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DIFFERENCE LABELLING OF CACTI

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Abstract

A graph G is a *difference graph* iff there exists $S \subset I\!N^+$ such that G is isomorphic to the graph $DG(S) = (V, E)$, where $V = S$ and $E = \{\{i, j\} : i, j \in V \wedge |i - j| \in V\}$.

It is known that trees, cycles, complete graphs, the complete bipartite graphs $K_{n,n}$ and $K_{n,n-1}$, pyramids and n-sided prisms ($n \geq 4$) are difference graphs (cf. [4]). Giving a special labelling algorithm, we prove that cacti with a girth of at least 6 are difference graphs, too.

Keywords: graph labelling, difference graph, cactus.

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1. Introduction and Basic Definitions

Harary [10] introduced the notion of sum graphs in 1988. In recent years, a lot of authors published papers dealing with sum graphs, e.g. [1, 2, 6, 8, 9], [11] – [19].

Moreover, in [10] Harary mentioned the concept of difference graphs. Some classes of difference graphs (paths, trees, cycles, special wheels, complete graphs, complete bipartite graphs etc.) were investigated by Bloom, Burr, Eggleton, Gervacio, Hell and Taylor in the undirected (cf. [3, 4, 7]) as well as in the directed case (cf. [5]). In the papers [3, 4, 7] undirected difference graphs were called *autographs* or *monographs*.

In the following, we will present an algorithm for the difference labelling of cacti with a girth of at least 6.

All graphs considered in this article are supposed to be nonempty and finite without loops and multiple edges.

Let $S \subset \mathbb{N}^+$ be finite. $DG(S) = (V, E)$ is the *difference graph of* S iff $V = S$ and $E = \{\{i, j\} : i, j \in V \wedge \ |i - j| \ \in V\}$.

Furthermore, a given graph G is a *difference graph* iff there exists $S \subset \mathbb{N}^+$ such that G is isomorphic to $DG(S)$.

Let $DG_{\mathbb{N}^+}$ be the class of all difference graphs. As an example, consider the wheel $W_4 \in DG_{\mathbb{N}^+}$ in Figure 1.

W_4 :

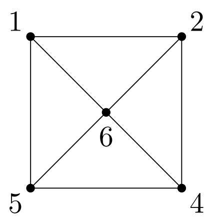


Figure 1

A mapping $r : \ V \longrightarrow \mathbb{N}^+$ is called a *difference labelling* of the difference graph $G = (V, E)$ iff G is isomorphic to $DG(S)$, where $S := \{r(v)|v \in V\}$. Obviously, every difference labelling is injective.

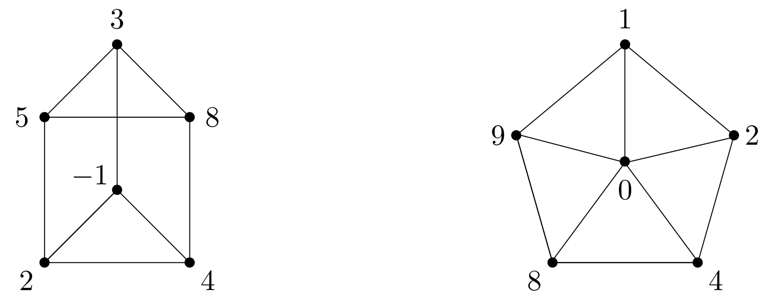
In [3] and [4] several modifications of the notion of difference graphs were investigated. In analogy with the notation $DG_{\mathbb{N}^+}$ the classes of all (generalized) difference graphs with vertex labels in $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}^+$ and $\mathbb{R}$ are denoted by $DG_{\mathbb{N}}$, $DG_{\mathbb{Z}}$, $DG_{\mathbb{R}^+}$ and $DG_{\mathbb{R}}$, respectively.

Bloom and Burr [3] proved $DG_{\mathbb{R}} = DG_{\mathbb{Z}}$ and $DG_{\mathbb{R}^+} = DG_{\mathbb{N}^+}$. On the other hand, it is known that $DG_{\mathbb{N}} \subset DG_{\mathbb{Z}}$ and $DG_{\mathbb{N}^+} \subset DG_{\mathbb{N}}$ (cf. Figure 2).

Another modification is to use non-injective difference labellings, i.e., we allow to give the same label to different vertices. E.g., $K_{m,n}$ for $m \geq 4$, $1 < n < m - 1$ can be given such a non-injective difference labelling, but $K_{m,n} \notin DG_{\mathbb{N}^+}$.

It is known that trees, cycles, complete graphs, the complete bipartite graphs $K_{n,n}$ and $K_{n,n-1}$, pyramids and n-sided prisms ($n \geq 4$) are difference graphs (cf. [4]). Gervacio [7] proved that W_3, W_4 and W_6 are the only wheels which are difference graphs.

In the following, we generalize the result of Bloom, Hell and Taylor that trees are difference graphs to the class of cacti with a girth of at least 6.



3-sided prism $\in DG_{\mathbb{Z}} \setminus DG_{\mathbb{N}}$ $\qquad$ $W_5 \in DG_{\mathbb{N}} \setminus DG_{\mathbb{N}^+}$

Figure 2

$K_{m,n}$:

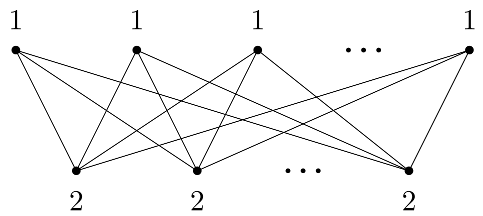


Figure 3

2. Cacti

A nonempty, finite and connected graph $G = (V, E)$ is called a *cactus* iff every edge $e \in E$ is contained in at most one cycle. In [4] Bloom et al. introduced an irreducibility concept for trees which is useful for cacti, too.

An end edge $e \in E$ is called a *prickle* of the cactus $G = (V, E)$. G is *irreducible* iff no vertex $v \in V$ is incident with more than one prickle.

Bloom, Hell and Taylor developed the following procedure to reduce the construction of difference labellings of (reducible) trees to irreducible trees. (We will apply this procedure to cacti.) Let $G = (V, E)$ be a reducible tree/cactus.

Remove prickles in pairs at each vertex $v \in V$ (with at least two prickles; e.g., $\{v, v'\}$ and $\{v, v''\}$) until G is irreducible (see Figure 4).

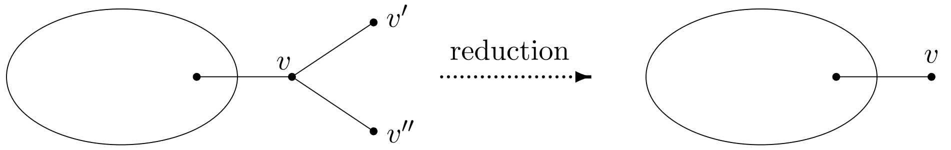


Figure 4

Construct a difference labelling r of the resulting graph.

Reinsert the prickles (e.g., $\{v, v'\}$ and $\{v, v''\}$) in pairs; if $r(v) = \alpha$ then put $r(v') := \frac{1}{p}$ and $r(v'') := \alpha - \frac{1}{p}$ for a prime p not previously used in G (cf. Figure 5).

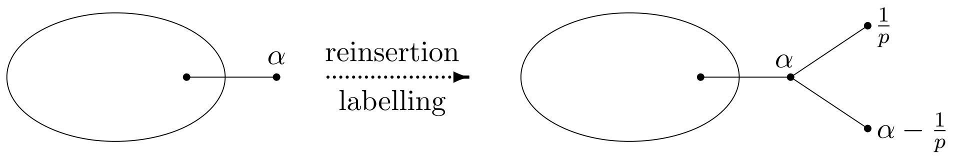


Figure 5

After reinserting and labelling all pairs of prickles, multiply the labels of all vertices of G by $\prod_{i \in I} p_i$, where $\{p_i | i \in I\}$ is the set of all prime values used to label end vertices of pairs of prickles as described above. This yields a difference labelling of the graph G.

Consequently, we can restrict on irreducible cacti.

2.1. Caterpillars and hedgehogs

A tree $T = (V, E)$ is called a *caterpillar* iff deleting all end vertices (and prickles) of T results in a path. A *k-caterpillar* is a caterpillar with a longest path of a length of $k-1$. A k-caterpillar can be considered as a path P_k (the *backbone* of the caterpillar) with additional prickles at some inner vertices. Note that the backbone of a caterpillar is possibly not unique. In this case let us choose one longest path of T as "the" backbone of T and call one of its end vertices the *initial* and the other one the *terminal vertex* of the caterpillar.

With regard to the labelling algorithm in Section 2.2, the *initial* and the *terminal edge* of a caterpillar (i.e., the edges incident to the initial and the terminal vertex, respectively) are not considered as prickles per definition.

A *k-hedgehog* is a k-cycle C_k with additional prickles at some vertices. Consequently, a k-hedgehog can be defined as a graph with the property that deleting all end vertices and prickles results in a C_k.

In a certain sense, a cactus has a tree-like structure, and it is possible to decompose it into hedgehogs and caterpillars. (In general, this decomposition is not unique.)

To construct a difference labelling of an irreducible cactus we choose such a decomposition. Then, step by step we construct special labellings (so-called *(x,t)-labellings*) of the hedgehogs and caterpillars and combine these labellings to obtain a difference labelling of the cactus.

To avoid undesired edges (between different hedgehogs or caterpillars) induced by vertex labels, we construct the labelling in a way that guarantees large differences between vertex labels of different hedgehogs and caterpillars, respectively.

Definition. r is an *(x,t)-labelling* of $G=(V,E)$ with initial vertex v iff $x,t \in \mathbb{N}^+$, $t > 2x$, $v \in V$ and $r : V \longrightarrow \{x\} \cup \{n | n \in \mathbb{N}^+ \wedge \ n \geq t\}$ is a difference labelling of G with $r(v) = x$ and $\forall a,b \in V : \ a \neq b \longrightarrow |r(a)-r(b)| = x \vee |r(a)-r(b)| > \frac{t-1}{2}$.

(x,t)-Lemma.

(C) *Let $k \geq 3$, $G=(V,E)$ an (irreducible) k-caterpillar and $v \in V$ the initial vertex of G,*

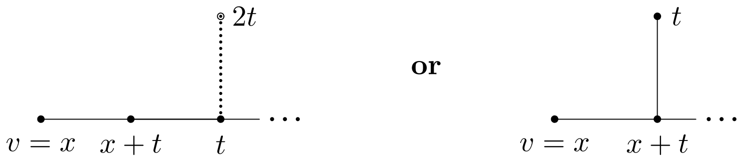
or

(H) *let $k \geq 6$, $G=(V,E)$ an (irreducible) k-hedgehog, $v \in V$ an end vertex or a cycle vertex without prickle.*

Then, for arbitrary $x \in \mathbb{N}^+$ and $t > 2x$, there exists an (x,t)-labelling r of G with initial vertex v.

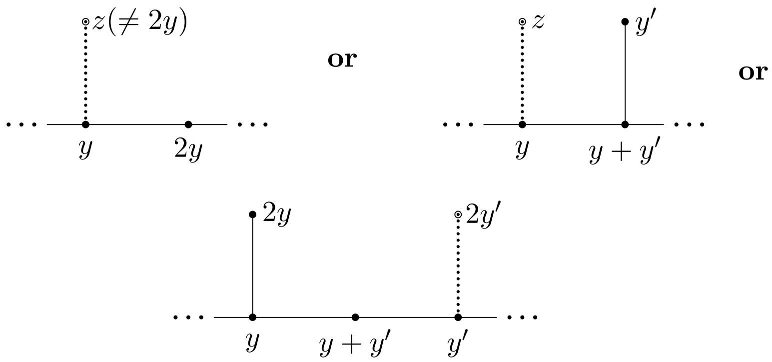
Proof. For simplification, in most cases we identify vertices $u \in V$ with their label $r(u)$.

Case (C). We start at the initial vertex v with $r(v) := x$ and label the caterpillar along its backbone. Depending on the local structure of the caterpillar (i.e., whether or not there are prickles at the vertices of the backbone) we have to use different labelling principles, which we sketch in Figures 6–8. In these sketches we label from the left to the right; dotted lines and hollow dots will be used for prickles which need not — but may — exist.



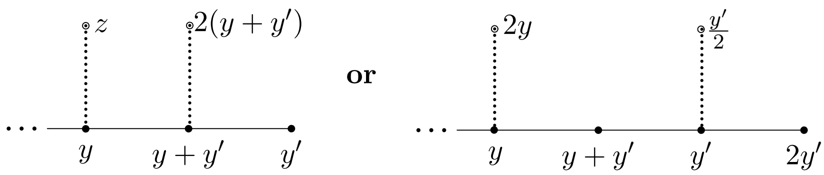
initial part of the caterpillar

Figure 6



middle part of the caterpillar

Figure 7



terminal part of the caterpillar

Figure 8

Note that the numbers y' must be sufficiently large, i.e., if $w \in V$ is the next vertex to be labelled with such a "sufficiently large" number y', we can use $y' := 2 \cdot \max\{r(a) | a \in V \wedge a \text{ is a labelled vertex}\} + 1$; only in the last case (cf. the right picture in Figure 8), when there exists the prickle $\{y', \frac{y'}{2}\}$, we need the larger label $y' := 4 \cdot \max\{r(a) | a \in V \wedge a \text{ is a labelled vertex}\} + 2$.

Case (H). Let v be a cycle vertex (Figure 9.1) or an end vertex (Figure 9.2).

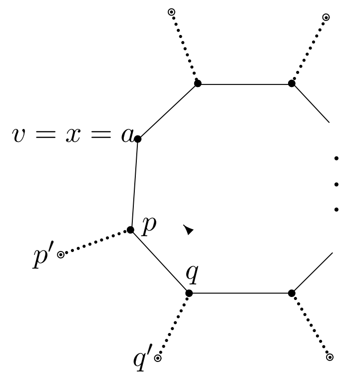


Figure 9.1

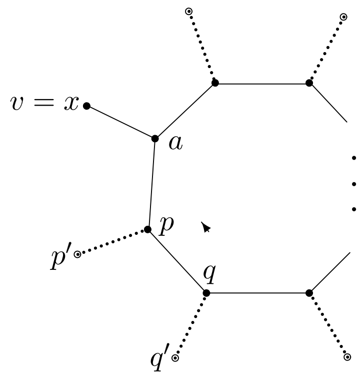


Figure 9.2

In order to label the hedgehog, we follow the direction sketched in the figures starting at v. Let a be the first cycle vertex which we reach (i.e., $a = v$ or — if v is an end vertex — a is incident with the prickle with end vertex v) and q be the predecessor of the last cycle vertex p (which is the predecessor of a). At first, consider the caterpillar which we obtain by deleting p (and its prickle, if exists). This caterpillar has the initial vertex v and the terminal vertex q or q' (if there is a prickle $\{q, q'\}$ at q). Using the labelling method described in Case (C), we construct an (x, t)-labelling of this caterpillar starting with $r(v) = x$.

Now the labels of a and q are determined, and for the vertex p we choose the label $p = a + q$. If there exists a prickle $\{p, p'\}$ the label $p' = 2 \cdot p$ can be used (cf. Figure 10).

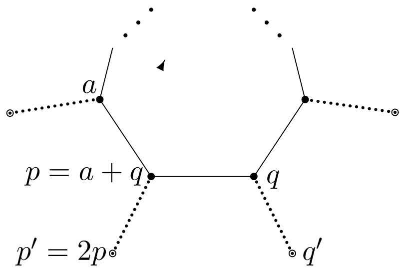


Figure 10

This completes our proof. ∎

2.2. Main Theorem

Following the idea described at the beginning of Section 2 (decompose an irreducible cactus into hedgehogs and caterpillars, construct (x, t)-labellings of them and combine these labellings to obtain a difference labelling of the cactus), we are able to prove our main theorem.

Theorem. *Cacti with a girth of at least* 6 *are difference graphs.*

Proof. Without loss of generality, we can restrict our investigations on irreducible cacti $G = (V, E)$ (with a girth of at least 6).

Using an algorithm we will construct a difference labelling of a given cactus $G = (V, E)$. This algorithm makes use of the (x, t)-Lemma for the construction of (x, t)-labellings of certain hedgehogs and caterpillars (in steps 4 and 5). To describe the decomposition of G into these hedgehogs and caterpillars we need the following notation:

Consider a vertex $v \in V$ which is contained in a path w [a cycle c] of the irreducible cactus $G = (V, E)$. v is called a *branch vertex* iff after the construction of an (x, t)-labelling (with suitable x, $t \in \mathbb{N}^+$) of w [of c] (including the prickles of w [of c]) v is incident with at least two [at least one] unlabelled edges [edge]. Of course, these edges cannot be prickles but if v lies on a path w, two of them can be in a common cycle.

Algorithm

1. $L := \emptyset$.
2. **Let** $v \in V$ be an end vertex, if G contains one, or a cycle vertex, otherwise; $r(v) := 1$.

3. $x := r(v)$, $t := 2 \cdot \max\{r(u) \mid u \in V \wedge u \text{ is labelled}\} + 1$.
4. **If** the distance of v to any unlabelled cycle is greater than 1,
   then, starting at v, construct an (x, t)-labelling along an (unlabelled) path $w = (v = v_0, v_1, \ldots, v_k)$ of maximum length, where w must not contain edges of cycles, **and**
   set $L := L \cup \{u \mid u \in V(w) \wedge u \text{ is a branch vertex}\}$.
5. **If** the distance of v to an unlabelled cycle c is at most 1,
   then, starting at v, construct an (x, t)-labelling along c **and**
   set $L := L \cup \{u \mid u \in V(c) \wedge u \text{ is a branch vertex}\}$.
6. **If** $L = \emptyset$, **then** stop.
7. **Let** $v \in L$.
8. **If** v is incident with exactly one unlabelled edge
   or with exactly two unlabelled edges contained in a common cycle,
   then $L := L \setminus \{v\}$.
9. **Go to** 3.

Of course, if we construct (x, t)-labellings *along paths* and *cycles* in steps 4 and 5, we mean that we construct such labellings of the corresponding caterpillars and hedgehogs, respectively.

Because G is connected, every vertex has got a label after applying the algorithm: In the algorithm the set L picks up every branch vertex v of G, and (immediately after the removal of v from L) all unlabelled vertices, which are adjacent to v or contained in the caterpillar w/hedgehog c (see steps 4/5), get their label. This way, all caterpillars and hedgehogs get their (x, t)-labelling step by step.

Considering a single caterpillar and hedgehog G' labelled in step 4 and 5, respectively, the (x, t)-Lemma ensures that we obtain a difference labelling of G'. During this labelling procedure it is important that at the beginning of step 4 and 5, respectively, only the "initial vertex" v of G' has already a label; because G is a cactus, all vertices of $V(G') \setminus \{v\}$ must be unlabelled.

Furthermore, for choosing t "sufficiently large" in step 3 we obtain the fact that the labels of the caterpillar/hedgehog G' being constructed in the following steps 4/5 generate no "undesired" edges in the difference graph.

Hence the algorithm provides a difference labelling of the cactus $G = (V, E)$. ■

Simple examples show that it is impossible to use the concept of (x, t)-labelling in the same way for cacti with short cycles (i.e., with k-hedgehogs with $k \in \{3, 4, 5\}$) in many cases. E.g., the only difference labelling of a cycle of a length of 3, 4 and 5 uses the label set $\{x, 2x, 3x\}$, $\{x, 2x, 4x, 5x\}$ and $\{x, 2x, 4x, 8x, 9x\}$ (with arbitrary $x \in \mathbb{N}^+$), respectively. On the other hand, it seems to be difficult to label such short hedgehogs immediately when they are reached in the labelling procedure ("in passing"). Trying this a vast number of cases has to be considered.

But no example is known in which a cactus (with short cycles) is not a difference graph. Thus the question arises whether or not all cacti have a difference labelling.

Conjecture. Cacti are difference graphs.

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