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LABELING THE VERTEX AMALGAMATION OF GRAPHS

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Dedicated to Professor Miguel Angel Fiol

Abstract

A graph G of size q is graceful if there exists an injective function $f : V(G) \rightarrow \{0, 1, \ldots, q\}$ such that each edge uv of G is labeled $|f(u) - f(v)|$ and the resulting edge labels are distinct. Also, a (p, q) graph G with $q \geq p$ is harmonious if there exists an injective function $f : V(G) \rightarrow \mathbb{Z}_q$ such that each edge uv of G is labeled $f(u)+f(v) \pmod q$ and the resulting edge labels are distinct, whereas G is felicitous if there exists an injective function $f : V(G) \rightarrow \mathbb{Z}_{q+1}$ such that each edge uv of G is labeled $f(u) + f(v) \pmod q$ and the resulting edge labels are distinct. In this paper, we present several results involving the vertex amalgamation of graceful, felicitous and harmonious graphs.

Further, we partially solve an open problem of Lee et al., that is, for which m and n the vertex amalgamation of n copies of the cycle C_m at a fixed vertex $v \in V(C_m)$, $\mathrm{Amal}(C_m, v, n)$, is felicitous? Moreover, we provide some progress towards solving the conjecture of Koh et al., which states that the graph $\mathrm{Amal}(C_m, v, n)$ is graceful if and only if $mn \equiv 0$ or $3 \pmod 4$. Finally, we propose two conjectures.

Keywords: felicitous labellings, graceful labellings, harmonious labellings.

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1. Introduction

We follow Chartrand and Lesniak [1] for most of the graph theory terminology and notation used in this paper. In particular, we will consider a graph to be finite and without loops or multiple edges. Further, the vertex set of a graph G is denoted by $V(G)$, whereas the edge set of G is denoted by $E(G)$.

To formalize our presentation of this paper, we will introduce some definitions not found in [1].

In 1967, Rosa [6] initiated the study of β-valuations, which were subsequently named graceful labelings by Golomb [2]. The term, graceful labeling, is now the popular one in the literature of graph labeling. A graph G of size q is *graceful* if there exists an injective function $f : V(G) \to \{0, 1, \ldots, q\}$ such that each edge uv of G is labeled $|f(u) - f(v)|$ and the resulting edge labels are distinct. Such a function is said to be a *graceful labeling*. In [6], Rosa also introduced the notion of an α-valuation. A graceful labeling f of a graph G is an *α-valuation* if there exists an integer k, called the *characteristic* of f, so that $\min\{f(u), f(v)\} \leq k < \max\{f(u), f(v)\}$ for each edge uv of G.

In 1980, Graham and Sloane [3] introduced the notion of harmonious labelings. A (p, q) graph G with $q \geq p$ is *harmonious* if there exists an injective function $f : V(G) \to \mathbb{Z}_q$ such that each edge uv of G is labeled $f(u) + f(v) \pmod q$ and the resulting edge labels are distinct. Such a function is called a *harmonious labeling*. If G is a tree (so that $q = p - 1$) exactly two vertices are labeled the same; otherwise, the definition is the same.

We now consider a type of graph labeling proposed by Shee [7] as a possible generalization of harmonious labelings. A graph G of size q is *felicitous* if there exists an injective function $f : V(G) \to \mathbb{Z}_{q+1}$ such that each edge uv of G is labeled $f(u) + f(v) \pmod q$ and the resulting edge labels are distinct. Such a function is called a *felicitous labeling.*

Here, we define a felicitous labeling f of a graph G to be *strongly felicitous* if there exists an integer k, called the *characteristic* of f, so that $\min\{f(u), f(v)\} \leq k < \max\{f(u), f(v)\}$ for each edge uv of G. Thus, a *strongly felicitous graph* is a graph that admits a strongly felicitous labeling.

The following definition due to Lee, Schmeichel and Shee [5] will be useful for the remainder of this paper. For any two graphs G and H, let u and v be fixed vertices of G and H, respectively. Then the *vertex amalgamation* of G and H is the graph obtained from G and H by identifying G and H at the vertices u and v. In particular, the vertex amalgamation of n copies of G at the fixed vertex $v \in V(G)$ is denoted by $\mathrm{Amal}(G, v, n)$ for every integer $n \geq 2$.

In this paper, we first present a number of methods to preserve felicitousness and harmoniousness from the vertex amalgamation of felicitous and harmonious graphs, respectively. We also establish a way of producing an α-valuation from the vertex amalgamation of strongly felicitous graphs. We next consider an open problem posed by Lee, Schmeichel and Shee [5], that is, for which m and n is the graph $\mathrm{Amal}(C_m, v, n)$ felicitous? Moreover, we provide some progress towards settling the conjecture by Koh, Rogers, Lee and Toh [4] that the graph $\mathrm{Amal}(C_m, v, n)$ is graceful if and only if $mn \equiv 0$ or $3 \pmod 4$. Finally, we propose two conjectures.

To conduct our study of α-valuations of the vertex amalgamation of graphs, the following result will prove to be useful.

Lemma 1.1. *A (p, q) graph G with $q \geq p - 1$ is strongly felicitous if and only if G admits an α-valuation.*

Proof. Assume that a (p, q) graph G with $q \geq p - 1$ admits a strongly felicitous labeling f with characteristic k. Then G is bipartite with partite sets

$$V_1 = \{v \in V(G) : f(v) \leq k\} \text{ and } V_2 = \{v \in V(G) : f(v) > k\}.$$

Now, define the vertex labeling $g : V(G) \to \{0, 1, \ldots, q\}$ such that

$$g(x) = \begin{cases} f(x) & \text{if } x \in V_1, \\ k + q + 1 - f(x) & \text{if } x \in V_2. \end{cases}$$

Next, we verify that g is an α-valuation of G with characteristic k. Here, notice that

$$g(V_1) \subseteq \{0, 1, \ldots, k\} \text{ and } g(V_2) \subseteq \{k+1, k+2, \ldots, q\},$$

implying that g is injective and

$$\begin{aligned} |g(u) - g(v)| &= g(u) - g(v) \\ &= k + q + 1 - (f(u) + f(v)) \end{aligned}$$

for each edge uv of G, where $u \in V_2$ and $v \in V_1$. Thus

$$1 \leq |g(u) - g(v)| \leq q$$

since $k + 1 \leq f(u) + f(v) \leq k + q$.

Finally, observe that $\{f(u) + f(v) \pmod q : uv \in E(G)\}$ is a set of q consecutive integers; hence, $\{|g(u) - g(v)| : uv \in E(G)\} = \{1, 2, \ldots, q\}$.

The converse is analogous in that it requires the inverse of the transformation between f and g used above, and a similar verification of conditions. ■

The preceding lemma leads to some evidence of the validity of conjecture by Lee, Schmeichel and Shee [5] that every tree is felicitous since a number of techniques to construct trees from smaller ones with α-valuations have been shown to yield α-valuations in the resulting trees. The reader is referred to the survey paper by Gallian [2] for references on these methods.

2. General Results

In this section, we present some general results involving the vertex amalgamation of graphs admitting α-valuations, and felicitous and harmonious labelings.The first result shows that the graph obtained from amalgamating two strongly felicitous graphs at the vertices labeled 0 preserves strongly felicitousness.

Theorem 2.1. *Assume that G_i is a strongly felicitous graph with a strongly felicitous labeling f_i for $i = 1, 2$. Then the graph H obtained from G_1 and G_2 by amalgamating G_1 and G_2 at the vertices $u \in V(G_1)$ and $v \in V(G_2)$ such that $f_1(u) = 0$ and $f_2(v) = 0$ is strongly felicitous.*

Proof. Let k_i be the characteristic of G_i for $i = 1$, 2. Then, without loss of generality, assume that $k_1 \geq k_2$.

Now, let $q(G_i)$ denote the size of G_i, and define the vertex labeling

$$g : V(G_1) \cup V(G_2) \to \mathbb{Z}_{q(G_1)+q(G_2)+1}$$

such that

$$g(x) = \begin{cases} k_1 - f_1(x) & \text{if } x \in V_1(G_1), \\ k_1 + k_2 + q(G_1) + 1 - f_1(x) & \text{if } x \in V_2(G_1), \\ k_1 + f_2(x) & \text{if } x \in V_1(G_2), \\ q(G_1) + f_2(x) & \text{if } x \in V_2(G_2), \end{cases}$$

where $V_1(G_i) = \{v \in V(H) : f_i(v) \leq k_i\}$ and
$V_2(G_i) = \{v \in V(H) : f_i(v) > k_i\}$ for $i = 1, 2$.

Then

$$V_1(G_1) \cup V_1(G_2) = \{v \in V(H) : g(v) \leq k_1 + k_2\}$$

and

$$V_2(G_1) \cup V_2(G_2) = \{v \in V(H) : g(v) > k_1 + k_2\}\,.$$

Therefore, g extends to a strongly felicitous labeling of H with characteristic $k_1 + k_2$. ■

In light of Lemma 1.1 and Theorem 2.1, we now obtain the next corollary.

Corollary 2.2. *Assume that G_i is a strongly felicitous graph with a strongly felicitous labeling f_i for $i = 1, 2$. Then the graph H obtained from G_1 and G_2 by amalgamating G_1 and G_2 at the vertices $u \in V(G_1)$ and $v \in V(G_2)$ such that $f_1(u) = 0$ and $f_2(v) = 0$ admits an α-valuation.*

Whereas Theorem 2.1 concerns the vertex amalgamation of two strongly felicitous graphs, our next result involves the vertex amalgamation of n copies of harmonious graphs when n is odd.

Theorem 2.3. *Assume that G is a harmonious graph with aharmonious labeling f, and let n be odd. Then the graph $H \cong Amal(G, v, n)$ obtained from n copies of G by amalgamating at the vertex $v \in V(G)$ such that $f(v) = 0$ is harmonious.*

Proof. First, let $V(G) = \{v_1, v_2, \ldots, v_p\}$. Then, without loss of generality, assume that $f(v_1) = 0$.

Now, let $G_j \cong G$ with $V(G_j) = \{v_1^j, v_2^j, \ldots, v_p^j\}$ for every positive integer j such that $1 \leq j \leq n$. Then the vertex labeling $g : V(H) \to \mathbb{Z}_{nq(G)}$ such that

$$g(x) = \begin{cases} 0 & \text{if } x = v_1^j \text{ and } 1 \leq j \leq n, \\ f(v_i) + q(G)(j-1) & \text{if } x = v_i^j, 2 \leq i \leq p \text{ and } 1 \leq j \leq n, \end{cases}$$

extends to a harmonious labeling of H, where $q(G)$ is the size of G. ∎

The previous theorem easily extends to the graphs with felicitous labelings; hence, we analogously obtain the next result.

Theorem 2.4. *Assume that G is a felicitous graph with a felicitous labeling f, and let n be odd. Then the graph Amal(G, v, n) obtained from n copies of G by amalgamating at the vertices $v \in V(G)$ such that $f(v) = 0$ is felicitous.*

3. Results on Vertex Amalgamation of Cycles

With the results of the previous section in hand, we investigate α-valuations, felicitous and harmonious labelings of the graph Amal(C_m, v, n) obtained from n copies of the cycle C_m by amalgamating at the fixed vertex $v \in V(C_m)$.

Now, we present a strongly felicitous labeling of Amal$(C_m, v, 2)$ when $m \geq 4$ is even. This labeling is of interest as the central vertex of Amal$(C_m, v, 2)$ in it receives the label 0, which implies that two copies of Amal$(C_m, v, 2)$ can be amalgamated at their central vertices to show that Amal$(C_m, v, 4)$ is strongly felicitous as well.

Theorem 3.1. *If $m \geq 4$ is even, then the graph $G \cong$ Amal$(C_m, v, 2)$ is strongly felicitous.*

Proof. Assume that $m \geq 4$ is even, and let $G_1 \cong G_2 \cong C_m$. Then amalgamate the vertices u_1 and v_1 of G_1 and G_2, respectively, to obtain the graph G such that

$$V(G) = \{u_i \in V(G_1) : 1 \leq i \leq m\} \cup \{v_i \in V(G_2) : 1 \leq i \leq m\},$$

where $u_1 = v_1$, and

$$E(G) = \{u_1 u_m, v_1 v_m\} \cup \{u_i u_{i+1}, v_i v_{i+1} : 1 \leq i \leq m-1\}.$$

Now, consider three cases for the vertex labeling $f : V(G) \to \mathbb{Z}_{2m+1}$.

Case 1. For $m = 4$, let

$$f(u_1) = 0, \quad f(u_2) = 3, \quad f(u_3) = 1, \quad f(u_4) = 5,$$
$$f(v_1) = 0, \quad f(v_2) = 7, \quad f(v_3) = 2, \quad f(v_4) = 8.$$

Then f is certainly a strongly felicitous labeling of $\mathrm{Amal}(C_4, v, 2)$ with characteristic 2.

Case 2. Assume that $m = 4k+2$, where k is a positive integer, then let

$$f(x) = \begin{cases} i-1 & \text{if } x = u_{2i-1} \text{ and } 1 \le i \le 2k+1, \\ 4k+i & \text{if } x = u_{2i} \text{ and } 1 \le i \le k+1, \\ 4k+i+2 & \text{if } x = u_{2i} \text{ and } k+2 \le i \le 2k+1, \\ 2k+i-1 & \text{if } x = v_{2i-1} \text{ and } 2 \le i \le 2k+1, \\ 6k+i+3x & \text{if } x = v_{2i} \text{ and } 1 \le i \le 2k+1. \end{cases}$$

Consequently, the vertex labels are distinct without using the integers $5k+2$ and $5k+3$, which shows that f is injective.

To verify that f is indeed a strongly felicitous labeling of G, we compute the induced edge labels.

For the edges of G in the set $\{v_1 v_{4k+2}\} \cup \{v_i v_{i+1} : 2 \le i \le 4k+1\}$, we have that

$$f(v_1) + f(v_{4k+2}) = 8k+4$$

and

$$f(v_i) + f(v_{i+1}) = 8k+i+3, \text{ if } 2 \le i \le 4k+1,$$

which provides the set of integers $\{8k+i+3 : 1 \le i \le 4k+1\}$.

For the remaining edges of G, we have that

$$\begin{aligned} f(u_i) + f(u_{i+1}) &= 4k+i, \text{ if } 1 \le i \le 2k+2; \\ f(u_1) + f(u_{4k+2}) &= 6k+3; \\ f(v_1) + f(v_2) &= 6k+4; \\ f(u_i) + f(u_{i+1}) &= 6k+i, \text{ if } 2k+3 \le i \le 4k+1. \end{aligned}$$

Hence the induced edge labels are distinct in $\mathbb{Z}_{8k+4}$.

Finally, notice that G is bipartite with partite sets

$$V_1 = \{u_{2i-1}, v_{2i-1} : u_1 = v_1 \text{ and } 1 \leq i \leq 2k+1\}$$

and

$$V_2 = \{u_{2i}, v_{2i} : 1 \leq i \leq 2k+1\}.$$

Thus f is a strongly felicitous labeling of G with characteristic $4k$.

Case 3. Assume $m = 4k+4$, where k is a positive integer, then let

$$f(x) = \begin{cases} i-1 & \text{if } x = u_{2i-1} \text{ and } 1 \leq i \leq 2k+2, \\ 4k+i+3 & \text{if } x = u_{2i} \text{ and } 1 \leq i \leq k, \\ 5k+5 & \text{if } x = u_{2k+2}, \\ 4k+i+5 & \text{if } x = u_{2i} \text{ and } k+2 \leq i \leq 2k+1, \\ 4k+3 & \text{if } x = u_{4k+4}, \\ 2k+i & \text{if } x = v_{2i-1} \text{ and } 2 \leq i \leq 2k+2, \\ 6k+i+6 & \text{if } x = v_{2i} \text{ and } 1 \leq i \leq 2k+2. \end{cases}$$

Consequently, the vertex labels are distinct without using the integers $5k+4$ and $5k+6$, which shows that f is injective.

To verify that f is indeed a strongly felicitous labeling of G, we compute the induced edge labels.

For the edges in the set $\{v_1 v_{4k+4}\} \cup \{v_i v_{i+1} : 2 \leq i \leq 4k+3\}$, we have that

$$f(v_1) + f(v_{4k+4}) = 8k+8$$

and

$$f(v_i) + f(v_{i+1}) = 8k+i+7, \text{ if } 2 \leq i \leq 4k+3,$$

which provide the set of integers $\{8k+i+7 : 1 \leq i \leq 4k+3\}$.

For the remaining edges of G, we have that

$$\begin{aligned} f(u_1) + f(u_{4k+4}) &= 4k+3; \\ f(u_i) + f(u_{i+1}) &= 2k+i+5, \text{ if } 1 \leq i \leq 2k; \\ f(u_{4k+3}) + f(u_{4k+4}) &= 6k+4; \\ f(u_i) + f(u_{i+1}) &= 4k+i+4, \text{ if } 2k+1 \leq i \leq 2k+2; \\ f(v_1) + f(v_2) &= 6k+7; \\ f(u_i) + f(u_{i+1}) &= 4k+i+5, \text{ if } 2k+3 \leq i \leq 4k+2. \end{aligned}$$

Hence the above induced edge labels are distinct in $\mathbb{Z}_{8k+8}$.

Finally, observe that G is bipartite with partite sets

$$V_1 = \{u_{2i-1}, v_{2i-1} : u_1 = v_1 \text{ and } 1 \leq i \leq 2k+2\}$$

and

$$V_2 = \{u_{2i}, v_{2i} : 1 \leq i \leq 2k+2\}.$$

Thus f is a strongly felicitous labeling of G with characteristic $4k+2$. Therefore, with these three cases in hand, our proof concludes. ■

In light of Theorems 2.4 and 3.1, we have the corollary promised in the paragraph preceding the above theorem.

Corollary 3.2. *If $m \geq 4$ is even, then the graph Amal$(C_m, v, 4)$ is strongly felicitous.*

If we apply Theorem 3.1 to Theorem 2.4, then we obtain the following corollary.

Corollary 3.3. *If $m \geq 4$ is even and $n \equiv 2 \pmod 4$, then the graph Amal(C_m, v, n) is felicitous.*

The next corollary is an immediate consequence of Theorem 2.4 and Corollary 3.2.

Corollary 3.4. *If $m \geq 4$ is even and $n \equiv 2 \pmod 8$, then the graph Amal(C_m, v, n) is felicitous.*

In [5], Lee, Schmeichel and Shee showed that the cycle C_m is felicitous if and only if $m \equiv 0$, 1 or 3 $\pmod 4$. Further, they proved that Amal (C_m, v, n) is not felicitous when $mn \equiv 2 \pmod 4$. Thus, by Theorem 2.4, we obtain the next result.

Corollary 3.5. *If $m \equiv 0$, 1 or 3 $\pmod 4$ and n is odd, then the graph Amal(C_m, v, n) is felicitous.*

As a corollary to the results found in [5] and this paper, all that remains to be settled are the cases when $m \geq 3$ is odd and $n \equiv 0 \pmod 4$, or $m \geq 4$ is even and $n \equiv 0 \pmod 8$.

The above results lead us to propose the following conjecture.

Conjecture 3.6. *The graph Amal*(C_m, v, n) *is felicitous if and only if* $mn \equiv 0, 1$ *or* $3 \pmod 4$.

Before presenting the next result, it is worthwhile to mention that Grahamand Sloane [3] proved that Amal(C_3, v, n) is harmonious if and only if $n \equiv 0$, 1 or 3 (mod 4), while Shee [8] found a harmonious labeling of Amal(C_4, v, n) for every positive integer n. Consequently, we have that Amal(C_3, v, n) is felicitous if and only if $n \equiv 0, 1$ or $3 \pmod 4$, whereas Amal(C_4, v, n) is felicitous for every positive integer n. In [3], Graham and Sloane proved that the cycle C_m is harmonious if and only if $m \geq 3$ is odd. Thus, by Theorem 2.3, we obtain the next result.

Theorem 3.7. *If both* $m \geq 3$ *and* $n \geq 1$ *are odd, then the graph Amal*(C_m, v, n) *is harmonious.*

We now consider the conjecture of Koh, Rogers, Lee and Toh [4] that the vertex amalgamation of n copies of the cycle C_m, Amal(C_m, v, n), is graceful if and only if $mn \equiv 0$ or $3 \pmod 4$.

For $m \equiv 0 \pmod 4$, the felicitous labeling of C_m provided in [5] is, in fact, strongly felicitous; hence, by Lemma 1.1, Corollary 2.2 and Theorem 3.1, we obtain the next result.

Theorem 3.8. *If* $m \equiv 0 \pmod 4$, *and* $n = 1, 2, 3$ *or* 4, *then the graph Amal*(C_m, v, n) *admits an* α*-valuation.*

Utilizing Corollary 2.2 and Theorem 3.1, we can easily obtain the next result.

Theorem 3.9. *If* $m \equiv 2 \pmod 4$, *and* $n = 2$ *or* 4, *then the graph Amal*(C_m, v, n) *admits an* α*-valuation.*

We close this section with the following conjecture.

Conjecture 3.10. *The graph Amal*(C_m, v, n) *admits an* α*-valuations if and only if* $mn \equiv 0 \pmod 4$.

4. Conclusions

In this paper, we partially solved an open problem stated in [5] that asks for which m and n the graph Amal(C_m, v, n) is felicitous.

Moreover, we advanced the conjecture of Koh, Rogers, Lee and Toh [4] that the graph Amal(C_m, v, n) is graceful if and only if $mn \equiv 0$ or $3 \pmod 4$. Finally, if one wants to explore the conjecture by Lee, Schmeichel and Shee [5] that all trees are felicitous, our above result on amalgamating strongly felicitous graphs with K_2 at the vertex labeled 0 offers a possible insight into the problem. Indeed, if one could find an analogous result on amalgamating felicitous graphs at the vertices labeled 0, the fact in [5] on the rotability of this labeling would allow us to growany tree by attaching pendant edges to any vertex of a felicitous tree. However, as it stands, our result only allows us to grow trees that admits α-valuations, and Rosa [6] showed that not all trees do so.

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