Discussiones Mathematicae
Graph Theory 23 (2003) 141–158

THE RAMSEY NUMBER $r(C_7, C_7, C_7)$

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Abstract

Bondy and Erdös [2] have conjectured that the Ramsey number for three cycles C_k of odd length has value $r(C_k, C_k, C_k) = 4k-3$. We give a proof that $r(C_7, C_7, C_7) = 25$ without using any computer support.

Keywords: Ramsey numbers, extremal graphs.

2000 Mathematics Subject Classification: 05C55, 05C35.

1. Introduction and Main Theorem

Let $G = (V(G), E(G))$ be an undirected finite graph without any loops or multiple edges, where $V(G)$ denotes its vertex set and $E(G)$ its edge set. In the following we will often consider the complete graph K_p on p vertices and the cycle C_p on p vertices. A k-coloring $(F_1, F_2, \ldots, F_k)$ of a graph G is a coloring of the edges of G with at most k different colors $F_1, \ldots, F_k$.

The graph $\langle F_i \rangle = (V(G), E(F_i))$ denotes the subgraph of G which consists of all vertices of G and all edges which are colored with color F_i. We say that $K_p \longrightarrow (G_1, G_2, \ldots, G_k)$, if in each k-coloring of K_p the subgraph $\langle F_i \rangle$ contains a graph isomorphic to G_i for at least one i with $1 \leq i \leq k$. Now the Ramsey Number of k graphs $G_1, G_2, \ldots, G_k$ is defined as the minimal integer p such that $K_p \longrightarrow (G_1, \ldots, G_k)$, that is $r(G_1, \ldots, G_k) := min\{p \,|\, K_p \longrightarrow (G_1, \ldots, G_k)\}$.

In the sequal we consider the case $k = 3$ and use the colors red, green and black. For short we say that the set X, $X \subseteq V(G)$, spans or contains a red (green, black) graph H if the red (green, black) edges between the vertices in X contain a subgraph which is isomorphic to H. This subgraph does not have to be an induced one.

A good and detailed overview about known estimations and exact values is given in Radziszowski's survey 'Small Ramsey Numbers', [11]. Most of the known results hold for the case $k = 2$. In the case $k = 3$ it becomes even more complicated to find general results or to determine exact Ramsey Numbers. Up to now there is only one known exact value for the so called multicolored ($k \geq 3$) classical Ramsey Numbers, namely $r(K_3, K_3, K_3) = r(C_3, C_3, C_3) = 17$ [9]. Considering cycles instead of complete graphs the following Ramsey Numbers are proved:
$r(C_4, C_4, C_4) = 11$ [11], [6], $r(C_5, C_5, C_5) = 17$ [13] and $r(C_6, C_6, C_6) = 12$ [14]. These last two numbers are determined by using computer support.

Example 1. Let the complete graph on $4 \cdot (l-1)$ vertices be colored as follows: There are 4 subgraphs K^1, K^2, K^3 and K^4, each of order $l-1$ and completely colored with color F_1. All edges between K^1 and K^2 and all edges between K^3 and K^4 are colored by F_2. Further all the remaining edges are colored with a third color. If l is odd, this coloring contains no monochromatic C_l. Hence we conclude $r(C_l, C_l, C_l) \geq 4 \cdot (l-1) + 1$ for odd l. The above quoted result shows that this bound is sharp for $l = 5$. Bondy and Erdös conjectured in 1973 that this is true for all odd natural numbers m.

Conjecture 1 (Bondy, Erdös [2], [7])**.** *For all odd natural numbers* $m \geq 5$,

$$r(C_m, C_m, C_m) = 4m - 3.$$

Recently Łuczak [10] has shown that this holds asymptotically.

Theorem 1 (Łuczak [10])**.** *For all natural numbers* $m \geq 5$,

$$r(C_m, C_m, C_m) = (4 + o(1))m.$$

In this paper we will proof the following Main Theorem:

Theorem 2.
$$r(C_7, C_7, C_7) = 25 = 4 \cdot (7-1) + 1.$$

Hence the Ramsey Numbers $r(C_l, C_l, C_l)$ are completely determined for all $l \leq 7$. Our method works without any computer support and using analogous arguments we confirm that $r(C_5, C_5, C_5) = 17$. Also there is some hope that it helps to determine an upper bound for larger odd l.

2. IDEA OF THE PROOF

In this part we motivate and sketch the idea of the proof of Theorem 2. Our example gives for $l = 7$ a 3-colored complete graph on 24 vertices, which contains no monochromatic cycle C_7. Hence it remains to prove that there is no 3-coloring of K_{25}, which avoids a monochromatic cycle of length 7. We consider any 3-coloring of K_{25}. Altogether there are $\binom{25}{2} = 300$ edges. Therefore one of the color classes contains at least 100 edges.

Definition 1. By $ext(H, n)$ (ext for extremal) we denote the maximal number of edges a graph of order n may contain, if it does not contain a subgraph isomorphic to H.

Showing $ext(C_7, n) \leq 99$ would prove our Theorem. But any graph which contains a bipartite part $K_{10,10}$ already has 100 edges and not necessarily a cycle of length 7.

Hence we will extend this definition and prove that any graph on 100 edges which avoids some special graph structures (for instance not being bipartite) contains a cycle of length 7. This forbidden graph classes will be considered separately and we prove that either the graph itself or one color class of its 3-colored complement contains a monochromatic C_7.

These reflections motivate the following definitions and lemmas.

Definition 2. By K_{12}^* we denote any graph of order 12 missing at most four edges.

In the following we sometimes ask that $\overline{K_{12}^*} \not\subseteq G$. This means that any subset of 12 vertices contains at least five edges.

Definition 3. A $B_{7,7}$ is a bipartite graph with partite sets X and Y with $|X| = 7 = |Y|$, such that

- any two vertices in different partite sets are connected by a path of length 5 and by a path of length 3,
- any two vertices in the same partite set are connected by a path of length 4.

Later we will prove the following lemmas.

Lemma 1. *Any* 3*-coloring of* K_{25} *which contains a* 2*-colored* K_{12}^* *also contains a mono-chromatic* C_7.

Lemma 2. *Any* 3*-colored* K_{25}, *which contains a monochromatic* $B_{7,7}$, *also contains a monochromatic cycle of length 7.*

Last we need an extended definition of $ext(C_7, n)$.

Definition 4.

$$ext'(C_7, n) := \max\{|E(G)| \mid |V(G)| = n, C_7 \not\subseteq G,\ B_{7,7} \not\subseteq G,$$

$$\overline{K_{12}^*} \not\subseteq G, G \text{ is not bipartite}\}.$$

We say that any graph with n vertices and m edges, which is not bipartite, contains no $B_{7,7}$ and the complement of which does not contain a K_{12}^*, is an $[n, m]$-graph.

To prove Theorem 2 it would suffice to show that $ext'(C_7, 25) < 100$. But we will prove this recursively; that means to determine an upper bound for $ext'(C_7, n)$ we always need the upper bound for $ext'(C_7, n-1)$. In additon we need the exact value for $ext'(C_7, 12)$ and $ext'(C_7, 13)$ for proving Lemma 1 and Lemma 2. The following lemmas give the exact values $ext'(C_7, n)$ for $n = 7, \ldots, 13$ and upper bounds for $n = 14, \ldots, 25$.

For the presentation of Lemma 4 we need one more notation.

Definition 5.

(a) By $K_{p_1} * K_{p_2} * \ldots * K_{p_i}$ we denote a blockgraph, which consists of i complete blocks $K_{p_1}, \ldots, K_{p_i}$ such that exactly one vertex is contained in any of these complete subgraphs. Using this notation, for three graphs G, H and I, the graph $G * (H * I)$ consists of two graphs G and $H * I$, which have exactly one common vertex.

(b) $G_1 \bigotimes G_2$ denotes the graph, which consists of the two subgraphs G_1 and G_2 such that all vertices in G_1 are adjacent to all vertices in G_2.

Lemma 3. *If* $n = 7, 8, 9, 10$ *and* 11 *we have the following exact values for* $ext'(C_7, n)$:

n	7	8	9	10	11
$ext'(C_7, n) =$	16	18	21	25	30

Lemma 4. $ext'(C_7, 12) = 31$, *and there are exactly three* C_7*-free* $[12, 31]$*-graphs, namely* $K_6 * K_6 * K_2$, $K_6 * (K_6 * K_2)$ *and the graph, which consists of two* K_6, *connected by one edge.*

Lemma 5. $ext'(C_7, 13) = 33$.

Lemma 6. *For* $n \in \{14, 15, 16, \ldots, 25\}$ *the following upper bounds hold:*

n	14	15	16	17	18	19	20	21	22	23	24	25
$ext'(C_7, n) \leq$	41	46	51	56	61	66	71	76	81	87	93	99

Using these lemmas the proof of Theorem 2 reduces as follows:

***Proof of Theorem* 2.** We assume that the complete graph K_{25} is 3-colored with the colors red, green and black. Because of $\binom{25}{2} = 300$ at least one of the color classes contains at least 100 edges.

1. One of the induced color classes is bipartite. Without loss of generality assume this is $\langle R \rangle$. One of the partite sets has at least $\lceil \frac{25}{2} \rceil = 13$ vertices and because of $r(C_7, C_7) = 2 \cdot 7 - 1 = 13$ ([8]) it contains a monochromatic (green or black) C_7.
2. If $K_{12}^* \subseteq K_{25}$, we get a monochromatic C_7 by Lemma 1.
3. If $B_{7,7} \subseteq K_{25}$, we get the monochromatic C_7 by Lemma 2.

Lemma 6 gives $ext'(C_7, 25) \leq 99 < 100$ and hence we get a contradiction or a monochromatic C_7. ■

3. Proof of the Lemmas

To give detailed proofs of these lemmas would significantly expand the paper. Hence we only outline the ideas of the proofs and demonstrate some cases.

All the missing parts can be found in the PhD-thesis of the second author [12]. The proofs of Lemma 1 and Lemma 2 use the numbers $ext'(C_7, n)$ for some n. Hence we will first proof Lemma 3 – Lemma 6.

Lemma 3. *If* $n = 7, 8, 9, 10$ *and* 11 *we have the following exact values for* $ext'(C_7, n)$:

n	7	8	9	10	11
$ext'(C_7, n) =$	16	18	21	25	30

To prove this Lemma we need some more definitions and theorems about pancyclicity and hamiltonicity. We say that a graph G is *Hamiltonian* if it contains a spanning cycle and *pancyclic* if it contains all cycles C_k for $k = 3, 4, \ldots, n$. The graph G is called *weakly pancyclic* if it contains cycles of each length between the length of a shortest and a longest cycle. The *circumference* of a graph G is the length of its longest cycle. To construct the *p-closure* of a graph G we add successively all edges vw where $d(v)+d(w) \geq p$ holds.

The property "G has circumference k" is n-stable [4] means that, if the n-closure of a graph of order n has circumference k, then so does the graph itself. S. Brandt proved that a graph on n vertices and more than $\frac{(n-1)^2}{4} + 1$ edges is weakly pancyclic and contains a triangle (that means it contains all cycles of length between 3 and the length of the longest cycle) [3]. Hence we conclude, that a graph contains a cycle of length 7, if it has n vertices, at least $m \geq \frac{(n-1)^2}{4} + 2$ edges and an n-closure that has cicumference k for $k \geq 7$.

Proof. First, we have to show that any graph G with $m \geq ext'(C_7, n) + 1$ contains a cycle of length 7. Second, we have to find a graph G with $m = ext'(C_7, n)$ edges that contains no C_7.

• $\underline{ext'(C_7, 7) = 16}$: Assume $m \geq 17$. Since G is simple, any set of 6 vertices contains at most 15 edges. Hence we conclude $\delta(G) \geq 2$. For $\delta(G) \geq 4$ the n-closure is complete and G contains a C_7. For $2 \leq \delta(G) \leq 3$ we either have a complete closure or there are vertices v_1, v_2 with $d(v_1) + d(v_2) \leq 6$ and $v_1v_2 \notin E(G)$. But then the fact that there are at most 10 edges between the remaining 5 vertices contradicts $m = 17$ $(17 - 2 \cdot 3 = 11 > 10)$. Using the same arguments we find exactly one $[7, 16]$-graph which contains no C_7, namely the $K_6 * K_2$.

• $\underline{ext'(C_7, 8) = 18}$: Assume $m \geq 19$. If there is a vertex v_1 with $d(v_1) \leq 2$ the graph $G - v_1$ has 7 vertices and $19 - 2 = 17$ edges and hence contains a C_7. For $\delta(G) = d(v_1) = 3$ the graph $G - v_1$ already contains a C_7 or is equal to $K_6 * K_2$. But then there is a C_7 in G itself. Therefore we may assume $\delta \geq 4$, which directly implies a complete n-closure, and with $19 \geq \frac{(8-1)^2}{4} + 1 = 14$ we obtain a C_7. For $m = 18$ there are two counterexamples: the $K_3 \bigotimes \overline{K_5}$ and the $K_6 * K_3$.

• $\underline{ext'(C_7, 9) = 21}$: Let $m \geq 22$. For $\delta(G) = d(v_1) \leq 3$ there is already a C_7 in $G - v_1$ and for $\delta(G) \geq 5$ the complete n-closure and $22 \geq \frac{(9-1)^2}{4} + 1 = 17$ lead to a cycle of length 7. The case that remains is G is not hamiltonian and $\delta(G) = 4$. For this case we assume $m = 21$ and prove in addition, that there is no $[9, 21]$-graph containing no C_7.

Chvátal [5] proved that any graph with vertex degrees $d(v_1) \leq d(v_2) \leq \ldots \leq d(v_n)$, such that $i < n/2$ implies $d(v_i) > i$ or $d(v_{n-i}) \geq n - i$, is hamiltonian. This gives $d(v_1) = d(v_2) = d(v_3) = d(v_4) = d(v_5) = 4$. Because $42 - 5 \cdot 4 = 22$ and $22/4 > 5$ we have $d(v_9) \geq 5$, which implies that $d(v_9) = 8$ in the n-closure of G. From $\frac{22-8}{3} > 4$ the same follows for $d(v_8)$. It is easy to see that each closure has circumference k, $k \geq 7$, if 2 vertices have degree 8 and the remaining 7 vertices have at least degree 4.

This time we find two $[9, 21]$-graphs without any C_7. Obviously both have minimum degree at most 3: Those are the $K_3 \bigotimes \overline{K_6}$ and the $K_6 * K_4$.

• $\underline{ext'(C_7, 10) = 25}$: Analogously we conclude that there is an unique $[10, 25]$-graph with no C_7, namely the $K_6 * K_5$, and none $[10, 26]$-graph.

• $\underline{ext'(C_7, 11) = 30}$: This time we find $K_6 * K_6$ as the unique $[11, 30]$-graph containing no C_7. ∎

Lemma 4. *There are exactly three C_7-free $[12, 31]$-graphs, namely $K_6 * K_6 * K_2$, $K_6 * (K_6 * K_2)$ and the graph, which consists of two K_6, connected by one edge, and $ext'(C_7, 12) = 31$.*

Sketch of the proof. Considering not 2-connected graphs — which means blocks — it is easy to see, that the given 3 graphs (the $K_6 * K_6 * K_2$, the $K_6 * (K_6 * K_2)$ and the graph, which consists of two K_6, connected by one edge) are the only $[12, 31]$-graphs which are C_7-free.

Now we prove that any 2-connected $[12, 31]$-graph contains a cycle of length 7. Let G be any 2-connected non-bipartite graph. Also let $B_{7,7} \nsubseteq G$

and $\overline{K_{12}^*} \not\subseteq G$. If G is in addition C_7-free, then we will prove that there are at most 30 edges.

Since G is not bipartite, there is a longest odd cycle C_p. We denote this length by p. In each case we consider such a cycle C_p. Knowing that C_p is a longest odd cycle we first examine how many edges are possibly between $V(C)$ and the remaining vertices and in the set of the remaining vertices. Then we find out how many further neighbors on the cycle any vertex $v \in V(C_p)$ may have. Last we always get a cycle C_7, a longer odd cycle or a contradiction to the number of edges. We demonstrate the case $p = 11$. The remainig cases are similar, but not identical.

$\underline{p = 11}$. Let $C = (v_1, v_2, \ldots, v_{11}, v_1)$ be a cycle of length 11. Since there is no cycle of length 7, the remaining vertex v_{12} has at most 5 neighbors on the cycle. Now we consider how many additional edges are possibly within the cycle. A vertex v_i has at most 3 other neighbors (except v_{i-1} and v_{i+1}). Having 31 edges in G there is one vertex, say v_1, with exactly 3 other neighbors. We distinguish between the different possibilities and check how that influence the possible neighbors of v_7.

- $v_1v_3, v_1v_4, v_1v_5 \in E(G)$: In this case there is either a C_7 or v_7 has no further neighbor on the cycle.
- $v_1v_3, v_1v_4, v_1v_{10} \in E(G)$: In this case we again find either a cycle C_7 or v_7 has no further neighbor on the cycle.
- $v_1v_3, v_1v_5, v_1v_9 \in E(G)$: This time the vertex v_7 has at most 1 further neighbor, namely v_4 or v_{10} or we get a cycle of length 7.
- $v_1v_4, v_1v_5, v_1v_8 \in E(G)$: Now v_7 has at most 2 neighbors in addition.

Altogether this means: If a vertex v_i has 3 further neighbors in $V(C)$, then the vertex on the opposite side, v_{i+6}, has at most 2 further neighbors on the cycle. Hence either G contains a cycle C_7 or there are at most $(\frac{11}{2} \cdot 3 + \frac{11}{2} \cdot 2)/2 < 14$ further edges between the vertices $V(C)$. Hence the graph G itself consists of at most $5 + 11 + 14 = 30 < 31$ edges. ∎

Lemma 5. $ext'(C_7, 13) = 33$.

Sketch of the proof. For example the graphs $K_6 * K_6 * K_3$, $K_6 * (K_6 * K_3)$ and $K_3 \bigotimes \overline{K}_{10}$ consist of 13 vertices, 33 edges, are not bipartite, contain no $B_{7,7}$ no cycle of length 7 and also no K_{12}^* in their complement. Hence $ext'(C_7, 13) \geq 33$.

Now we have to prove that for any $[13, m]$-graph, which contains no C_7 it is $m \le 33$.

Because lack of space we also skip this proof. You can verify the result by using methods similar to those of the proof of Lemma 4. Also here we distinguish between the lengths of the longest odd cycle and consider all possibilities. ■

To prove the following Lemma 6 we need one corollary.

Corollary 1. *Let G be a bipartite graph with partite sets X and Y with $|X| = |Y| = 7$. If the graph G contains at least $m = 35$ edges and has minimum degree $\delta(G) \ge 2$, then any pair of vertices in different partite sets is connected by a path of length 5.*

Sketch of the proof. We show that there is a path of length 5 between $x_1 \in X$ and $y_1 \in Y$. Therefore we let $x_1 y_2 \in E(G)$. To distinguish after the degree of y_1 and y_2 gives the desired result. ■

Lemma 6. *For $n \in \{14, 15, 16, \dots, 25\}$ the following upper bounds hold:*

n	14	15	16	17	18	19	20	21	22	23	24	25
$ext'(C_7, n) \le$	41	46	51	56	61	66	71	76	81	87	93	99

Proof. To prove this Lemma we will use completely different techniques than those used in the proofs of Lemma 3, Lemma 4 or Lemma 5. First we find the lower bound for the minimum degree of any graph G on n vertices and $ext'(C_7, n)$ edges, which is not bipartite and contains neither a C_7, a $B_{7,7}$, or a $\overline{K_{12}^*}$. Then we consider the following cases: The graph G contains a K_4, a $K_4 - e$, but no K_4, a K_3, but no $K_4 - e$, or last G is trianglefree.

Again we have not enough place for the whole proof. We demonstrate the ideas by considering the case that G contains a K_4 step by step. The remaining cases are very long, but mainly use similar ideas.

If G is not 2-connected we consider the different possible blocks and in each case we get one of the forbidden graphs or a contradiction to the number of edges. Hence we may assume that G is 2-connected.

1. The minimum degrees

We determine the minimum degree of G for all n with $n = 14, 15, \dots, 25$ simultaneously. Hence we always assume — when considering the graph on n vertices — that the upper bound for $ext'(C_7, n-1)$ is already known.

Let $n = 14$ and $m = 42$: Assume $\delta(G) \leq 5$. We consider the graph G^* which results from G by removing a vertex v with minimum degree. Because of $ext'(C_7, 13) = 33$ we conclude that G^* either contains a C_7, a $B_{7,7}$, a $\overline{K^*_{12}}$ or is bipartite. In the first three cases so does the graph G itself which is a contradiction to the definition of $ext'(C_7, n)$. Thus we assume that G^* is bipartite. Since G is not bipartite the vertex v has at least one neighbor in each partite set of G^*, which implies that v has at most four neighbors in one partite set. Now we consider the different possibilities of the order of the partite sets. Let $G_{p,q}$ denote any bipartite graph with partite sets X and Y where $|X| = p$ and $|Y| = q$. If G^* is isomorphic to a $G_{1,12}$, a $G_{2,11}$, a $G_{3,10}$ or a $G_{4,9}$, it consists of less than $4 \cdot 9 = 36 = 37 = 42 - 5$ edges or G contains a C_7. In each case this is a contradiction. If $\delta(G^*) \geq 2$ then in any $G_{5,8}$ and any $G_{6,7}$ on 37 edges each pair of vertices in different partite sets is connected by a path of length 5. Hence we again find a C_7 in G and conclude $\delta(G) \geq 6$.

In Corollary 1 we have proved that in a graph $G_{7,7}$ with $\delta(G) \geq 2$ and 35 edges any pair of vertices in different partite sets is connected by a path of length 5. Because of $\delta(G) \geq 2$ then either this also holds for $G_{p,q}$ with $p, q \geq 7$ or the graph $G_{p,q}$ contains at most $34+4(p-7)+4(q-7)+(p-7)(q-7)$ edges.

Hence, since G contains a $\overline{K^*_{12}}$ if the graph G^* is isomorphic to a $G_{p,q}$ with $q \geq 11$ ($p \geq 11$ respectively) we have $\delta(G) \geq 6$ for $n \geq 15$.

2. <u>G contains a K_4</u>
Let a, b, c, d span the K_4. Since G is 2-connected, there are two vertices x_1 and x_2 with $ax_1, bx_2 \in E(G)$. Either these two vertices are connected by another path P in $V(G) \setminus K_4$, or all vertices of $V(G) \setminus K_4$ have only neighbors in K_4. But this already contradicts $\delta(G) \geq 6$. We consider all possible lengths of such a path P.

(a) <u>$P = P_2$:</u> Let $X := K_4 \cup \{x_1, x_2\}$ and $Y := V(G) \setminus X$. Now the following hold:

(i) It is not possible that a vertex $y \in Y$ has more than two neighbors in X and there are only three possibilities such that a vertex y has two neighbors in X, namely $yx_1, yb \in E(G)$, $yx_2, ya \in E(G)$ or $ya, yb \in E(G)$.

(ii) If there are two vertices y_1 and $y_2 \in Y$ each of which having two neighbors in X, then there are the following possibilities:

– $y_1a, y_1b, y_2x_1, y_2b \in E(G)$, where y_1 and y_2 are not adjacent and not connected by a path of length 2 or 3.
– $y_1a, y_1b, y_2x_2, y_2a \in E(G)$, where y_1 and y_2 are not adjacent and not connected by a path of length 2 or 3.
– $y_1a, y_1b, y_2a, y_2b \in E(G)$, where y_1 and y_2 are not connected by any path of length 2 or 3.
– $y_1x_1, y_1b, y_2x_1, y_2b \in E(G)$, where y_1 and y_2 are not adjacent and not connected by any path of length 2 or 3.
– $y_1x_2, y_1a, y_2x_2, y_2a \in E(G)$, where y_1 and y_2 are not adjacent and not connected by any path of length 2 or 3.

(iii) If there are two vertices y_1 and y_2 with $y_1y_2 \in E(G)$ and both have at least one neighbor in X, then they either have the same neighbor or $y_1c, y_2d \in E(G)$ or $y_1a, y_2b \in E(G)$.

Since $\delta(G) \geq 6$ each of the vertices x_1, x_2, c and d has a neighbor in Y. Because of i. these four neighbors are distinct. Let x_1y_1, x_2y_2, cy_3 and $dy_4 \in E(G)$. The vertices y_3 and y_4 must not have another neighbor in X, but possibly we have $y_3y_4 \in E(G)$. Hence both have four neighbors in Y. To avoid a C_7 we have $y_5, y_6, y_7, y_8 \in N(y_3)$ and $y_9, y_{10}, y_{11}, y_{12} \in N(y_4)$. One of the vertices y_1 and y_2 may have a second neighbor in X. Nevertheless both have at least four neighbors in Y and again we either find a C_7 or there are eight additional vertices. But then we would have at least $6+10+5+5 = 26$ vertices, a contradiction to $n \leq 25$.

(b) $\underline{P = P_3,\ P = P_4,\ P = P_5}$: This directly gives a C_7.

(c) $\underline{P = P_6}$: Let $x_1x_3x_4x_5x_6x_2$ be the path between x_1 and x_2 and $X := \{a, b, c, d, x_1, \dots, x_6\}$. Avoiding a C_7 and a contradiction to one of the previous cases both, x_6 and x_3, have no other neighbor in X and also no common neighbor in Y. Hence $y_1, y_2, y_3, y_4 \in N(x_3)$ and $y_5, y_6, y_7, y_8 \in N(x_6)$. Now we consider the vertices c and d. Both are connected to at most one of x_1 and x_2 and must not be connected to any vertex of $\{x_3, x_4, x_5, x_6, y_1 \dots, y_8\}$. Hence there are two further vertices y_9 and y_{10}. Let $cy_9, cy_{10} \in E(G)$. Then only a, b and d might be additional neighbors of y_9 and y_{10} in X. Now we distinguish between $y_9, y_{10} \in N(d)$ and $y_9, y_{10} \notin N(d)$.

- $dy_9, dy_{10} \in E(G)$: This implies $y_9y_{10} \notin E(G)$. If $y_9a \in E(G)$ then $y_{10}b \notin E(G)$ and vice versa. Otherwise we would find the case that $P \cong P_2$. This also implies that y_9 (y_{10}) has at most one neighbor in $\{a, b\}$. Hence both y_9 and y_{10} have at least three further neighbors and in addition these have to be distinct. But this implies that we need at least $10 + 10 + 6 = 26$ vertices, a contradiction to $n \leq 25$.

- $dy_9 \in E(G)$, $dy_{10} \notin E(G)$: This gives the existence of a further vertex y_{11} which is adjacent to d. Avoiding a C_7 the vertices y_{10} and y_{11} have no common neighbor and also avoiding the case $P \cong P_2$ they are not adjacent to y_9. Thus we need six further vertices, which again contradicts $n \leq 25$.
- $dy_9, dy_{10} \notin E(G)$: As above we need at least six additional vertices and hence get a contradiction.

(d) $\underline{P = P_k,\ k \geq 7}$: Analogously to the previous case we need more than 25 vertices because of $\delta \geq 6$. ■

Lemma 1. *Any 3-coloring of K_{25} which contains a 2-colored K_{12}^* also contains a monochromatic C_7.*

Proof. Assume that the edges of the complete graph on 25 vertices are colored with the three colors red, green and black, and that there is a subset of 12 vertices which contains at most four green edges. Since $ext'(C_7, 12) = 31$ and $31+31+4 = 66 = \binom{12}{2}$ we conculde that one of the color classes (red or black) of this set is bipartite, or that both color classes consist of exactly 31 edges.

In the first case let the red graph be the bipartite one. If one of the partite sets has seven or more vertices we directly find a monochromatic black C_7. Hence both partite sets are of order 6. We may partition the vertex set $V(G)$ into three sets A, B and X, such that $|A| = |B| = 6$ and $|X| = 13$. All red edges are between A and B, and within the sets A and B there are only black and green edges (remember that altogether there are at most four green edges).

In the second case we consider the three different possible $[12, 31]$-graphs and conclude, that also then there is either a monochromatic C_7 or a bipartite color class. Again we have three vertex sets A, B and X with $|A| = |B| = 6$ and $|X| = 13$. Also this time almost all (except at most four) edges in A and B are black and all red edges are between A and B.

Case 1. First we assume that each of the sets A and B contains at most two green edges. Now the following hold:

(a) We consider the black subgraph. Any two adjacent vertices within A (B resp.) are connected by a black path of any length up to P_6 within A (B), and between any two non adjacent (in black) vertices only the P_2 is missing: We consider the subgraph H which consists of the vertex set A and all black edges between these vertices.

We know that the property of Hamiltonian-connected is $(n+1)$-stable. This means that a graph G is Hamiltonian-connected if its $(n+1)$-closure is Hamiltonian-connected. Since H misses at most two edges, we have $d_H(a_i) + d_H(a_j) \geq 5 + 5 - 3 = 7$ for all non adjacent vertices a_i and a_j. We get a complete 7-closure, and hence H is Hamiltonian-connected.

Similar we prove the existence of the paths of length 4: Let a_1 and a_2 be any two vertices in H. We consider the graph $H' := H - \{a_k\}$, where $k \neq 1, 2$ and $d_{H'}(a_k)$ is minimum. Hence, $d_{H'}(a_i) + d_{H'}(a_j) \geq 4+4-2 = 6 = n(H')+1$ for all non adjacent pairs of vertices. Also H' is Hamiltonian-connected and there is a P_5 between a_1 and a_2. In a direct way the existence of the remaining paths P_3 and P_4 between any two vertices in H can be proved.

(b) Any two vertices in A (B resp.) are connected by a red P_3 and a red P_5 within $A \cup B$.

Because of (a) there is at most one black edge between A and B. Hence there are at most five non red edges between A and B. Considering the different possibilities for the vertex degrees we conclude (b) and also (c).

(c) Any two vertices $a \in A$ and $b \in B$ are connected by a red P_4 and a red P_6, except for the case that a (b respectively) has exactly four green and one black neighbor in B (A respectively). We will consider this case $(*)$ separately at the end of the proof. Hence we may assume now that any $a \in A$ and $b \in B$ are connected by a red P_4 and a red P_6.

These previous remarks imply the following coloring:

- Each vertex in X has either no red neighbor in A or no red neighbor in B. Hence let $X = C \cup D$ and we may assume that there are no red edges between A and C and no red edges between B and D (but maybe between A and D and between B and C). Without loss of generality let $|C| \geq 7$.
- Because of (a) there are at most $|C|$ ($|D|$ resp.) black edges between A and C (B and D). In particular each vertex in C (D) has at most one black neighbor in A (B).
- Hence most of the edges (at least $6 \cdot 7 - 7 = 35$) between A and C are green and also most of the edges (except $|D|$) between B and D.

– Any two vertices in C have at least four common green neighbors in A, and with a third (fourth) vertex they still have three (two) in common. Hence any green edge in C would give a green C_7. We consider the set $C \cup B \setminus \{b\}$, where b is the vertex in B that has the most green neighbors in B. That implies that $B \setminus \{b\}$ contains at most one green edge. Since $|C \cup B \setminus \{b\}| \geq 12$, $ext'(C_7, 12) = 31$, $2 \cdot 31 = 62$ and $\frac{11 \cdot 12}{2} = 66$ there are at least three green edges between C and B.

We now distinguish the cardinality of D. Since there are only red and black edges in C the case $|D| \leq 1$ contradicts $ext'(C_7, 12) = 31$. Hence let $|D| \geq 2$. In the case $|D| \geq 4$ the set D — and also C — must not contain a green edge. Otherwise we would find a monochromatic C_7, constructed with one green edge in D and six further edges between B and D. But again because of $ext'(C_7, 12) = 31$, $2 \cdot 31 = 62$ and $\frac{11 \cdot 12}{2} = 66$ there is a green edge $c_1 d_1$ between C and D. If $D = \{d_1, d_2\}$ ($D = \{d_1, d_2, d_3\}$ resp.) we consider the sets $C \cup \{d_1\}$ ($C \cup \{d_1, d_2\}$ resp.), and it follows that there is a green edge $c_1 d_1$.

Now we focus on the set $C \setminus \{c_1\} \cup B$. The set B contains at most two green edges. Since $ext'(C_7, 12) = 31$ and $C \setminus \{c_1\} \cup B \geq 12$ there is also a green edge $c_2 b_1$ between B and C. The vertices c_1 and c_2 have a common green neighbor in A, let this be a_1. In addition there is a green edge $d_1 b_2$, and the vertices b_1 and b_2 also have a green neighbor in $D \setminus \{d_1\}$ in common. But now we have a green cycle C_7, namely $(a_1 c_1 d_1 b_2 d_2 b_1 c_2 a_1)$.

Now we consider the case $(*)$. We assume that $a \in A$ (without loss of generality) has exactly four green and one black neighbor in B. If nevertheless there is no vertex in $C \cup D$ having a red neighbor in A and B, everything works as above. Hence let us assume that there are five vertex sets A, B, C', D and C'', such that A and B are as before, C' contains all vertices without any red neighbor in A, D contains all vertices with no red neighbor in B, and C'' contains all vertices having red neighbors in A and B. Avoiding a monochromatic C_7 all vertices $c \in C''$ have exactly one red neighbor in A, namely a, and exactly one in B, namely the red neighbor of a. Let $C := C' \cup C''$. If $|D| \geq 7$ there is again no green edge in D, and with D instead of C we find a green C_7. Now let $|C| \geq 7$. If we could prove that also in this case there is no green edge in C, then the existence of a monochromatic C_7 follows as above. Thus let the edge $c_1 c_2$ be green. Obviously the vertices c_1 and c_2 have three green neighbors in common in $A \setminus \{a\}$, say a_1, a_2 and a_3. Of course this also holds for the vertices c_3 and c_4, where at least one of these neighbors is in $\{a_1, a_2, a_3\}$. Now these vertices span a green C_7, a contradiction.

Case 2. Last we consider the case that one of the sets A or B contains three or four green edges. Without loss of genenerality we assume this is the set A. Again we denote by H the subgraph which consists of the vertex set A and all black edges between these vertices. Hence, $\overline{H}$ consists of the vertex set A and all green edges between these vertices. As above we can prove that any two vertices in H are connected by a path P_3, P_4 and a path P_5. In addition we know that the 7-closure of H is either complete (and hence H is Hamiltonian-connected), or $\overline{H}$ contains a triangle or a $K_{1,3}$. We now consider these last possibilities in the appendix. ■

Lemma 2. *Any* 3*-colored* K_{25}*, which contains a monochromatic* $B_{7,7}$*, also contains a monochromatic cycle of length* 7*.*

Proof. Again we will use the three colors red, green and black and denote them by R, G and B. With no loss of generality we will assume that the $B_{7,7}$ is in $\langle R\rangle$ and that the parts of the $B_{7,7}$ will be denoted by U and V. We will assume that there is no monochromatic C_7, and show that this leads to a contradiction.

No vertex outside of $B_{7,7}$ can have a red adjacency in both U and V, because this would imply a C_7 in $\langle R\rangle$. Therefore, the vertices of $V(K_{25}) \setminus V(B_{7,7})$ can be partitioned into two sets X and Y, such that the vertices of X have no red neighbor in V and the vertices of Y have no red neighbor in U as well. We have $|X| + |Y| = 11$, and with no loss of generality we can assume that $|X| \geq |Y|$, and that X will be chosen as large as possible. This implies that each vertex of Y must have a red neighbor in V, for otherwise it could be moved to X. The general structure of the proof will be to appropriately select a set Z of 13 vertices that has no more than 11 red edges. Then, since $ext'(13, C_7) = 33$ and $2 \cdot 33 + 11 = 77 < \binom{13}{2}$, either $\langle B\rangle$ or $\langle G\rangle$ must be bipartite to avoid a monochromatic C_7. If, say $\langle B\rangle$ is bipartite, then there is a set of 7 vertices that has only green and black edges. If the red edges of Z have the additional property that the graph induced by any set of 7 vertices has a C_7 in its complement, then there is a monochromatic C_7 in G and this gives a contradiction that completes the proof. The set of 13 vertices will normaly consist of six vertices from U, four vertices from X, and three vertices from Y, but this will vary in some cases. The remainder of the proof will be broken into 6 cases, depending on $|X|$. We here only consider the case $|X| = 6$.

$\underline{|X| = 6 :}$ We first claim that there cannot be a red C_6 in X. Therefore we assume this were true. If there are two vertices in X without a red neighbor in U, then these two vertices could be moved to Y, which contradicts the maximality of $|X|$. If a pair of consecutive vertices on the C_6 have a red neighbor in U, then there is a red C_7.

In V and between V and X there are no red edges. Because of Lemma 1 any 2-colored $K_{12} - 4e$ gives a monochromatic cycle C_7. Hence any five vertices in X are connected by at least five red edges. By $\langle R\rangle_X$ we denote the red subgraph which is induced by all vertices in X. The set X consists of six vertices. If there is a vertex with m red neighbors, then $\langle R\rangle_X$ contains at least $m + 5$ edges. If $m \leq 2$ for all vertices $x \in X$ we may find a set of five vertices with less than five red edges. Hence we can conclude that there are at least $3 + 5 = 8$ edges in $\langle R\rangle_X$. In addition $\langle R\rangle_X$ is either connected or we have $\langle R\rangle_X \cong K_1 \cup (K_5 - e)$ or $\langle R\rangle_X \cong K_1 \cup K_5$.

First let $\langle R\rangle_X$ be connected. As before at least 5 of the 6 vertices in X have a red neighbor in U. In addition adjacent vertices must not have distinct red neighbors in U. In both cases failur would imply a monochromatic C_7.

If $\langle R\rangle_X$ is 2-connected, then there is only one vertex $u \in U$ with red neighbors in X. Let $U' := U\setminus\{u\}$. In addition there is no red edge between X and Y, since otherwise we could find a red path P_5 starting in u through two vertices in X and one vertex in Y to some $v \in V$. This would imply a red cycle of length 7. Since $\langle R\rangle_X$ is not complete, there is a subset on four vertices in X which does not form a K_4. We will denote such a subset by X'. Also we assume that Y' consists of three vertices in Y. The set $Z := X' \cup Y' \cup U'$ contains $|Z| = 6 + 4 + 3 = 13$ vertices and has at most $5 + 3 = 8$ red edges. In addition a subgraph which is induced by seven vertices, contains a cycle C_7 that avoids any red edges. These required properties are fulfilled by Z and we get a contradiction.

Using the same arguments we find a contradiction if there is a subset of X on four vertices which are at most in red adjacent to one vertex $u \in U$ and are not completely connected by red edges.

If $\langle R\rangle_X$ is not 2-connected, then there is only the possibility that $\langle R\rangle_X \cong K_5 * K_2$, where the cutvertex x_s has no red neighbor in U. By x we denote the second vertex of the K_2 (the vertices x and x_s form this K_2). As before all vertices in the K_5 — except the cutvertex — have the same vertex $u \in U$ as a red neighbor. Also there is no red edge between these five vertices and Y. Because of the maximality of X we conclude that the vertex x has at least one red neighbor in $U \setminus \{u\}$ and possibly it is red adjacent to all vertices in $U \setminus \{u\}$. In additon there may be red edges to Y. We will consider these possibilities.

First we assume that x has no red neighbor in Y. Let $U_R(x) := N_R(x) \cap U$. If $|U_R(x)| \leq 4$, then we may choose the set X' as x and three further vertices in $V(K_5) \setminus \{x_s\}$ and get a contradiction. If $|U_R(x)| \geq 5$, then we conclude by the definition of $B_{7,7}$ that $N_R(u) \cap N_R(u') \cap V \neq \emptyset$ for at least one neighbor u' of x. Thus we find a red C_7, that uses four vertices in X, two vertices in U and one vertex in V.

If there is a set of four vertices in Y that does not build a red K_4 then we choose this set as Y'. The set X' consists of three vertices in $X \setminus \{x\}$, and again we get a contradiction with $Z = X' \cup Y' \cup U'$.

Hence the vertices in Y span a complete red graph K_5 and all have the same vertex $v \in V$ as unique red neighbor in V. If x has a neighbor in Y, then there is a red path of length 4 starting in $U \setminus \{u\}$ through $X \cup Y$ to $v \in V$ and thus there also is a cycle of length 7.

If $\langle R \rangle_X$ is not connected, that means $\langle R \rangle_X \cong K_1 \cup K_5 - e$ or $\langle R \rangle_X \cong K_1 \cup K_5$, then we get a last contradiction — analogously to the case where $\langle R \rangle_X = K_5 * K_2$. ■

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Received 30 July 2001
Revised 18 January 2002