Discussiones Mathematicae
Graph Theory 23 (2003) 67–83

VERTEX-ANTIMAGIC TOTAL LABELINGS OF GRAPHS

Martin Bača

Department of Applied Mathematics
Technical University, 04200 Košice, Slovak Republic

e-mail: Martin.Baca@tuke.sk

François Bertault

Department of Computer Science and Software Engineering
University of Newcastle, NSW 2308, Australia

e-mail: francois@cs.newcastle.edu.au

James A. MacDougall

Department of Mathematics
University of Newcastle, NSW 2308, Australia

e-mail: jmacd@math.newcastle.edu.au

Mirka Miller, Rinovia Simanjuntak and Slamin

Department of Computer Science and Software Engineering
University of Newcastle, NSW 2308, Australia

e-mail: {mirka,rino,slamin}@cs.newcastle.edu.au

Abstract

In this paper we introduce a new type of graph labeling for a graph $G(V, E)$ called an (a, d)-vertex-antimagic total labeling. In this labeling we assign to the vertices and edges the consecutive integers from 1 to $|V| + |E|$ and calculate the sum of labels at each vertex, i.e., the vertex label added to the labels on its incident edges. These sums form an arithmetical progression with initial term a and common difference d.

We investigate basic properties of these labelings, show their relationships with several other previously studied graph labelings, and

show how to construct labelings for certain families of graphs. We conclude with several open problems suitable for further research.

Keywords: super-magic labeling, (a, d)-vertex-antimagic total labeling, (a, d)-antimagic labeling.

2000 Mathematics Subject Classification: 05C78, 05C05, 05C38.

1. Introduction

All graphs considered here are finite, simple, and undirected. The graph G has vertex set $V = V(G)$ and edge set $E = E(G)$ and we let $|V| = v$ and $|E| = e$. For a general reference for graph theoretic notions, see [15].

A *labeling* (or *valuation*) of a graph is any mapping that sends some set of graph elements to a set of numbers (usually positive or non-negative integers). If the domain is the vertex-set or the edge-set, the labelings are called respectively *vertex-labelings* or *edge-labelings*. In this paper we deal with the case where the domain is $V \cup E$, and these are called *total* labelings. A general survey of graph labelings is found in [5]. Various authors, beginning with Sedláček [12] have introduced labelings that generalize the idea of a magic square. *Magic* labelings are one-to-one maps onto the appropriate set of consecutive integers starting from 1, satisfying some kind of "constant-sum" property. A *vertex-magic* labeling is one in which the sum of all labels associated with a vertex is a constant independent of the choice of vertex. *Edge-magic* labelings are defined similarly. *Vertex-magic total labelings* were first introduced in [10]. Such a labeling is a one-to-one mapping $\lambda : E \cup V \to \{1, 2, \ldots, v + e\}$ with the property that there is a constant k such that at any vertex x

$$\lambda(x) + \sum \lambda(xy) = k$$

where the sum is over all vertices y adjacent to x. For any labeling we call the sum of the appropriate labels at a vertex the *weight* of the vertex, denoted $wt(x)$; so for vertex-magic total labelings we require that the weight of all vertices be the same, namely k and this number is called the *magic constant* for the labeling.

Edge-magic total labelings have been studied recently in [14] and readers are referred to [14] and [10] for more background on these subjects and a standardization of the terminology.

Hartsfield and Ringel [6] introduced the concept of an antimagic graph. In their terminology, an *antimagic* labeling is an edge-labeling of the graph with the integers $1, 2, \ldots, e$ so that the weight at each vertex is different from the weight at any other vertex. It is an easy exercise to write down many antimagic labelings for most graphs, so some further restriction on the vertex-sums is usually introduced. Thus Bodendiek and Walther [3] defined the concept of an (a, d)-*antimagic* labeling as an edge-labeling in which the vertex weights form an arithmetic progression starting from a and having common difference d.

In this paper we introduce the notions of the *vertex-antimagic total labeling* and the (a, d)-*vertex-antimagic total labeling.* For a vertex-antimagic total labeling we label all vertices and edges with the numbers from 1 to $v+e$ and require that the weights of the vertices be all distinct. For an (a, d)-vertex-antimagic total labeling we impose the restriction that the vertex weights form an arithmetic progression. More formally, we have:

Definition 1. A bijection $\lambda : V \cup E \to \{1, 2, \ldots, v + e\}$ is called a *vertex-antimagic total labeling* of $G = G(V, E)$ if the weights of vertices $wt(x)$, $x \in V$ are distinct.

Definition 2. A bijection $\lambda : V \cup E \to \{1, 2, \ldots, v + e\}$ is called an (a, d)-*vertex-antimagic total labeling* (VATL) of G if the set of vertex weights is $W = \{wt(x)|x \in V\} = \{a, a+d, \ldots, a+(v-1)d\}$ for some integers a and d.

Figure 1 gives an example of $(10, 4)$-VATL of $K_4 - e$.

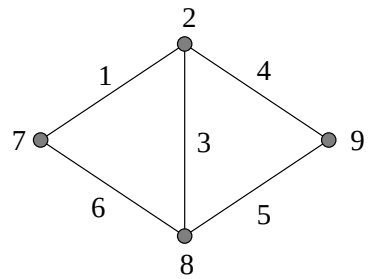

Figure 1. $(10, 4)$-VATL of $K_4 - e$

Unless some further restriction is imposed, VATLs are too plentiful to be of much interest. Consequently, in this paper we investigate the basic properties of (a, d)-VATLs. We point out connections with some other previously

studied types of graph labelings, and construct (a, d)-VATLs for certain families of graphs. The paper concludes with several open problems which bear further investigation.

2. General Properties

2.1. Basic Counting

Set $M = e + v$ and let S_v be the sum of the vertex labels and S_e the sum of the edge labels. Since the labels are the numbers $1, 2, \ldots, M$, we have as the sum of all labels:

$$S_v + S_e = \sum_{1}^{M} i = \binom{M+1}{2}.$$

If we let $wt(x_i) = a + id$, then summing the weights over all vertices adds each vertex label once and each edge label twice, so we get:

$$S_v + 2S_e = \frac{v}{2}(2a + (v-1)d).$$

Combining these two equations gives us

$$S_e + \binom{M+1}{2} = va + \binom{v}{2}d.$$

The edge labels could conceivably receive the e smallest labels or, at the other extreme, the e largest labels, or anything between. Consequently, we have

$$\sum_{1}^{e} i \leq S_e \leq \sum_{v+1}^{M} i.$$

A corresponding result holds for S_v. Combining these last two equations results in the inequalities

$$\binom{M+1}{2} + \binom{e+1}{2} \leq va + \binom{v}{2}d \leq 2\binom{M+1}{2} - \binom{v+1}{2}$$

which restrict the feasible values for a and d. For particular graphs, however, we can often exploit the structure to get considerably stronger restrictions.

We note that if δ is the smallest degree in G, then the minimum possible weight on a vertex is at least $1 + 2 + \ldots + (\delta + 1)$, consequently

$$a \geq \frac{(\delta+1)(\delta+2)}{2}.$$

Similarly, if Δ is the largest degree, then the maximum vertex weight is no more than the sum of the $\Delta + 1$ largest labels. Thus

$$\begin{aligned} a + (v-1)d &\leq \sum_{i=M-\Delta}^{M} i \\ &= \frac{(2M-\Delta)(\Delta+1)}{2}. \end{aligned}$$

Combining these two inequalities gives the following upper bound on values of d:

$$d \leq \frac{(2M-\Delta)(\Delta+1) - (\delta+1)(\delta+2)}{2(v-1)}. \tag{1}$$

2.2. New Labelings from Old

Given one VATL on a graph, it may be possible to construct other VATLs from it. Let $\lambda : V \cup E \to \{1, 2, \ldots, e+v\}$ be a one-to-one map. We define the map λ' on $V \cup E$ by

$$\begin{aligned} \lambda'(x) &= M + 1 - \lambda(x), \quad x \in V, \\ \lambda'(xy) &= M + 1 - \lambda(xy), \quad xy \in E. \end{aligned}$$

Clearly, λ' is also a one-to-one map from the set $V \cup E$ to $\{1, 2, \ldots, e+v\}$; we say λ' is the *dual* of λ.

Theorem 1. *The dual of an (a, d)-vertex-antimagic total labeling for a graph G is an (a', d)-vertex-antimagic total labeling for some a' if and only if G is regular.*

Proof. Suppose λ is an (a, d)-VATL for G and let $w_\lambda(x)$ be the weight of vertex x under the labeling λ. Then $W = \{w_\lambda(x) | x \in V\} = \{a, a+d, \ldots, a+(v-1)d\}$ is the set of vertex weights of G. For any vertex $x \in V$ we have

$$\begin{aligned} w_{\lambda'}(x) &= \lambda'(x) + \sum_{xy \in E} \lambda'(xy) \\ &= M + 1 - \lambda(x) + \sum_{xy \in E} [M + 1 - \lambda(xy)] \\ &= (r_x + 1)(M + 1) - w_\lambda(x), \end{aligned}$$

where r_x is the number of edges incident to the given vertex x. Clearly, the set $W' = \{w_{\lambda'}(x) | x \in V\}$ consists of an arithmetic progression with difference $d' = d$ if and only if r_x is constant for every x, that is, if and only if G is regular. ■

Corollary 1.1. *Let G be a regular graph of degree r. Then G has an (a, d)-vertex-antimagic total labeling if and only if G has an (a', d)-vertex-antimagic total labeling where $a' = (r + 1)(M + 1) - a - (v - 1)d$.*

Proof. Let G be a regular graph of degree r and λ be an (a, d)-VATL for G. If λ' is the dual labeling of λ then for every vertex $x \in V$ we have $w_{\lambda'}(x) = (r + 1)(e + v + 1) - w_\lambda(x)$, where $w_\lambda(x)$ is the weight of the vertex x under the labeling λ. We have $w_\lambda(x) = a + (v - 1)d$ as the maximum vertex weight under the labeling λ if and only if $w_{\lambda'}(x) = (r + 1)(e + v + 1) - a - (v - 1)d$ is the minimum vertex weight under the labeling λ'. ■

Can one use a VATL on a graph G to derive a VATL for a subgraph of G? This seems to be a difficult question in general. The following theorem provides one case in which it is possible.

Theorem 2. *Let G be a regular graph of degree r labeled in such a way that some edge z receives the label 1. Then G has an (a, d)-vertex-antimagic total labeling if and only if $G - \{z\}$ has an (a', d)-vertex- antimagic total labeling with $a' = a - r - 1$.*

Proof. Assume that G is an r-regular graph and λ is the (a, d)-VATL on G. Define a new mapping μ by

$$\begin{aligned} \mu(x) &= \lambda(x) - 1, \quad x \in V, \\ \mu(xy) &= \lambda(xy) - 1, \quad xy \in E. \end{aligned}$$

Clearly, the map μ is a one-to-one and the label 0 is assigned to edge z by μ. Then we have

$$\begin{aligned} w_\mu(x) &= \mu(x) + \sum_{xy \in E} \mu(xy) \\ &= \lambda(x) - 1 + \sum_{xy \in E} [\lambda(xy) - 1] \\ &= \lambda(x) + \sum_{xy \in E} \lambda(xy) - r - 1 \\ &= w_\lambda(x) - r - 1, \end{aligned}$$

(where the above summations are taken over all vertices adjacent to x). Clearly, the minimum value of $w_\mu(x)$ occurs when $w_\lambda(x) = a$.

If we delete the edge z from G, we obtain a graph $G - \{z\}$ and the restriction of the mapping μ to $G - \{z\}$ is an $(a - r - 1, d)$-VATL.

The proof of the converse is as follows. Let λ be the VATL for $G - \{z\}$. Define a new mapping μ in G by

$$\begin{aligned} \mu(z) &= 1, \\ \mu(x) &= \lambda(x) + 1 \qquad \text{for all } x \in V, \\ \mu(xy) &= \lambda(xy) + 1 \qquad \text{for all } xy \neq z \in E. \end{aligned}$$

Then it is easy to check that μ is the appropriate VATL for G. ■

3. RELATIONS WITH OTHER LABELINGS

As described in the introduction, other related types of labelings have been studied previously. In this section, we show that it is possible in some cases to derive a VATL from some other appropriate labeling of the graph. In particular, much work has been done on various kinds of edge labelings. Unfortunately, the terminology used by the various authors is not standard, so we repeat here the relevant definitions.

Some of the earlier work on edge labeling permitted the labels to belong to any set of positive integers. The following definition has been used:

Definition 3. If there exists a one-to-one map $f : E \to Z^+$ such that all vertices have the same weight $w(x)$, then the graph G is called *magic* and the map f is called a *magic labeling* of G.

In our terminology, this is a *vertex-magic edge labeling.* A characterization of regular magic graphs is given in [4]. Several necessary and sufficient conditions for the existence of magic graphs can be found in [7] and [8].

Stewart apparently was the first to impose the restriction on the magic labeling that the labels belong to the set $\{1, \ldots, e\}$. He made the following definition which appeared in [13]:

Definition 4 (Stewart [13])**.** If there exists a bijection $f : E \to \{1, 2, \ldots, e\}$ such that all vertices have the same weight $w(x)$, then the graph G is called *super-magic* and the map f is called a *super-magic labeling* of G.

Stewart [13] showed that the complete graph K_n is super-magic when $n = 2$ or $n > 5$ and $n \not\equiv 0 \pmod 4$. For K_n we have $v = n$ and $e = \frac{n(n-1)}{2}$. Let $f : E(K_n) \to \{1, 2, \ldots, e\}$ be the super-magic labeling of K_n. Thus the sum of all edge labels is equal to

$$\frac{(n^2 - n + 2)(n^2 - n)}{8}$$

and, since each label is used by two vertices, the magic constant (the constant sum at each vertex) is

$$k = \frac{(n^2 - n + 2)(n - 1)}{4}.$$

If we now label the vertices in G with $\{e + 1, e + 2, \ldots, v + e\}$ then these labels together with the edge labels from f combine to give an (a, d)-vertex antimagic total labeling where

$$a = k + e + 1 = \frac{n^3 + n + 2}{4} \quad \text{and} \quad d = 1.$$

A similar argument applies for any graph G that has a super-magic labeling and so, more generally, we have

Theorem 3. *Every super-magic graph G has an $(a, 1)$-vertex-antimagic total labeling.*

From [13] we know that both K_n and $K_{n,n}$ have super-magic labelings; consequently, we have the following two corollaries:

Corollary 3.1. *If $n = 2$ or $n > 5$ and $n \not\equiv 0 \pmod 4$ then the complete graph K_n has an $(a, 1)$-vertex-antimagic total labeling.*

Corollary 3.2. *There is an $(a, 1)$-vertex-antimagic total labeling for $K_{n,n}$ for all $n \geq 3$.*

Super-magic labelings have been described by Bača [1] for a family of quartic graphs R_n when $n = 4k$ or $n = 4k + 2$, $k \geq 1$. Therefore the next corollary also follows from the Theorem 3.

Corollary 3.3. *If $n = 4k$ or $n = 4k + 2$, $k \geq 1$, then the quartic graphs R_n have an $(a, 1)$-vertex-antimagic total labeling.*

As noted in the introduction, Bodendiek and Walther [3] introduced the notion of the (a, d)-*antimagic labeling*, an edge labeling in which the vertex weights form an arithmetic progression. They made the following definition:

Definition 5. A graph $G = (V, E)$ is said to be an (a, d)-*antimagic graph* if there exist positive integers a, d and a bijection $f : E \to \{1, 2, \ldots, e\}$ such that the set of vertex weights is $W = \{w(v)|v \in V\} = \{a, a + d, \ldots, a + (v - 1)d\}$. The map f is called an (a, d)-*antimagic labeling* of G.

These labelings have been investigated by Bača and others; see, for example, [2] and [11].

Theorem 4. (i) *If $d > 1$ then every (a, d)-antimagic graph G has an $(a + v + e, d - 1)$-vertex-antimagic total labeling.*
(ii) *Every (a, d)-antimagic graph G has an $(a+e+1, d+1)$-vertex-antimagic total labeling.*

Proof. We assume that graph G is (a, d)-antimagic with $d > 1$ and let $f : E \to \{1, 2, \ldots, e\}$ be an (a, d)-antimagic labeling of G. Then $W = \{w_f(x)|x \in V\} = \{a, a + d, \ldots, a + (v - 1)d\}$ is the set of vertex weights of G. For $i = 0, \ldots, (v - 1)$, let x_i be the vertex with weight $w_f(x_i) = a + id$. Define two sets of labels on the vertices

$$f', f'' : V \to \{e + 1, e + 2, \ldots, e + v\}$$

as follows:

$$\begin{aligned} f'(x_i) &= e + i + 1, \\ f''(x_i) &= v + 2e + 1 - f'(x_i). \end{aligned}$$

Then the labelings f and f' combine to give an $(a+e+1, d+1)$-VATL for G and f and f'' to give an $(a+v+e, d-1)$-VATL for G. ■

Readers should note that the term *magic labeling* of a graph G has been used by Kotzig and Rosa [9] and others to mean a total labeling, specifically, a bijection f from $V \cup E$ to $\{1, 2, \ldots, v+e\}$ such that for all edges xy, $f(x) + f(y) + f(xy)$ is constant. In our terminology, this is an edge-magic total labeling of G.

The notion of vertex-magic total labeling has recently been introduced [10]; the total labeling in which the vertex weights are constant. In fact that may be considered a special case of the (a, d)-vertex-antimagic total labeling in which $d = 0$. In subsequent papers we hope to explore the relationship between these two types of graph labelings. The next theorem gives an example of how one may construct a VATL from a vertex-magic total labeling.

Theorem 5. *Let G be a graph with a total labeling whose vertex labels constitute an arithmetic progression with difference d. Then G has a vertex-magic total labeling with magic constant k if and only if G has an $(a', 2d)$-vertex-antimagic total labeling where $a' = k + (1-v)d$.*

Proof. Let λ be a vertex-magic total labeling of G and k the magic constant for λ. Suppose that, under the labeling λ, the vertex labels of G constitute an arithmetic progression with difference d; in other words,

$$\begin{aligned} \{\lambda(x_i) \mid x_i \in V\} &= \{p + (i-1)\ d \mid i = 1, 2, \ldots, v\} \\ &= \{p, p+d, \ldots, p+(v-1)\ d\}, \quad p \in \mathbf{Z}^+. \end{aligned}$$

Then, under the edge labeling λ_E induced by λ, the weights of vertices constitute an arithmetic progression; specifically

$$\begin{aligned} \{w_{\lambda_E}(x_i) \mid x_i \in V\} &= \{w_\lambda(x_i) - \lambda(x_i) \mid x_i \in V\} \\ &= \{k - p - (i-1)\ d \mid i = 1, 2, \ldots, v\} \\ &= \{k-p, k-p-d, \ldots, k-p-(v-1)d\}. \end{aligned}$$

Define a new mapping μ by

$$\begin{aligned} \mu(z) &= \lambda(z) && \text{for } z \in E, \text{ and} \\ \mu(x_i) &= p + (v-i)d && \text{for } x_i \in V. \end{aligned}$$

It can be seen that the weights of vertices, under the new mapping μ, constitute the set

$$\begin{aligned} W &= \{w_\mu(x_i) \mid x_i \in V\} \\ &= \{k + (v + 1 - 2i)\ d \mid i = 1, 2, \ldots, v\} \\ &= \{k + (v - 1)\ d, k + (v - 3)\ d, \ldots, k + (1 - v)\ d\}, \end{aligned}$$

i.e., the weights of vertices constitute an arithmetic progression with difference $2d$ and the minimum value of weight is $k+(1-v)d$. Hence μ is a VATL on G.

The proof of the converse is similar and is omitted. ■

4. PATHS AND CYCLES

Among the graphs for which it is easiest to find VATLs are the cycles and paths. In this section we provide labelings for both families of graphs. For the n-cycle C_n we have $v = e = n$, so that the label set is $\{1, \ldots, 2n\}$.

Applying inequality 1 with $\Delta = \delta = 2$ calculate the maximum feasible value for d. We get $a + (n - 1) \leq 6n - 3$ where $6 \leq a$. Consequently we find

$$d \leq \frac{6n - 9}{n - 1} = 6 - \frac{3}{n - 1}.$$

Thus $d \leq 5$ for all $n \geq 4$ and $d \leq 4$ for $n = 3$. In Figure 2, we give examples of C_3 for each feasible value of d.

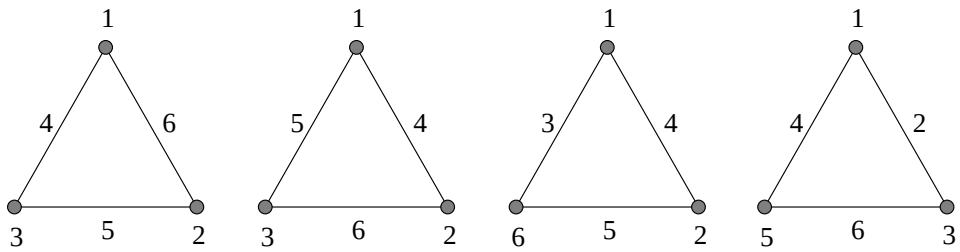

Figure 2. VATLs of C_3 for all feasible d

We proved in Theorem 2 that every VATL for a graph of the form $G - \{z\}$, where G is regular and in which an edge z has the label 1, is obtained from

a VATL of G. Since a path P_n is the cycle C_n with an edge removed, then every VATL for the path P_n is obtained from a corresponding VATL for C_n (note that the converse is not necessarily true).

Theorem 6. *Every odd cycle C_n, $n \geq 3$, has a $(\frac{3n+5}{2}, 2)$-vertex-antimagic total labeling and $(\frac{5n+5}{2}, 2)$-vertex-antimagic total labeling.*

Proof. Wallis *et al.* [14] proved that every odd cycle has an edge-magic total labeling with magic constant $k = \frac{5n+3}{2}$. For cycles (and only for cycles), an edge-magic total labeling is equivalent to a vertex-magic total labeling (see [10]) and, moreover, the vertex labels of the considered vertex-magic total labeling constitute an arithmetic progression with difference $d = 1$. Thus, by Theorem 5, the odd cycle C_n has a $(\frac{3n+5}{2}, 2)$-VATL.
To prove that C_n has $(\frac{5n+5}{2}, 2)$-VATL, we make use of Corollary 1.1 and the fact that C_n is a 2-regular graph. It is simple to verify that the minimal vertex weight is $\frac{5n+5}{2}$. ■

The following is an easy consequence of the Theorem 2.

Corollary 6.1. *For n odd and $n \geq 3$, the path P_n has a $(\frac{3n-1}{2}, 2)$-vertex-antimagic total labeling.*

Proof. The cycle C_n is a 2-regular graph and by Theorem 6 admits a $(\frac{3n+5}{2}, 2)$-VATL in which the label 1 is assigned to an edge z. Theorem 2 now guarantees that the path P_n has a $(\frac{3n-1}{2}, 2)$-VATL. ■

In the following theorems we provide examples of VATLs with various values of d for the cycles C_n and the paths P_n.

Theorem 7. *Every cycle C_n, $n \geq 3$ has a $(3n+2, 1)$-vertex-antimagic total labeling and a $(2n+2, 1)$-vertex-antimagic total labeling.*

Proof. Let the cycle C_n be $(x_1, \ldots, x_n)$. If we label the vertices and edges in C_n by

$$\begin{aligned} \lambda(x_i) &= i && \text{for } i = 1, \ldots, n, \\ \lambda(x_i x_{i+1}) &= 2n - i && \text{for } i = 1, \ldots, n-1, \\ \lambda(x_n x_1) &= 2n, \end{aligned}$$

then the vertex weights will be

$$w_\lambda(x_i) = \begin{cases} 4n + 1 - i & \text{for } i = 1, \ldots, n-1, \\ 4n + 1 & \text{for } i = n, \end{cases}$$

and these clearly form the arithmetic progression $3n+2, 3n+3, \ldots, 4n+1$. Thus C_n has a $(3n+2,1)$-VATL.

Combining this with Corollary 1.1, it is easy to see that C_n also has a $(2n+2,1)$-VATL. ■

Since the cycle C_n has a $(2n+2,1)$-VATL in which the label 1 is assigned to an edge, by Theorem 2 we have

Corollary 7.1. *Every path P_n, $n \geq 3$, has a $(2n-1,1)$-vertex-antimagic total labeling.*

Theorem 8. *Every cycle C_n, $n \geq 3$ has a $(2n+3,2)$-vertex-antimagic total labeling and a $(2n+2,2)$-vertex-antimagic total labeling.*

Proof. Let the cycle C_n be $(x_1, \ldots, x_n)$. If we label the vertices and edges in C_n by

$$\begin{aligned} \lambda(x_i) &= 2i-1 && \text{for } i = 1, \ldots, n, \\ \lambda(x_i x_{i+1}) &= 2(n+1-i) && \text{for } i = 1, \ldots, n-1, \\ \lambda(x_n x_1) &= 2, \end{aligned}$$

then the vertex weights are

$$w_\lambda(x_i) = \begin{cases} 4n+5-2i & \text{for } i = 2, \ldots, n, \\ 2n+3 & \text{for } i = 1, \end{cases}$$

and these form the arithmetic progression $2n+3, 2n+5, \ldots, 4n+1$. Thus C_n has a $(2n+3,2)$-VATL.

Combining this with Corollary 1.1, it is easy to see that C_n also has a $(2n+2,2)$-VATL. ■

Since the cycle C_n has a $(2n+2,2)$-VATL in which the label 1 is assigned to an edge, by Theorem 2 we have

Corollary 8.1. *Every path P_n, $n \geq 3$, has a $(2n-1,2)$-vertex-antimagic total labeling.*

Theorem 9. *Every cycle C_n, $n \geq 3$ has a $(2n+2,3)$-vertex-antimagic total labeling and an $(n+4,3)$-vertex-antimagic total labeling.*

Proof. As before, the cycle C_n is $(x_1, \ldots, x_n)$. Label the vertices and edges in C_n as follows:

$$\begin{aligned}
\lambda(x_i) &= i && \text{for } i = 1, \ldots, n-1, \\
\lambda(x_n) &= 2n, \\
\lambda(x_i x_{i+1}) &= n + i && \text{for } i = 1, \ldots, n-1, \\
\lambda(x_n x_1) &= n,
\end{aligned}$$

then the vertex weights are

$$w_\lambda(x_i) = 2n - 1 + 3i, \;\; 1 \le i \le n,$$

clearly making a $(2n+2, 3)$-VATL.

Combining this with Corollary 1.1, it is easy to see that C_n also has a $(n+4, 3)$-VATL. ■

Theorem 10. *Every odd cycle C_n, $n \ge 3$ has an $(n+4, 4)$-vertex-antimagic total labeling and a $(n+3, 4)$-vertex-antimagic total labeling.*

Proof. Letting C_n be $(x_1, \ldots, x_n)$, we label the vertices and edges as follows:

$$\begin{aligned}
\lambda(x_i) &= 2i - 1 && \text{for } i = 1, \ldots, n, \\
\lambda(x_i x_{i+1}) &= i + 1 && \text{for } i \text{ odd}, \; i \ne n, \\
\lambda(x_i x_{i+1}) &= n + i + 1 && \text{for } i \text{ even}, \\
\lambda(x_n x_1) &= n + 1,
\end{aligned}$$

then the vertex weights are

$$w_\lambda(x_i) = n + 4i, \;\; 1 \le i \le n,$$

which clearly constitutes an $(n+4, 4)$-VATL for C_n.

Combining this with Corollary 1.1, it is easy to see that C_n also has a $(n+3, 4)$-VATL. ■

Since the cycle C_n has an $(n+3, 4)$-VATL in which the label 1 is assigned to an edge, by Theorem 2 we have

Corollary 10.1. *Every odd path P_n, $n \ge 3$, has an $(n, 4)$-vertex-antimagic total labeling.*

Theorem 11. *The path P_n has a $(2n-1,1)$-vertex-antimagic total labeling for any $n \geq 2$.*

Proof. Name the vertices in P_n as $x_1, \ldots, x_n$ and the set of edges is $E(P_n) = \{x_i x_{i+1} | i = 1, \ldots, n-1\}$. Then attach labels to all the vertices and edges as follows:

$$\lambda(x_i) = \begin{cases} n & \text{for } i = 2, \\ 2n-i & \text{for } i = 3, \ldots, n-1, \\ 2n-2 & \text{for } i = n, \\ 2n-1 & \text{for } i = 1, \end{cases}$$

$$\lambda(x_i x_{i+1}) = \begin{cases} 1 & \text{for } i = n-1, \\ i & \text{for } i = 2, \ldots, n-2, \\ n-1 & \text{for } i = 1. \end{cases}$$

Under this labeling we have the vertex weights:

$$w_\lambda(x_i) = \begin{cases} 3n-1-i & \text{for } i = n-1, n, \\ 2n-1+i & \text{for } i = 2, \ldots, n-2, \\ 3n-2 & \text{for } i = 1. \end{cases}$$

These form the arithmetic progression $2n-1, 2n, \ldots, 3n-2$ and so, λ is a $(2n-1,1)$-VATL. ■

As mentioned at the beginning of this section, a VATL for the path P_n, for $n \geq 3$, provides a corresponding VATL for the cycle C_n. Therefore we have the following corollary.

Corollary 11.1. *Every cycle C_n, $n \geq 3$ has a $(2n+2,1)$-vertex-antimagic total labeling and an $(3n+2,1)$-vertex-antimagic total labeling.*

Interestingly, this labeling and the labeling produced by Theorem 7 are both $(2n+2,1)$-VATL, but they are different. Here is an example of different VATLs on the same graph achieving the same values of a and d.

5. OPEN QUESTIONS

In a subsequent paper, we will provide constructions for VATLs for a variety of families of graphs. But there are many graphs we have not studied, and

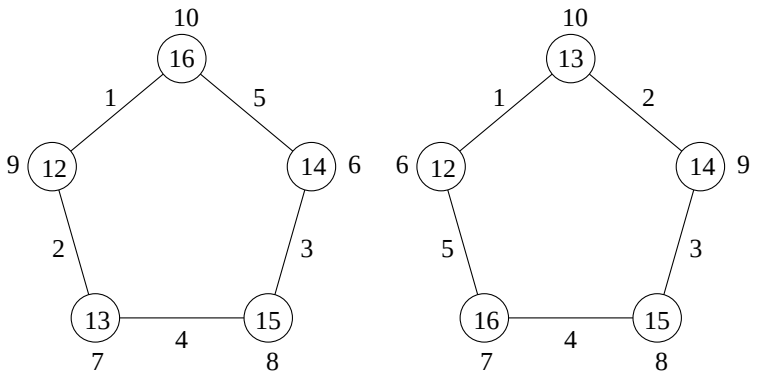

Figure 3. Two different (12,1)-VATLs for C_5

several families of graphs that we have studied for which we have not found VATLs. We list here several problems for further investigation.

Open problem 1. For the paths P_n and the cycles C_n, determine if there is a vertex-antimagic total labeling for every feasible pair (a, d).

Open problem 2. Apart from duality, how can a vertex-antimagic total labeling for a graph be used to construct another vertex-antimagic total labeling for the same graph, preferably with different a and d?

Open problem 3. In Theorem 5, we found a way to construct VATL for a graph G from a vertex-magic total labeling of G. Are there other ways to do this?

Open problem 4. Find, if possible, some structural characteristics of a graph which make a vertex-antimagic total labeling impossible.

References

[1] M. Bača, I. Holländer and Ko-Wei Lih, *Two classes of super-magic quartic graphs*, JCMCC **13** (1997) 113–120.

[2] M. Bača and I. Holländer, *On (a, d)-antimagic prisms*, Ars. Combin. **48** (1998) 297–306.

[3] R. Bodendiek and G. Walther, *Arithmetisch antimagische graphen*, in: K. Wagner and R. Bodendiek, Graphentheorie III, (BI-Wiss. Verl., Mannheim-Leipzig-Wien-Zürich, 1993).

[4] M. Doob, *Generalizations of magic graphs*, J. Combin. Theory (B) **17** (1974) 205–217.

[5] J.A. Gallian, *A dynamic survey of graph labeling*, Electronic. J. Combin. **5** (1998) #DS6.

[6] N. Hartsfield and G. Ringel, Pearls in Graph Theory (Academic Press, Boston-San Diego-New York-London, 1990).

[7] R.H. Jeurissen, *Magic graphs, a characterization*, Report 8201 (Mathematisch Instituut, Katholieke Universiteit Nijmegen, 1982).

[8] S. Jezný and M. Trenkler, *Charaterization of magic graphs*, Czechoslovak Math. J. **33** (1983) 435–438.

[9] A. Kotzig and A. Rosa, *Magic valuations of finite graphs*, Canad. Math. Bull. **13** (1970) 451–461.

[10] J.A. MacDougall, M. Miller, Slamin, and W.D. Wallis, *Vertex-magic total labelings of graphs*, Utilitas Math., to appear.

[11] M. Miller and M. Bača, *Antimagic valuations of generalized Petersen graphs*, Australasian J. Combin. **22** (2000) 135–139.

[12] J. Sedláček, Problem 27 in Theory of Graphs and its Applications, Proc. Symp. Smolenice, June 1963, Praha (1964), p. 162.

[13] B.M. Stewart, *Supermagic complete graphs*, Can. J. Math. **19** (1967) 427–438.

[14] W.D. Wallis, E.T. Baskoro, M. Miller and Slamin, *Edge-magic total labelings of graphs*, Australasian J. Combin. **22** (2000) 177–190.

[15] D.B. West, An Introduction to Graph Theory (Prentice-Hall, 1996).

Received 15 June 2001
Revised 20 October 2001