contradiction.

Now, suppose that $f(0) \neq f(2)$. If $f(4s) = f(4s+2)$ then using the automorphism $x \mapsto x - 4s$ we reduce the proof to the first case. By the same argument $f(8s) \neq f(8s+2)$. Moreover,

$$\left.\begin{array}{r}\varepsilon_1 = \{0, 8s, 2\} \in \widetilde{H}_n,\\ f(0) = R,\ f(8s) = Y,\ f(2) \neq R.\end{array}\right\} \Rightarrow f(2) = Y,$$

$$\left.\begin{array}{r}\varepsilon_1 + 4s = \{4s, 0, 4s+2\} \in \widetilde{H_n},\\ f(0) = R,\ f(4s) = B,\ f(4s+2) \neq B.\end{array}\right\} \Rightarrow f(4s+2) = R,$$

$$\left.\begin{array}{r}\varepsilon_1 + 8s = \{8s, 4s, 8s+2\} \in \widetilde{H}_n,\\ f(4s) = B,\ f(8s) = Y,\ f(8s+2) \neq Y.\end{array}\right\} \Rightarrow f(8s+2) = B$$

and

$$\left.\begin{array}{r}\varepsilon_1 + 2 = \{2, 8s+2, 4\}\,,\ \varepsilon_2 = \{0, 8s, 4\} \in \widetilde{H}_n,\\ f(0) = R,\ f(2) = Y,\ f(8s) = Y,\ f(8s+2) = B.\end{array}\right\} \Rightarrow f(4) = Y.$$

Again, using the automorphism $x \mapsto x - 2$ we reduce the proof to the first case. ∎

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Received 26 November 2001
Revised 6 May 2002