Discussiones Mathematicae
Graph Theory 23 (2003) 163–175

UPPER BOUNDS FOR THE DOMINATION NUMBERS OF TOROIDAL QUEENS GRAPHS

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Abstract

We determine upper bounds for $\gamma(Q_n^t)$ and $i(Q_n^t)$, the domination and independent domination numbers, respectively, of the graph Q_n^t obtained from the moves of queens on the $n \times n$ chessboard drawn on the torus.

Keywords: queens graph, toroidal chessboards, queens domination problem.

2000 Mathematics Subject Classification: 05C69.

1. Introduction

The study of combinatorial problems on chessboards dates back to 1848, when German chess player Max Bezzel [2] first posed the n-queens problem, that is, the problem of placing n queens on an $n \times n$ chessboard so that no two queens attack each other. The study of chessboard domination problems dates back to 1862, when C.F. de Jaenisch [9] first considered the queens domination problem, that is, the problem of determining the minimum number of queens required to cover every square on an $n \times n$ chessboard. Since then many papers concerning combinatorial problems on chessboards have appeared in the literature. See [10] for a survey of the topic; recent results not mentioned there can be found in [4, 5, 6, 11, 13, 15, 16].

The n-queens problem for chessboards drawn on the torus was solved by Monsky [12], while the study of the queens domination problem on the torus was initiated in [3]. The results obtained in these papers, some of which are mentioned in Section 2, clearly show that the n-queens problem and the queens domination problem on the torus differ substantially from the corresponding problems for plane chessboards.

When drawn on the torus, the rows and columns of the chessboard are rings round the torus. We cut the torus along arbitrary lines separating two rows and two columns, and draw the $n \times n$ toroidal chessboard in the plane, numbering its rows and columns from 0 to $n-1$, beginning at the bottom left hand corner. Thus each square has *co-ordinates* (x, y), where x and y are the column and row numbers of the square, respectively. The *lines* of the board are the rows, columns, *sum diagonals*, abbreviated *s-diagonals* (*i.e.*, sets of squares such that $x + y \equiv k (\operatorname{mod} n)$, where k is a constant) and *difference diagonals*, abbreviated *d-diagonals* (sets of squares such that $y - x \equiv k (\operatorname{mod} n)$). Note that there are n s-diagonals and n d-diagonals, and each contains n squares. Rows and columns are collectively called *orthogonals*.

The vertices of Q_n^t, the *queens graph obtained from an $n \times n$ chessboard on the torus*, are the n^2 squares of the chessboard, and two squares are adjacent if they are collinear. It is easy to verify that for any $a, b \in \{0, 1, \ldots, n-1\}$, the mapping defined by $\tau_{a,b}(x, y) = (x + a, y + b)$ is a graph automorphism of Q_n^t, so Q_n^t is vertex-transitive. A queen on a square (x, y) of Q_n^t is said to *cover* or *dominate* (x, y) and any square adjacent to (x, y). A set D of squares is a *dominating set* of Q_n^t if every square of Q_n^t is either in D or adjacent to a square in D, *i.e.,* if a set of queens, one on each square in D, covers the board. If no two squares of the dominating set D are adjacent, then D is an *independent dominating set.* As is standard in domination theory we denote the domination, independent domination and independence numbers of Q_n^t by $\gamma(Q_n^t)$, $i(Q_n^t)$ and $\beta(Q_n^t)$, respectively. The n-queens problem on the torus is thus equivalent to determining whether $\beta(Q_n^t) = n$ and finding such a solution (the set of squares which contain the n non-attacking queens) if it exists, and the queens domination problem on the torus is the problem of determining $\gamma(Q_n^t)$.

In this paper we determine upper bounds for $\gamma(Q_n^t)$ and $i(Q_n^t)$, where $n = 3k$ and $k \equiv 0, 3, 4, 6, 8, 9 \ (\operatorname{mod} 12)$, or $n = 2k$ and $k \equiv 2, 4 (\operatorname{mod} 6)$.

2. Previous Results

Denote the graph obtained from the moves of queens on the ordinary (plane) $n \times n$ chessboard by Q_n. Ahrens [1] showed that $\beta(Q_n) = n$ for all $n \geq 4$. In contrast, it is not always possible to place n mutually non-attacking queens on Q_n^t — Monsky [12] showed that

$$\beta(Q_n^t) = \begin{cases} n & \text{if} \quad n \equiv 1, 5, 7, 11 (\text{mod}\, 12), \\ n-1 & \text{if} \quad n \equiv 2, 10 (\text{mod}\, 12), \\ n-2 & \text{if} \quad n \equiv 0, 3, 4, 6, 8, 9 (\text{mod}\, 12). \end{cases} \tag{1}$$

As shown in [3] there is a correspondence between independent sets of cardinality n of Q_n^t and dominating sets of cardinality n of the larger graph Q_{3n}^t.

Theorem 1 [3]. *For any n, the set S with $|S| = n$ is an independent set of Q_n^t if and only if $\{(3x, 3y) : (x, y) \in S\}$ is a dominating set of Q_{3n}^t.*

Since it was also shown in [3] that $\gamma(Q_n^t) \geq \lceil n/3 \rceil$ for all $n \geq 1$, and since the dominating set in Theorem 1 is independent, it follows that

Corollary 2 [3]. *If $n \equiv 1$ or $5 (\text{mod}\, 6)$, then $\gamma(Q_{3n}^t) = i(Q_{3n}^t) = n$, otherwise $i(Q_{3n}^t) \geq \gamma(Q_{3n}^t) \geq n + 1$.*

Similarly, using the fact that $\beta(Q_n^t) = n - 1$ for $n \equiv 2, 10 (\text{mod}\, 12)$, it was also shown that

Theorem 3 [3]. *If $n \equiv 2$ or $10 (\text{mod}\, 12)$, then $\gamma(Q_{3n}^t) = n + 1$ and $i(Q_{3n}^t) \leq n + 3$.*

The method used in Theorem 1 also gives the following bound for $\gamma(Q_k^t)$, where k is even but not divisible by 3 or 4.

Proposition 4 [3]. *If $n \equiv 1$ or $5 (\text{mod}\, 6)$, then $\gamma(Q_{2n}^t) \leq i(Q_{2n}^t) \leq n$.*

The only other exact values known for $\gamma(Q_n^t)$ and $i(Q_n^t)$ were determined by exhaustive search and are given in Tables 1 and 2, where the boldface entries show that neither $\gamma(Q_n^t)$ nor $i(Q_n^t)$ is monotone. (In the case of Q_n, the queens graph for ordinary chessboards, the corresponding questions of monotonicity remain unresolved.) Any dominating set of Q_n also dominates Q_n^t and so $\gamma(Q_n^t) \leq \gamma(Q_n)$ for all n. However, an independent set of queens on Q_n is not necessarily independent on Q_n^t and so $i(Q_n^t)$ and $i(Q_n)$ are not comparable.

Table 1. $\gamma(Q_n^t)$ for small values of n

n	$\gamma(Q_n)$	$\gamma(Q_n^t)$	Solutions
4	2	2	(0,0),(1,2)
5	3	3	(0,0),(2,2),(4,4)
6	3	3	(0,0),(2,4),(4,2)
7	4	4	(0,0),(1,1),(2,2),(5,5)
8	5	4	(0,0),(3,7),(4,3),(7,4)
9	5	5	(0,0),(1,3),(3,7),(5,1),(7,5)
10	5	5	(0,0),(2,4),(4,8),(6,2),(8,6)
11	5	5	(0,0),(2,6),(4,2),(6,9),(8,4)
12	6	**6**	(0,0),(2,2),(4,4),(6,10),(8,8),(10,6)
13	7	≤ 7	
14	8*	≤ 7	Proposition 4
15	9*	**5**	(0,0),(3,6),(6,12),(9,3),(12,9)

*See [11]

Table 2. $i(Q_n^t)$ for small values of n

n	$i(Q_n)$	$i(Q_n^t)$	Solutions
4	3	2	See γ
5	3	**5**	(0,0),(1,2),(2,4),(3,1),(4,3)
6	4	**4**	(0,0),(2,3),(3,5),(5,2)
7	4	5	(0,0),(1,4),(2,6),(3,1),(4,5)
8	5	4	See γ
9	5	5	See γ
10	5	5	See γ
11	5	5	See γ
12	7	6	(0,0),(1,2),(2,11),(6,5),(7,3),(8,6)
13	7	7	(0,0),(1,2),(2,4),(3,12),(4,1),(5,3),(6,11)
14	8	7	(0,0),(2,4),(4,8),(6,12),(8,2),(10,6),(12,10)
15	9	5	See γ
16	9	8	(0,0),(1,4),(2,8),(3,5),(4,9),(5,13),(8,3),(13,10)

For ordinary chessboards, P.H. Spencer (see [8]) showed that $\gamma(Q_n) \geq \lfloor \frac{n}{2} \rfloor$, $n \geq 1$. Weakley [14] improved this bound to $\gamma(Q_n) \geq \lceil \frac{n}{2} \rceil$ for $n \equiv 1 (\mathrm{mod}\, 4)$, and in fact $\gamma(Q_n) = \lceil \frac{n}{2} \rceil$ for all values of $n \equiv 1 (\mathrm{mod}\, 4)$ up to $n = 129$ (see [4, 10, 11, 13]). All other known values (finitely many) are in the range $\lfloor \frac{n}{2} \rfloor \leq \gamma(Q_n) \leq \lceil \frac{n}{2} \rceil + 1$. The best upper bound is given by $\gamma(Q_n) \leq \frac{101}{195} n + O(1)$ for all n large enough [6]. Thus it seems reasonable to conjecture that $\gamma(Q_n) \approx \frac{n}{2}$ for all n. This is again in contrast to the situation for toroidal boards, where $\gamma(Q_n^t) = \frac{n}{3}$ for $n \equiv 3, 15 (\mathrm{mod}\, 18)$ (thus infinitely many exact values are known), and small exact values indicate that $\gamma(Q_n^t) \approx \frac{n}{2}$ if n is not divisible by 3.

3. Upper Bounds for $\gamma(Q_{3n}^t)$ and $i(Q_{3n}^t)$

Exact values of $\gamma(Q_{3n}^t)$ for $n \equiv 1, 2, 5, 7, 10, 11 (\mathrm{mod}\, 12)$ are given in Corollary 2 and Theorem 3. In this section we determine an upper bound for $\gamma(Q_{3n}^t)$ for all other values of n. Let S be a set of squares of Q_n^t. A line (row, column, diagonal) which contains (respectively does not contain) a square of S is called an *occupied*(respectively *empty*) line (row, column, diagonal). We begin by finding a sufficient condition on S such that $T = \{(3x, 3y) : (x, y) \in S\}$ dominates Q_{3n}^t.

Proposition 5. *If S is a set of squares on Q_n^t such that $S \cap l \neq \phi$ for each line l of Q_n^t, then $T = \{(3a, 3b) : (a, b) \in S\}$ dominates Q_{3n}^t.*

Proof. Consider any square (x, y) of Q_{3n}^t. Suppose firstly that $x = 3x'$ for some $x' \in N$. By assumption S contains a square in column x'; say $(x', y') \in S$. Then $(3x', 3y') \in T$ dominates (x, y) by column. A similar argument holds if y is a multiple of 3. Suppose neither x nor y is a multiple of 3; say $x = 3l + m$ and $y = 3p + q$, where $m, q \in \{1, 2\}$. If $m = q$, then (x, y) lies on the d-diagonal $d \equiv 3(p - l) (\mathrm{mod}\, 3n)$ of Q_{3n}^t. But by hypothesis the d-diagonal $(p - l) (\mathrm{mod}\, n)$ of Q_n^t contains a square (a, b) of S with $b - a \equiv (p - l) (\mathrm{mod}\, n)$ and so the square $(3a, 3b) \in T$ dominates (x, y) diagonally. Similarly, if $m \neq q$, then (x, y) lies on the s-diagonal $s \equiv 3(l + p + 1) (\mathrm{mod}\, 3n)$ of Q_{3n}^t. The s-diagonal $(l + p + 1) (\mathrm{mod}\, n)$ of Q_n^t contains a square (i, j) of S with $i + j \equiv (l + p + 1) (\mathrm{mod}\, n)$, and the square $(3i, 3j) \in T$ dominates (x, y) diagonally. ■

Theorem 6. *If* $n \equiv 0, 3, 4, 6, 8, 9 (\mathrm{mod}\, 12)$*, then* $\gamma(Q^t_{3n}) \leq n + 2$ *and* $i(Q^t_{3n}) \leq n + 6$*.*

Proof. Let S' be a maximum independent set of Q^t_n. By (1), $|S'| = n - 2$ for the values of n under consideration. Since S' is independent, S' contains at most one square in each line, hence there are exactly two empty lines of each type (row, column, s- and d-diagonals). Let r_i, c_i, s_i and d_i, $i = 1, 2$, be the empty lines of Q^t_n. For $i = 1, 2$, let (x_i, y_i) be the (unique) square where row r_i intersects s-diagonal s_i, (x'_i, y'_i) the square where column c_i intersects d-diagonal d_i, and $S = S' \cup \{(x_1, y_1), (x_2, y_2), (x'_1, y'_1), (x'_2, y'_2)\}$. Note that S is well-defined, $|S| = n + 2$ and $S \cap l \neq \phi$ for each line l of Q^t_n. By Proposition 5, $T = \{(3x, 3y) : (x, y) \in S\}$ dominates Q^t_{3n} and thus $\gamma(Q^t_{3n}) \leq n + 2$.

Let $T' = \{(3x, 3y) : (x, y) \in S'\}$ and $\mathcal{L} = \{3c_1, 3c_2, 3r_1, 3r_2, 3s_1, 3s_2, 3d_1, 3d_2\}$. The independence of S' ensures the independence of T' in Q^t_{3n}. Also, it follows similar to the proof of Proposition 5 that T' dominates most squares of Q^t_{3n}, the exceptions being some of the squares on lines in $\mathcal{L}$, that is, some squares (a, b) for which $\{a, b, a+b, b-a\} \cap \mathcal{L} \neq \phi$. We now successively adjoin undominated squares to T' until an independent dominating set I of Q^t_{3n} is obtained. By each time choosing a square on a different line in $\mathcal{L}$, we see that this can be achieved by using at most $|\mathcal{L}| = 8$ additional squares. Hence $i(Q^t_{3n}) \leq |I| \leq |T'| + 8 = n + 6$. ∎

Combined with the results of [3] we thus have

$$
(2)\quad
\begin{array}{rcccll}
 & & \gamma(Q^t_{3n}) & = & \begin{cases} n & \text{if } n \equiv 1, 5, 7, 11 (\mathrm{mod}\, 12), \\ n+1 & \text{if } n \equiv 2, 10 (\mathrm{mod}\, 12), \end{cases} \\
n+1 & \leq & \gamma(Q^t_{3n}) & \leq & n+2 \quad \text{if } n \equiv 0, 3, 4, 6, 8, 9 (\mathrm{mod}\, 12), \\
 & & i(Q^t_{3n}) & = & n \quad \text{if } n \equiv 1, 5, 7, 11 (\mathrm{mod}\, 12), \\
n+1 & \leq & i(Q^t_{3n}) & \leq & \begin{cases} n+3 & \text{if } n \equiv 2, 10 (\mathrm{mod}\, 12), \\ n+6 & \text{if } n \equiv 0, 3, 4, 6, 8, 9 (\mathrm{mod}\, 12). \end{cases}
\end{array}
$$

If S is any dominating set of the ordinary queens graph Q_n, and (n, n) is the corner square in the $(n+1)^{\text{st}}$ row and column of Q_{n+1}, then $S \cup \{(n, n)\}$ is a dominating set of Q_{n+1}. Hence if $\gamma(Q_n) = k$, then $\gamma(Q_{n+m}) \leq k + m$ for any $m \geq 1$. However, since the s-diagonal s_i, $i \neq n - 1$ (d-diagonal d_i, $i \neq 0$) of Q^t_n is not contained in the s-diagonal s_i (d-diagonal d_i) of Q^t_{n+1}, the same result does not hold for queens graphs on the torus. Therefore the values of $\gamma(Q^t_{3n})$ and $i(Q^t_{3n})$ above do not give upper bounds for the other cases.

4. Upper Bounds for $\gamma(Q^t_{2n})$ and $i(Q^t_{2n})$

By Proposition 4, $\gamma(Q^t_{2n}) \leq n$ if n is not divisible by 2 or 3. We now show that the same bound holds if n is even. (Note that the construction is also valid if $3|n$, but by (2) does not yield best possible bounds in this case.) We begin with a result similar to Proposition 5.

Proposition 7. *If n is even and X is a set of squares on Q^t_n such that $X \cap l \neq \phi$ for each orthogonal l of Q^t_n and $X \cap l_e \neq \phi$ for each even diagonal l_e of Q^t_n, then $T = \{(2a, 2b) : (a, b) \in X\}$ dominates Q^t_{2n}.*

Proof. For any square (x, y) of Q^t_{2n}, if x or y is even, then there is a square $(a, b) \in X$ such that $x = 2a$ or $y = 2b$. Hence $(2a, 2b) \in T$ dominates (x, y) by column or row. Suppose x and y are both odd. If $x \equiv y (\operatorname{mod} 4)$, then, since $2n \equiv 0 (\operatorname{mod} 4)$, $(y - x)(\operatorname{mod} 2n) \equiv 0(\operatorname{mod} 4)$; say $(y - x)(\operatorname{mod} 2n) = 4l$ for some integer $l \in \{0, \ldots , \frac{n}{2} - 1\}$. By hypothesis there is a square (a, b) of X on the even d-diagonal $d = 2l$ of Q^t_n. Then $(2a, 2b) \in T$ lies on the d-diagonal $d = 4l$ of Q^t_{2n} and dominates (x, y). Similarly, if $x \not\equiv y(\operatorname{mod} 4)$, then $(x + y)(\operatorname{mod} 2n) \equiv 0(\operatorname{mod} 4)$ and there is a square of T that dominates (x, y) along an s-diagonal. ■

Theorem 8. *If n is even, then $\gamma(Q^t_{2n}) \leq n$.*

Proof. If $n \leq 6$ the result follows from the values of $\gamma(Q^t_{2n})$ listed in Table 1; hence we assume $n \geq 8$. We give two constructions, depending on whether $n \equiv 0$ or $2(\operatorname{mod} 4)$. First let $n \equiv 0(\operatorname{mod} 4)$ and consider the set $X = X_1 \cup X_2 \cup X_3 \cup X_4 \cup X_5$ of squares of Q^t_n, where

$$\begin{aligned}
X_1 &= \{(i, i) : i \in \{0, 1, \ldots , \tfrac{n}{4} - 1\}\},\\
X_2 &= \{(i, i) : i \in \{\tfrac{n}{4} + 1, \ldots , \tfrac{n}{2}\}\},\\
X_3 &= \{(\tfrac{n}{2} + i, n - i) : i \in \{1, 2, \ldots , t\tfrac{n}{4} - 1\}\},\\
X_4 &= \{(\tfrac{n}{2} + i, n - i) : i \in \{\tfrac{n}{4} + 1, \ldots , \tfrac{n}{2} - 1\}\},\\
X_5 &= \{(\tfrac{n}{4}, \tfrac{3n}{4}), (\tfrac{3n}{4}, \tfrac{n}{4})\}.
\end{aligned}$$

(See Figure 1.) Then $|X| = n$ and X contains squares in each row and each column of Q_n^t. For $i = 1, \ldots, 5$, let $\mathcal{S}_i$ ($\mathcal{D}_i$, respectively) be the set of s-diagonals (d-diagonals) of Q_n^t occupied by squares in X_i, and let $\mathcal{S} = \cup_{i=1}^5 \mathcal{S}_i$ and $\mathcal{D} = \cup_{i=1}^5 \mathcal{D}_i$. Then (with arithmetic performed modulo n)

$$\begin{aligned}
\mathcal{S}_1 &= \{0, 2, \ldots, \tfrac{n}{2} - 2\},\\
\mathcal{S}_2 &= \{\tfrac{n}{2} + 2, \ldots, n - 2, 0\},\\
\mathcal{S}_3 &= \mathcal{S}_4 = \{\tfrac{n}{2}\},\\
\mathcal{S}_5 &= \{0\},
\end{aligned}$$

so that $\mathcal{S} = \{0, 2, \ldots, n-2\}$. Further,

$$\begin{aligned}
\mathcal{D}_1 &= \mathcal{D}_2 = \{0\},\\
\mathcal{D}_3 &= \{2, 4, \ldots, \tfrac{n}{2} - 2\},\\
\mathcal{D}_4 &= \{\tfrac{n}{2} + 2, \ldots, n - 2\},\\
\mathcal{D}_5 &= \{\tfrac{n}{2}\},
\end{aligned}$$

hence $\mathcal{D} = \{0, 2, \ldots, n-2\} = \mathcal{S}$. By Proposition 7, $T = \{(2a, 2b) : (a, b) \in X\}$ dominates Q_{2n}^t. (See Figure 2.)

Now let $n \equiv 2 (\operatorname{mod} 4)$ and $X = X_1 \cup X_2$, where

$$\begin{aligned}
X_1 &= \{(i, i) : i \in \{0, \ldots, \tfrac{n}{2} - 1\}\},\\
X_2 &= \{(\tfrac{n}{2} + i, n - 1 - i) : i \in \{0, \ldots, \tfrac{n}{2} - 1\}\}.
\end{aligned}$$

(See Figure 3.) Here too X contains squares in each row and column of Q_n^t. With $\mathcal{S}$, $\mathcal{S}_i$, $\mathcal{D}$, $\mathcal{D}_i$ as above, we have (arithmetic modulo n)

$$\begin{aligned}
\mathcal{S}_1 &= \{0, 2, \ldots, n - 2\},\\
\mathcal{S}_2 &= \{\tfrac{n}{2} - 1\},\\
\mathcal{D}_1 &= \{0\},\\
\mathcal{D}_2 &= \{\tfrac{n}{2} - 1, \tfrac{n}{2} - 3, \ldots, -\tfrac{n}{2} + 1\} = \{0, 2, \ldots, n - 2\}.
\end{aligned}$$

(Note that since $n \equiv 2 (\operatorname{mod} 4)$, $\frac{n}{2} - r$ is even for each odd r.) Thus $\mathcal{S} = \mathcal{D} = \{0, 2, \ldots, n-2\}$ and the result follows from Proposition 7. ∎

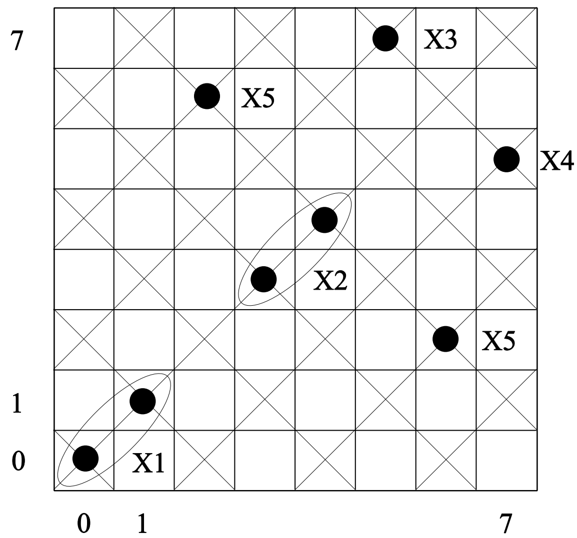

Figure 1. S contains squares on each orthogonal and each even diagonal of Q_8^t

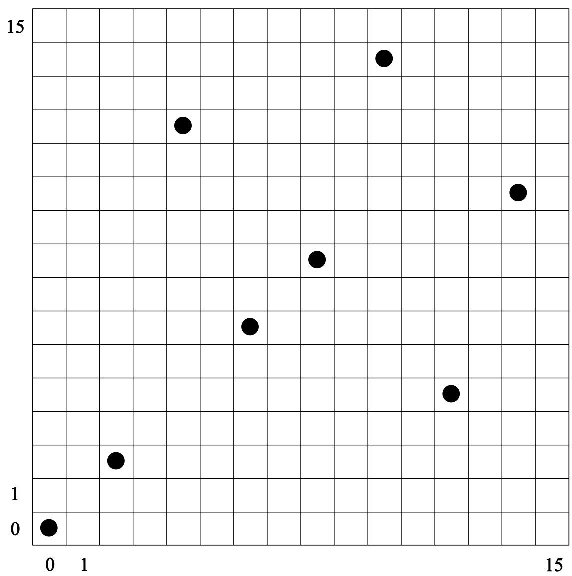

Figure 2. A dominating set of Q_{16}^t

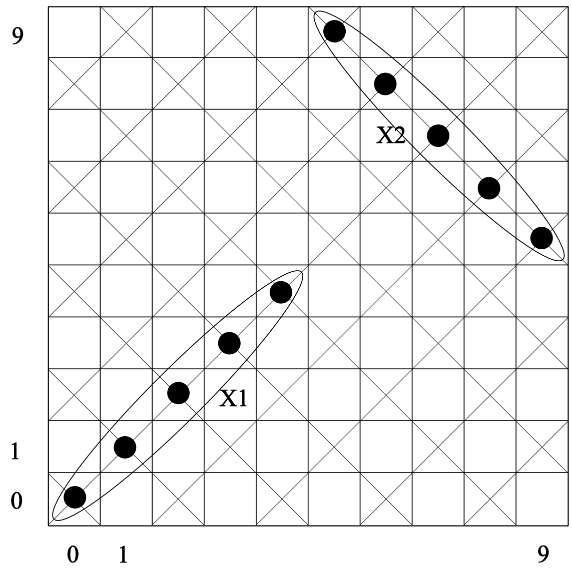

Figure 3. S contains squares on each orthogonal and each even diagonal of Q_{10}^t

The upper bounds obtained above are exact for all known values of $\gamma(Q_k^t)$ (where of course we use the upper bound obtained in Section 3 if this bound is lower than that of Theorem 8). However, only a few exact values have been determined, and it remains an intriguing question whether $\gamma(Q_k^t)$ can be equal to or of order $\frac{k}{2}$ for some (infinitely many) values of k, and equal to $\frac{k}{3}$ for others.

We next consider bounds for $i(Q_k^t)$ where k is even and not divisible by 3. If $n \equiv 1, 5, 7, 11 (\mathrm{mod}\, 12)$, then $i(Q_{2n}^t) \leq n$ by Proposition 4. Thus we need to consider $n \equiv 2, 10 (\mathrm{mod}\, 12)$ and $n \equiv 4, 8 (\mathrm{mod}\, 12)$.

Theorem 9. *If* $n \equiv 2, 10 (\mathrm{mod}\, 12)$*, then* $i(Q_{2n}^t) \leq n + 1$.

Proof. Consider a maximum independent set X of queens on Q_n^t. By (1), $|X| = n - 1$. Say $X = \{(x_i, y_i) : i = 1, \ldots, n-1\}$. Let $s_i \equiv (x_i + y_i)(\mathrm{mod}\, n)$ and $d_i \equiv (y_i - x_i)(\mathrm{mod}\, n)$ be the s- and d-diagonals containing (x_i, y_i). There is at most one square of X in every row, column, s-diagonal and d-diagonal of Q_n^t, so that there is one empty line of each type. By symmetry we may assume that row 0, column 0, s-diagonal p and d-diagonal q are empty. Now,

$$
\begin{array}{rrcll}
 & s_i & \equiv & (x_i + y_i)(\operatorname{mod} n) & \text{for each } i = 1, \dots, n-1, \\
\therefore & \sum_{i=1}^{n-1} s_i & \equiv & \sum_{i=1}^{n-1} (x_i + y_i)(\operatorname{mod} n) & \text{(summing over all queens in } X\text{)}, \\
\therefore & \sum_{i=1}^{n-1} i - p & \equiv & 2 \sum_{i=1}^{n-1} i (\operatorname{mod} n) & \text{(since } s\text{-diagonal } p \text{ is empty)}, \\
\therefore & \frac{n(n-1)}{2} - p & \equiv & n(n-1)(\operatorname{mod} n), & \\
\therefore & p & = & \frac{n}{2} & \text{(since } n \text{ is even).}
\end{array}
$$

Similarly, by considering d-diagonals $d_i \equiv (y_i - x_i)(\operatorname{mod} n)$, we find that $q = \frac{n}{2}$. But $n \equiv 2$ or $10(\operatorname{mod} 12)$, hence $\frac{n}{2}$ is odd. Thus the empty diagonal in each case is an odd diagonal, and it follows that $X \cap l_e \neq \phi$ for each even diagonal l_e of Q_n^t.

Let $T = \{(2x, 2y) : (x, y) \in X\}$. The independence of X ensures the independence of T in Q_{2n}^t. As in the proof of Proposition 7, T diagonally dominates all squares (a, b) of Q_{2n}^t with a and b both odd. Also, T dominates all squares of Q_{2n}^t in even rows or columns, except possibly squares in row and column 0. We now successively adjoin undominated squares to T to obtain an independent dominating set I of Q_{3n}^t by first choosing a square in (say) row 0 and then one in column 0, if required. Clearly I can be constructed by using at most two additional squares. Hence $i(Q_{2n}^t) \leq |I| \leq |T| + 2 = n + 1$. ∎

Theorem 10. *If* $n \equiv 4, 8(\operatorname{mod} 12)$*, then* $i(Q_{2n}^t) \leq n + 4$*.*

Proof. Maximum independent sets X of $n - 2$ queens on Q_n^t are given in [12] for the two cases $n \equiv 4(\operatorname{mod} 12)$ and $n \equiv 8(\operatorname{mod} 12)$. There are exactly two empty lines of each type; say columns c_1, c_2, rows r_1, r_2, diagonals s_1, s_2, d_1, d_2 are empty. It is a straightforward but tedious exercise to check that in each case the s_i are of different parity, as are the d_i. Say s_1 and d_1 are even. If $T = \{(2x, 2y) : (x, y) \in X\}$ and $\mathcal{L} = \{2c_1, 2c_2, 2r_1, 2r_2, 2s_1, 2d_1\}$, then T is independent in Q_{2n}^t and dominates all squares of Q_{2n}^t except for some squares on lines in $\mathcal{L}$. As before, T can be extended to an independent dominating set I by adding at most six squares on distinct lines in $\mathcal{L}$. Thus $i(Q_{2n}^t) \leq |I| \leq n + 4$. ∎

5. Problems

Problem 1. As stated in (2), if $n \equiv 0, 3, 4, 6, 8, 9 \pmod{12}$, then $n+1 \leq \gamma(Q^t_{3n}) \leq n+2$. Exact values given in Table 1 suggest that $\gamma(Q^t_{3n}) = n+2$. Can the lower bound be improved? [1]

Problem 2. If n is not a multiple of 3, then $\lceil \frac{2n}{3} \rceil \leq \gamma(Q^t_{2n}) \leq n$. Is the upper bound closer (or equal) to the correct answer?

Problem 3. Determining upper bounds for $\gamma(Q_n)$ (the ordinary queens graph) is a difficult problem — see [6] and [16] for recent bounds. In contrast the upper bounds in Sections 3 and 4 were easier to obtain, but when n is odd and not divisible by 3, the bounds for $\gamma(Q_n)$ are still the best general bounds for $\gamma(Q^t_n)$, while $\lceil \frac{n}{3} \rceil$ is the best general lower bound. The exact values of $\gamma(Q^t_n)$ given in Table 1 for such n are all equal to $\gamma(Q_n)$. Determine better bounds for $\gamma(Q^t_n)$, or show that $\gamma(Q^t_n) = \gamma(Q_n)$ for $n \equiv 1, 5 \pmod 6$.

Problem 4. The upper bounds for $i(Q^t_{3n})$ and $i(Q^t_{2n})$ given in Theorems 3, 9 and 10 are not exact for small n. The method of extending maximum independent sets of Q^t_n to independent dominating sets of the larger boards is not efficient enough. Find configurations that give better upper bounds.

Problem 5. As in the case of the domination number, if n is not divisible by 3, then the lower bound $\lceil \frac{2n}{3} \rceil \leq i(Q^t_{2n})$ is probably not very good. Determine better lower bounds for $i(Q^t_{2n})$.

Acknowledgement

This paper was written while the author enjoyed the hospitality of the Department of Mathematics and Statistics of the University of Victoria, Canada. Financial support from the University of South Africa and the NRF is gratefully acknowledged.

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[1] Note added in proof: This question was answered in the affirmative in [7].

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Received 2 October 2001
Revised 18 January 2002