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PRIME IDEALS IN THE LATTICE OF ADDITIVE INDUCED-HEREDITARY GRAPH PROPERTIES

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Abstract

An additive induced-hereditary property of graphs is any class of finite simple graphs which is closed under isomorphisms, disjoint unions and induced subgraphs. The set of all additive induced-hereditary properties of graphs, partially ordered by set inclusion, forms a completely distributive lattice. We introduce the notion of the join-decomposability number of a property and then we prove that the prime ideals of the lattice of all additive induced-hereditary properties are divided into two groups, determined either by a set of excluded join-irreducible properties or determined by a set of excluded properties with infinite join-decomposability number. We provide non-trivial examples of each type.

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1. Introduction

An *additive induced-hereditary property* of graphs is any class of finite simple graphs (undirected graphs without loops and multiple edges) which is closed under isomorphisms, disjoint unions and induced subgraphs. The set of all additive induced-hereditary properties of graphs, partially ordered by set inclusion, is a completely distributive algebraic lattice. The investigation of the structure of this lattice is motivated by generalized colourings of graphs (see [3, 2, 7]). We use the notation $\mathbb{M}^a$ to denote this lattice of properties. $\mathbb{M}^a$ is a bounded lattice, bounded above by $\mathcal{I}$, the class of all graphs, and bounded below by the empty class of graphs. In general, we follow the notation of [5] and [2].

The lattice $\mathbb{M}^a$ is closed under arbitrary intersections, so the meet of any two properties $\mathcal{P}$ and $\mathcal{Q}$ is $\mathcal{P} \cap \mathcal{Q}$. If $\mathcal{S}$ is any set of graphs, we denote the smallest additive induced-hereditary property containing every graph in $\mathcal{S}$ by $[\mathcal{S}]$, and call this property the *property generated by* $\mathcal{S}$. $[\mathcal{S}]$ is the intersection of all the properties which contain every graph in $\mathcal{S}$ and it consists of all graphs whose connected components are either in $\mathcal{S}$ or are induced subgraphs of graphs in $\mathcal{S}$. The join of two properties $\mathcal{P}$ and $\mathcal{Q}$ in $\mathbb{M}^a$, written $\mathcal{P} \vee \mathcal{Q}$, is then $[\mathcal{P} \cup \mathcal{Q}]$. In [6] Jakubík proved that the lattice $\mathbb{M}^a$ is completely distributive.

For any property $\mathcal{P}$ in $\mathbb{M}^a$, the set $\boldsymbol{C}(\mathcal{P})$ of *minimal forbidden induced subgraphs* of $\mathcal{P}$ is defined by $\boldsymbol{C}(\mathcal{P}) = \{G : G \notin \mathcal{P}$ but every proper induced subgraph of G is in $\mathcal{P}\}$. It is easy to see that, for an additive property of graphs, the set $\boldsymbol{C}(\mathcal{P})$ contains only connected graphs. The set $\boldsymbol{C}(\mathcal{P})$ characterizes $\mathcal{P}$ in the sense that a graph is in $\mathcal{P}$ if and only if it contains no graph from $\boldsymbol{C}(\mathcal{P})$ as an induced subgraph.

A proper non-empty subset I of a lattice L is called an *ideal* of L if I is closed under the meet and join operations of L (i.e., I is a sublattice of L), and if for any $i \in I$ and any $j \in L$, the lattice element $i \wedge j$ is again in I. An ideal I is called a *prime* ideal if whenever $j, k \in L$ with $j \wedge k \in I$, then $j \in I$ or $k \in I$. The complement of a prime ideal I, the set $L - I$, is a *prime dual ideal*, a sublattice of L satisfying that $j \vee k \in L - I$ for every $j \in L - I$ and every $k \in L$, and if $j \vee k \in L - I$, then j or k must be in $L - I$. A chain in $(X, \leq)$ is a subset of X in which every pair of elements a and b are *comparable*, that is, $a \leq b$ or $b \leq a$.

A partially ordered set $(X, \leq)$ is called *up-directed* if any two elements of X have a common upper bound in X, and *down-directed* if any two elements have a common lower bound in X.

Our aim in this paper is to characterize the prime ideals in the lattice $\mathbb{M}^a$. In Theorem 3.3 we divide the prime ideals into two types and then in Section 4 we give non-trivial examples of both types.

2. Meet- and Join-Irreducibility

As in the theory of lattices, a property $\mathcal{P} \in \mathbb{M}^a$ is called *join-irreducible* in $\mathbb{M}^a$ if and only if $\mathcal{P}$ cannot be written as the join of two properties properly contained in $\mathcal{P}$. Equivalently, $\mathcal{P}$ is join-irreducible if $\mathcal{P} = \mathcal{R} \vee \mathcal{S}$ implies that $\mathcal{R} = \mathcal{P}$ or $\mathcal{S} = \mathcal{P}$. Analogously, property $\mathcal{P}$ is called *meet-irreducible* if and only if $\mathcal{P}$ cannot be written as the meet of two properties which properly contain $\mathcal{P}$, i.e., if $\mathcal{P} = \mathcal{R} \cap \mathcal{S}$, then $\mathcal{P} = \mathcal{R}$ or $\mathcal{P} = \mathcal{S}$.

Following the notation of [5], if L is a lattice and $p \in L$ we denote by $(p]$ the *principal ideal* consisting of all elements x of L satisfying $x \leq p$ and the *principal dual ideal* consisting of all elements x of L satisfying $x \geq p$ by $[p)$. It is known (see e.g. [5]) that in any distributive lattice L, an element p is join-irreducible if and only if the ideal $L - [p)$ is prime, while p is meet-irreducible if and only if the ideal $(p]$ is prime. We will call an ideal of the type $L - [p)$ a *co-principal ideal.*

The meet- and join-irreducible elements in the lattice $\mathbb{M}^a$ have been characterized in [1]. There the authors prove the following two results:

Theorem 2.1. *Let $\mathcal{P} \neq \emptyset$ be an additive induced-hereditary property of graphs. Then the following are equivalent:*

1. *$\mathcal{P}$ is join-irreducible in $\mathbb{M}^a$.*
2. *The connected graphs in $\mathcal{P}$ form an up-directed set (under the ordering $G \leq H$ iff G is an induced subgraph of H).*
3. *There is a chain $\mathcal{C}$ (which may be finite or infinite) of connected graphs in $\mathcal{P}$ such that $\mathcal{P} = [\mathcal{C}]$.*
4. *$\mathbb{M}^a - [\mathcal{P})$ is a prime ideal of $\mathbb{M}^a$.* ■

Theorem 2.2. *Let $\mathcal{P} \neq \mathcal{I}$ be an additive induced-hereditary property of graphs. Then the following are equivalent:*

1. *$\mathcal{P}$ is meet-irreducible in $\mathbb{M}^a$.*
2. *There is a connected graph G such that $\boldsymbol{C}(\mathcal{P}) = \{G\}$.*
3. *$(\mathcal{P}]$ is a prime ideal of $\mathbb{M}^a$.* ■

Let us remark that a characterization of intersection classes similar to Theorem 2.1 has been given by Scheinerman (see [8, 9]). From Theorems 2.1 and 2.2 we can see that meet- and join- irreducible properties abound in $\mathbb{M}^a$ and $\mathbb{M}^a$ therefore has many prime ideals of the form $(\mathcal{P}]$ and of the form $\mathbb{M}^a - [\mathcal{P})$.

Since not all properties in $\mathbb{M}^a$ can be generated by chains of connected graphs, not all properties are join-irreducible. Some properties, for example the property generated by cycles C_3 and $C_4, \mathcal{P} = [C_3, C_4] = [C_3] \vee [C_4]$, are not join-irreducible but can be written as the join of finitely many join-irreducible properties, while other properties, for example the property $\mathcal{S}_2$ of graphs having maximum degree at most 2, which is generated by all cycles, are not join-irreducible and also cannot be written as the join of finitely many join-irreducible properties. Let $\mathcal{P} \in \mathbb{M}^a$. Define the *join-decomposability number* of $\mathcal{P}$ by $\vee\text{-}dc(\mathcal{P}) = \min\{m : \mathcal{P}$ can be written as the join of m join-irreducible properties$\}$. Clearly $\vee\text{-}dc(\mathcal{P})$ can be finite or infinite, and $\vee\text{-}dc(\mathcal{P}) = 1$ if and only if $\mathcal{P}$ is join-irreducible.

If $\mathcal{P}$ is a property with finite join-decomposability number m, then $\mathcal{P}$ has a unique expression as the join of m join-irreducible properties, as the following theorem shows.

Theorem 2.3. *Let $\mathcal{P} \in \mathbb{M}^a$ and let m be a positive integer. Then the following are equivalent:*

1. $\vee\text{-}dc(\mathcal{P}) = m$.
2. *m is the smallest integer so that $\mathcal{P}$ can be generated by union of m chains of connected graphs.*
3. *m is the smallest integer such that: for any finite set of connected graphs $G_1, G_2, \ldots G_n$ in $\mathcal{P}$, there exist m connected graphs in $\mathcal{P}$ whose union contains each of $G_1, G_2, \ldots G_n$ as induced subgraphs.*
4. *$\mathcal{P}$ has a unique expression as the supremum of m join-irreducible properties.*

Proof. (1) if and only if (2) follows immediately from Theorem 2.1.

(1) implies (4): Suppose that $\vee\text{-}dc(\mathcal{P}) = m$. Then there exist m join-irreducible properties $\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_m$ such that $\mathcal{P} = \mathcal{P}_1 \vee \mathcal{P}_2 \vee \cdots \mathcal{P}_m$. Suppose that $\mathcal{R}_1, \mathcal{R}_2, \ldots, \mathcal{R}_m$ are join-irreducible properties such that $\mathcal{P} = \mathcal{P}_1 \vee \mathcal{P}_2 \vee \cdots \mathcal{P}_m = \mathcal{R}_1 \vee \mathcal{R}_2 \vee \cdots \mathcal{R}_m$. Then $\mathcal{R}_1 = (\mathcal{P}_1 \vee \mathcal{P}_2 \vee \cdots \mathcal{P}_m) \cap \mathcal{R}_1 = (\mathcal{P}_1 \cap \mathcal{R}_1) \vee (\mathcal{P}_2 \vee \mathcal{R}_1) \vee \cdots (\mathcal{P}_m \cap \mathcal{R}_1)$. Since $\mathcal{R}_1$ is join-irreducible we must have $\mathcal{R}_1 = \mathcal{P}_i \cap \mathcal{R}_1$ for some $i = 1, 2, \ldots, m$ and so $\mathcal{R}_1 \subseteq \mathcal{P}_i$. Similarly $\mathcal{P}_i \subseteq \mathcal{R}_j$

for some $j = 1, 2, \ldots, m$. But by the minimality of m, $\mathcal{R}_1 \subseteq \mathcal{P}_i \subseteq \mathcal{R}_j$ implies that j must equal 1 and $\mathcal{R}_1 = \mathcal{P}_i$. Similarly, each of the other $\mathcal{R}$'s must be equal to one of the $\mathcal{P}$'s.

(4) implies (3): Let $\mathcal{P} = \mathcal{P}_1 \vee \mathcal{P}_2 \vee \cdots \mathcal{P}_m$ be the unique expression for $\mathcal{P}$ as a join of m join-irreducible properties, and let $G_1, G_2, \ldots, G_n$ be any finite set of connected graphs in $\mathcal{P}$. Since each G_i is connected, each G_i is contained in one of the $\mathcal{P}_j$. By Theorem 2.1, for each $j = 1, 2, \ldots, m$ we can find one connected graph in $\mathcal{P}_j$ containing all of the G_i in $\mathcal{P}_j$ as induced subgraphs, and so there exist m connected graphs in $\mathcal{P}$ whose union contains each of $G_1, G_2, \ldots, G_n$ as an induced subgraph.

By the uniqueness of the expression $\mathcal{P} = \mathcal{P}_1 \vee \mathcal{P}_2 \vee \cdots \mathcal{P}_m$, for each $j = 1, 2, \ldots, m$, $\mathcal{P}_j \not\subseteq \vee\{\mathcal{P}_i : i = 1, 2, \ldots, m \text{ and } i \neq j\}$ and so for each $j = 1, 2, \ldots, m$ there exists a connected graph G_j contained in $\mathcal{P}_j$ and not contained in any of the other $\mathcal{P}_i$. Clearly no set of fewer than m connected graphs in $\mathcal{P}$ can contain each of $G_1, G_2, \ldots, G_n$ as an induced subgraph, and so m is minimal.

(3) implies (2): Suppose that condition (3) holds. We can construct m chains of connected graphs generating $\mathcal{P}$ as follows: Let $\{G_1, G_2, \ldots,\}$ be a listing of all the connected graphs in $\mathcal{P}$. By (3) there exists an integer n such that the union of any set of fewer than m connected graphs in $\mathcal{P}$ does not contain all of $G_1, G_2, \ldots, G_n$ as induced subgraphs. Let $A_1, A_2, \ldots, A_m$ be m connected graphs whose union contains each of $G_1, G_2, \ldots, G_n$ as induced subgraphs. Now there exist connected graphs $B_1, B_2, \ldots, B_m$ such that each of $A_1, A_2, \ldots, A_m, G_{n+1}$ is an induced subgraph of one of the B's. By the choice of n we cannot have two of the A's contained in one B. Suppose w.l.o.g. that $A_1 \subseteq B_1, A_2 \subseteq B_2, \ldots, A_m \subseteq B_m$. We can now continue this process: There exist connected graphs $C_1, C_2, \ldots, C_m$ containing each of $B_1, B_2, \ldots, B_m, G_{n+2}$ as induced subgraphs with $B_1 \subseteq C_1, B_2 \subseteq C_2, \ldots, B_m \subseteq C_m$, etc. ■

Let $\mathcal{P} \in \mathbb{M}^a$ and suppose that $\mathcal{P} = \vee_{i \in I} \mathcal{P}_i$ where each $\mathcal{P}_i$ is a join-irreducible property. This expression for $\mathcal{P}$ is called *irredundant* if for every $j \in I$, the property $\mathcal{P}_j$ is not contained in $\vee\{\mathcal{P}_i : i \in I \text{ and } i \neq j\}$, i.e., for every $j \in I$, $\vee\{\mathcal{P}_i : i \in I \text{ and } i \neq j\} \neq \mathcal{P}$. Otherwise the expression is called *redundant.* From the previous theorem we can conclude that every property with finite join-decomposability number m has a unique irredundant expression as the supremum of m join-irreducible properties. A property with infinite join-decomposability number may have no irredundant expression as a supremum of join-irreducible properties, as the following example shows.

Let $\mathcal{P}$ be the property generated by all graphs formed by first selecting an arbitrary path and then attaching one cycle to each vertex of the path in such a way that the resulting graph can be drawn with the cycles in increasing order (not necessarily strictly increasing), when viewed in some order along the vertices of the path.

Suppose now that $\mathcal{P} = \vee_{i \in I} \mathcal{P}_i$ where each $\mathcal{P}_i$ is join-irreducible. Let G be one of the generators of $\mathcal{P}$, and suppose that the maximum order of a cycle in G is k and that G has a base path of length r. Then there exists $j \in I$ such that G is in $\mathcal{P}_j$. Let H be the graph formed from G by starting with a path of length $r+2$, attaching all the cycles of G, and then attaching a cycle of order $k+1$ to the second-last vertex and a cycle of order $k+2$ to the last vertex. Form graph H' by starting with a path of length $r+3$, attaching all the cycles of G, then attaching two cycles of order $k+1$, one to vertex $r+1$ and one to vertex $r+2$, and then attaching a cycle of order $k+2$ to the last vertex. Both H and H' cannot be in $\mathcal{P}_j$ since these two graphs do not have a common supergraph, so there exists some $k \in I$ with $k \neq j$ such that $\mathcal{P}_k$ contains exactly one of H or H'. Since G is an induced subgraph of both H and H', G is in $\mathcal{P}_k$.

We have shown that every generator of $\mathcal{P}$ is contained in at least two of the join-irreducible factors in the representation of $\mathcal{P}$, and so every graph in $\mathcal{P}$ is contained in at least two of these factors. Hence for every $j \in I$, we have $\mathcal{P}_j \subseteq \vee\{\mathcal{P}_i : i \in I \text{ and } i \neq j\}$. We conclude that $\mathcal{P}$ has no irredundant expression as a supremum of join-irreducible properties.

We remark that, using Zorn's Lemma it is easily proved that there are maximal join-irreducible properties contained in any property $\mathcal{P}$ with infinite join-decomposability number. For example the maximal join-irreducible properties contained in the property $\mathcal{S}_2$ are $[C_p], p \geq 3$ and the class $\mathcal{LF}$ of all linear forests, but the expression $\mathcal{S}_2 = \vee_{p \geq 3}[C_p] \vee \mathcal{LF}$ is redundant since the property $\mathcal{LF}$ is contained in $\vee_{p \geq 3}[C_p]$.

3. Prime Ideals

Our aim is to characterize all the prime ideals of $\mathbb{M}^a$. So far we have seen that $\mathbb{M}^a$ has many principal and co-principal prime ideals. The following result shows that in $\mathbb{M}^a$, every prime ideal of the form $(\mathcal{P}]$ for some (necessarily meet-irreducible) property $\mathcal{P}$ is also of the form $\mathbb{M}^a - [\mathcal{Q})$ for some (necessarily join-irreducible) property $\mathcal{Q}$, i.e., every principal prime ideal is a co-principal prime ideal.

Theorem 3.1. *Every principal prime ideal in* $\mathbb{M}^a$ *is also a co-principal prime ideal.*

Proof. Suppose that I is a prime ideal of the form $(\mathcal{P}]$. Then $\mathcal{P}$ is meet-irreducible. Suppose that $\boldsymbol{C}(\mathcal{P}) = \{G\}$. Define $\mathcal{Q}$ to be the property generated by graph G. Then $(\mathcal{P}] \subseteq \mathbb{M}^a - [\mathcal{Q})$, since if $\mathcal{R} \subseteq \mathcal{P}$, then we cannot have $\mathcal{Q} \subseteq \mathcal{R}$.

Conversely, if $\mathcal{Q} \not\subseteq \mathcal{R}$, then $G \notin \mathcal{R}$. Thus some induced subgraph of G is in $\boldsymbol{C}(\mathcal{R})$. Since $\boldsymbol{C}(\mathcal{R})$ has an induced subgraph of G as one of its elements, $\mathcal{R} \subseteq \mathcal{P}$, i.e., $\mathbb{M}^a - [\mathcal{Q}) \subseteq (\mathcal{P}]$. ■

Note that the converse of this result is not true: a prime ideal of the form $\mathbb{M}^a - [\mathcal{Q})$ need not be of the form $(\mathcal{P}]$. For example if $\mathcal{Q}$ is the join-irreducible property generated by the chain of stars, then $\mathbb{M}^a - [\mathcal{Q})$ has no largest element since if $\mathcal{R}$ is in $\mathbb{M}^a - [\mathcal{Q})$, and we choose any graph G which is not in $\mathcal{R}$, then $\mathcal{P} \subset \mathcal{P} \vee [G] \in \mathbb{M}^a - [\mathcal{Q})$. Hence $\mathcal{Q}$ cannot be of the form $(\mathcal{P}]$.

If L is any lattice, then the set of all ideals of L, ordered under inclusion, is again a lattice, the *ideal lattice* of L. The following lemma is not difficult (see for example [4]).

Lemma 3.2. *Let L be a distributive lattice. Then an ideal I of L is a prime ideal if and only if I is meet-irreducible in the ideal lattice of L.*

With the help of this result we can divide all prime ideals of $\mathbb{M}^a$ into two types as follows:

Theorem 3.3. *Let I be a prime ideal of $\mathbb{M}^a$. Then either there exists a set $\mathcal{T}_1$ of join-irreducible properties in the complement of I such that $I = \cap\{\mathbb{M}^a - [\mathcal{P}) : \mathcal{P} \in \mathcal{T}_1\}$, or there exists a set $\mathcal{T}_2$ of properties with infinite join-decomposability number in the complement of I, each of which contains no join-irreducible property in the complement of I, such that $I = \cap\{\mathbb{M}^a - [\mathcal{P}) : \mathcal{P} \in \mathcal{T}_2\}$.*

Proof. Let $\mathcal{T}_1 = \{\mathcal{P} \in \mathbb{M}^a : \mathcal{P} \notin I$ and $\vee\text{-}dc(\mathcal{P}) = 1\}$ and $\mathcal{T}_2 = \{\mathcal{P} \in \mathbb{M}^a : \mathcal{P} \notin I, \vee\text{-}dc(\mathcal{P}) = \infty$, and $\mathcal{P}$ does not contain any join-irreducible property in $\mathbb{M}^a - I\}$. We will prove that $J_1 = \cap\{\mathbb{M}^a - [\mathcal{P}) : \mathcal{P} \in \mathcal{T}_1\}$ and $J_2 = \cap\{\mathbb{M}^a - [\mathcal{P}) : \mathcal{P} \in \mathcal{T}_2\}$ are both ideals, and then that $I = J_1 \cap J_2$. Since I is meet-irreducible in the ideal lattice of $\mathbb{M}^a$ by Lemma 3.2, the result will then follow immediately.

J_1 is an ideal since it is an intersection of ideals.

If $\mathcal{Q} \in J_2$ then $\mathcal{Q}$ contains no element of $\mathcal{T}_2$ and hence if $\mathcal{R}$ is any property in $\mathbb{M}^a$, $\mathcal{Q} \cap \mathcal{R}$ can contain no element of $\mathcal{T}_2$ and hence is in J_2.

Now suppose $\mathcal{Q} \in J_2$ and $\mathcal{R} \in J_2$. Suppose there is a property $\mathcal{P}$ with infinite join-decomposability number coming from $\mathbb{M}^a - I$ and containing no join-irreducible properties from $\mathbb{M}^a - I$ with $\mathcal{P} = \vee_{i=1}^{\infty}\mathcal{P}_i \subseteq \mathcal{Q} \vee \mathcal{R}$, and with each $\mathcal{P}_i$ join-irreducible. Then each join-irreducible $\mathcal{P}_i$ must be in $\mathcal{Q}$ or in $\mathcal{R}$. Suppose $\vee_{i\in X}\mathcal{P}_i \subseteq \mathcal{Q}$ and $\vee_{i\in Y}\mathcal{P}_i \subseteq \mathcal{R}$, with $(\vee_{i\in X}\mathcal{P}_i) \vee (\vee_{i\in Y}\mathcal{P}_i) = \mathcal{P} \notin I$. Since $\mathbb{M}^a - I$ is a prime dual ideal, $\vee_{i\in X}\mathcal{P}_i \notin I$ or $\vee_{i\in Y}\mathcal{P}_i \notin I$. Say $\vee_{i\in X}\mathcal{P}_i \notin I$. If this property has infinite join-decomposability number, we contradict the fact that $\mathcal{Q} \in J_2$. Suppose then that $\vee_{i\in X}\mathcal{P}_i$ has finite join-decomposability number m. Let $\mathcal{Q}_1, \mathcal{Q}_2, \dots, \mathcal{Q}_m$ be join-irreducible properties such that $\mathcal{Q}_1 \vee \mathcal{Q}_2 \vee \cdots \mathcal{Q}_m = \vee_{i\in X}\mathcal{P}_i \notin I$. Again since $\mathbb{M}^a - I$ is a prime dual ideal, one of the $\mathcal{Q}_i$ must be in $\mathbb{M}^a - I$. But now we have $\mathcal{Q}_i \subseteq \mathcal{P} \in \mathcal{T}_2$, a contradiction to the definition of $\mathcal{T}_2$. Hence $\mathcal{Q} \vee \mathcal{R} \in J_2$ and so J_2 is an ideal.

We now show that $I = J_1 \cap J_2$. Clearly any property in I is contained in both J_1 and J_2, since a property from I cannot contain a property which is not in I, so $I \subseteq J_1 \cap J_2$. Now suppose $\mathcal{Q} \in J_1 \cap J_2$ but $\mathcal{Q} \notin I$. If $\mathcal{Q}$ has finite join-decomposability number, then since $\mathbb{M}^a - I$ is a prime dual ideal, one of the join-irreducible 'factors' of $\mathcal{Q}$ must be in $\mathbb{M}^a - I$, contradicting the fact that $\mathcal{Q}$ is in J_1. If $\mathcal{Q}$ has infinite join-decomposability number, $\mathcal{Q}$ can contain no join-irreducible element from $\mathbb{M}^a - I$ (since $\mathcal{Q} \in J_1$). But this contradicts the fact that $\mathcal{Q} \in J_2$. Hence we can conclude that $\mathcal{Q} \in I$, and so $I = J_1 \cap J_2$. ∎

Notice that the proof above did not make use of the particular lattice $\mathbb{M}^a$. Theorem 3.3 is valid for any distributive lattice. Every prime ideal I thus has associated with it either a set of join-irreducible properties in the complement of I, $\mathcal{T}_1$, such that I is the set of all properties containing no element of $\mathcal{T}_1$, or a set $\mathcal{T}_2$ of properties in the complement of I which contain no join-irreducible properties in the complement of I and which have infinite join-decomposability number, such that I is the set of all properties containing no element of $\mathcal{T}_2$. For convenience we will here say that a prime ideal is of *type one* in the first case and of *type two* in the second case.

Every co-principal prime ideal (and every principal prime ideal, by Theorem 3.1) is clearly of type one. Theorem 3.3 does not rule out the possibility that $\mathbb{M}^a - I$ has only co-principal prime ideals. However we now give two examples of a type one and a type two prime ideal that are not co-principal.

4. Examples

4.1. A non-co-principal prime ideal of type one

Let $\mathcal{Q} \in \mathbb{M}^a$ be any property such that $\boldsymbol{C}(\mathcal{Q})$ is an infinite set of graphs with no degree one vertices. For each non-empty finite $S \subset \boldsymbol{C}(\mathcal{Q})$, define the property $\mathcal{Q}_S$ by $\boldsymbol{C}(\mathcal{Q}_S) = S$. Note that each property $\mathcal{Q}_S$ defined in this way is join-irreducible by Theorem 2.1 since if G and H are two connected graphs in $\mathcal{Q}_S$, then by joining any vertex from G to any vertex from H by a long enough path, we construct a connected graph in $\mathcal{Q}_S$ which contains both G and H.

Let $I = \{\mathcal{P} \in \mathbb{M}^a :$ for every finite $S \subset \boldsymbol{C}(\mathcal{Q}),\ \mathcal{Q}_S \not\subseteq \mathcal{P}\}$. Clearly I is a proper non-empty subset of $\mathbb{M}^a$. We will show that I is a prime ideal of $\mathbb{M}^a$, and that $\mathbb{M}^a - I$ has no smallest element and hence is not co-principal.

If $\mathcal{P} \in I$, and $\mathcal{R} \in \mathbb{M}^a$, then clearly $\mathcal{P} \cap \mathcal{R}$ is again in I.

Let $\mathcal{P}, \mathcal{R} \in I$. If $\mathcal{P} \vee \mathcal{R} \notin I$, then there exists a $\mathcal{Q}_S$ with $\mathcal{Q}_S \subseteq \mathcal{P} \vee \mathcal{R}$. Since $\mathcal{Q}_S$ is join-irreducible, this implies that $\mathcal{Q}_S$ is contained in $\mathcal{P}$ or $\mathcal{R}$, a contradiction. Hence $\mathcal{P} \vee \mathcal{R} \in I$. So I is an ideal.

I is prime: Suppose that $\mathcal{P} \cap \mathcal{R} \in I$. If $\mathcal{P} \notin I$ and $\mathcal{R} \notin I$, then there exist finite $S, S' \subset \boldsymbol{C}(\mathcal{Q})$ such that $\mathcal{Q}_S \subseteq \mathcal{P}$ and $\mathcal{Q}_{S'} \subseteq \mathcal{R}$. But then $\mathcal{Q}_S \cap \mathcal{Q}_{S'} \subseteq \mathcal{P} \cap \mathcal{R}$, a contradiction, since $\mathcal{Q}_S \cap \mathcal{Q}_{S'} = \mathcal{Q}_{S''}$, where the graphs in S'' are the minimal elements of the set $S \cup S'$, under inclusion as an induced subgraph. (This result is straightfoward, and is proved for additive hereditary properties in [3].) So we can conclude that $\mathcal{P}$ or $\mathcal{R}$ is in I, and hence I is a prime ideal.

I is not co-principal since $\mathbb{M}^a - I$ has no smallest element: $\cap \mathcal{Q}_S = \mathcal{Q}$, and $\mathcal{Q} \in I$ while all the properties $\mathcal{Q}_S$ are in the complement of I.

Clearly by letting the set $\mathcal{T}_1$ be any down-directed set of join-irreducible properties with no smallest element we have a non-co-principal prime ideal of type one. Indeed a prime ideal I is of type one if and only if there exists a down-directed set $\mathcal{T}_1$ of join-irreducible properties such that $I = \cap_{\mathcal{P} \in \mathcal{T}_1} \mathbb{M}^a - [\mathcal{P})$.

4.2. A non-co-principal prime ideal of type two

Let B be the Boolean lattice of all subsets of the set $\{3, 4, 5, \ldots, \}$, and let K be a maximal ideal of B containing all finite subsets of $\{3, 4, 5, \ldots, \}$. The existence of K, which will be a prime ideal, is guaranteed by the prime ideal theorem. (See for example [5]). Now let $\mathcal{T}_2$ be the set of all properties generated by a set of cycles indexed by an element in the prime dual ideal $B - K$. Every property in $\mathcal{T}_2$ is generated by an infinite set of cycles and hence has infinite join-decomposability number. Let I be the set $I = \{\mathcal{Q} \in \mathbb{M}^a : \mathcal{Q}$ contains no element of $\mathcal{T}_2\} = \cap_{\mathcal{P}\in\mathcal{T}_2}\mathbb{M}^a - [\mathcal{P})$.

Every property in $\mathcal{T}_2$ is generated by an infinite set of cycles and hence has infinite join-decomposability number. Also, if $\mathcal{Q} \in \mathcal{T}_2$ then the only join-irreducible properties contained in $\mathcal{Q}$ are either generated by one cycle, generated by one path or generated by all paths, and all these properties are in I. Hence every property in $\mathcal{T}_2$ has infinite join-decomposability number and contains no join-irreducible properties in $\mathbb{M}^a - I$.

I as defined above is a prime ideal:

If $\mathcal{Q} \in I$ and $\mathcal{R} \in \mathbb{M}^a$, then clearly $\mathcal{Q} \cap \mathcal{R} \in I$, by the definition of I. Now suppose that $\mathcal{Q} \in I$ and $\mathcal{R} \in I$. If $\mathcal{Q} \vee \mathcal{R} \notin I$ then there exists a property generated by all cycles indexed by an element of $B-K$ contained in $\mathcal{Q}\vee\mathcal{R}$. Say $[\{C_z : z \in Z\}] \subseteq \mathcal{Q} \vee \mathcal{R}$ where $Z \in B - K$. Every cycle C_z is then contained in $\mathcal{Q}$ or $\mathcal{R}$. We can thus split Z into two sets, say X and Y such that $X \cup Y = Z$ and $[\{C_z : z \in X\}] \subseteq \mathcal{Q}$ and $[\{C_z : z \in Y\}] \subseteq \mathcal{R}$. But since $B - K$ is a prime dual ideal and $X \cup Y \in B - K$, either X or Y must be in $B - K$. Say $X \in B - K$. But this contradicts the fact that $\mathcal{Q} \in I$, and we conclude that $\mathcal{Q} \vee \mathcal{R} \in I$.

To see that I is prime, suppose that $\mathcal{Q} \cap \mathcal{R} \in I$. We must show that $\mathcal{Q}$ or $\mathcal{R}$ is in I. Suppose that this is false and that $[\{C_z : z \in X\}] \subseteq \mathcal{Q}$ and $[\{C_z : z \in Y\}] \subseteq \mathcal{R}$, where X and Y are elements of $B - K$. But then $[\{C_z : z \in X \cap Y\}] \subseteq \mathcal{Q} \cap \mathcal{R}$, and $X \cap Y \in B - K$, contradicting the fact that $\mathcal{Q} \cap \mathcal{R} \in I$. Hence $\mathcal{Q}$ or $\mathcal{R}$ must be in I, and so I is a prime ideal.

I is clearly not co-principal, since the complement of a co-principal prime ideal has a minimum element which is join-irreducible, and we know that each of the properties in the set $\mathcal{T}_2$ contains no join-irreducible elements in $\mathbb{M}^a - I$.

We remark that an analogous result can be obtained for other types of lattices, for example for the lattice of additive hereditary properties of posets or hypergraphs.

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