Discussiones Mathematicae
Graph Theory 23 (2003) 37–54

DECOMPOSITION OF COMPLETE GRAPHS INTO FACTORS OF DIAMETER TWO AND THREE

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Abstract

We analyze a minimum number of vertices of a complete graph that can be decomposed into one factor of diameter 2 and k factors of diameter at most 3. We find exact values for $k \leq 4$ and the asymptotic value of the ratio of this number and k when k tends to infinity. We also find the asymptotic value of the ratio of the number of vertices of the smallest complete graph that can be decomposed into p factors of diameter 2 and k factors of diameter 3 and number k when p is fixed.

Keywords: decomposition, graph.

2000 Mathematics Subject Classification: 05C70.

1. Introduction

Decompositions of graphs into factors with given diameters have been extensively studied for many years, cf. [3, 4, 5, 6, 8]. The problem of decomposition of the factors of equal diameters d, $d \geqslant 3$, has been solved in [4]. Several papers are devoted to the decomposition of a complete graph into factors of diameter 2 [6, 7, 8]. Denote by $f(k)$ the smallest natural number n such that a complete graph on n vertices can be decomposed into k factors of diameter 2. In [6] it is proved that

$$f(k) \leq 7k.$$

In [2] this is improved to

$$f(k) \leq 6k.$$

In [7], it is proved that this upper bound is quite close to the exact value of $f(k)$ since,

$$f(k) \geq 6k - 7, \; k \geq 664$$

and in [8] the correct value of $f(k)$ is given for large values of k, namely

$$f(k) = 6k, \quad k \geq 10^{17}.$$

In this paper we asymptotically solve the problem of decomposition of a complete graph into factors of diameters two and three.

Also, decompositions into small number of factors have been extensively studied. Specially, the case of decomposition of a complete graph into two factors with given diameters is solved completely in [3] and for the case of decomposition of a complete graph into three factors with given diameters is partially solved in [5]. Therefore, we shall pay some more attention to decompositions into small number of factors.

2. Definitions and Preliminaries

By a factor of graph G we mean a subgraph of G containing all the vertices of G. Two or more factors are called disjoint if every edge of G belongs to at most one of them. A set of pairwise disjoint factors such that their union is a complete graph is called a decomposition. The symbol K_n denotes the complete graph on n vertices, $d_G(x)$ — degree of a vertex x in G, the symbol $\Delta(G)$ — the maximum degree of G, the symbol $\delta(G)$ — the minimum degree of G, $e(G)$ — the number of the edges of G and $V(G)$ — the set of vertices of G. The distance of vertices x and y in a G is denoted by $d_G(x, y)$. We define the function $f : \cup_{k\in\mathbb{N}}\mathbb{N}^k \rightarrow \mathbb{N}$ with

$$f(d_1, \ldots, d_k) = \min\{n : \text{there is a decomposition of } K_n \text{ into } k \text{ factors such that the diameter of the } i\text{-th factor is } d_i\}.$$

The following theorem can be found in [1].

Theorem 1. *If $m \geq f(d_1, d_2, \ldots, d_k) \geq 2$, then K_m can be decomposed into k factors such that the diameter of the i-th factor is d_i.*

We also define the function $\phi : \mathbb{N} \to \mathbb{N}$ with

$$\phi(k) = \min\{n : \text{there is a decomposition of } K_n \text{ into } k+1 \text{ factors, one of diameter 2 and others of diameter 3}\}.$$

The following simple lemma will be useful in the sequel.

Lemma 2. *If in a decomposition of K_n, $n \in \mathbb{N}$, at least one of the factors has diameter* 2, *then all the factors of diameter* 3 *must have at least* n *edges.*

Proof. Suppose to the contrary, that there is a factor F of diameter three which is a tree and denote the factor of diameter two by F'. Distinguish two cases.

(1) Suppose that the length of the longest path in F is more than 3. Then there are two vertices connected in F by two different paths. Since F is a tree, this is impossible.

(2) Suppose that the longest path in F has length 3. Denote, the vertices of arbitrary path of length three, in order of their appearance, by a, b, c, d.

Let us prove that each of the vertices $V(K_n)$ is adjacent to either b or c. Suppose oppositely that there is a vertex $x \in V(K_n) \setminus \{a, b, c, d\}$ which is not adjacent to either of vertices b and c. Since the longest path in F has length 3 and F does not contain a cycle, it follows that b is the only neighbor of a and that c is the only neighbor of d. It follows that there is a path of length at most 2 from x to b and from x to c. Note that $\{b, c\}$ is not an edge of any of these two paths and that b and c have no common neighbors. But, then this two paths together with the edge $\{b, c\}$ form a cycle, a contradiction.

Therefore, each vertex from $V(K_n) \setminus \{a, b, c, d\}$ is adjacent to either b or c, but then b and c have no common neighbors in F' and they are not adjacent in F'. This is in contradiction with the fact that $\mathrm{diam}(F') = 2$, so our claim is proved. ■

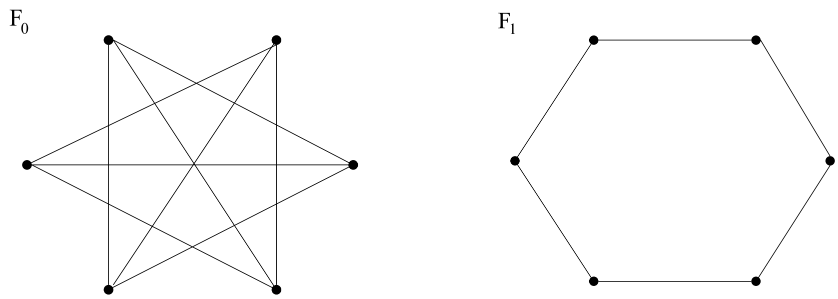
3. Small Values of k

Though the value of $\phi(1)$ follows from [3], for the sake of completeness we state

Proposition 3. $\phi(1) = 6$.

Proof. First, we prove that $\phi(1) \geq 6$. Suppose $\phi(1) \leq 5$. Then we can decompose K_5 into two factors, one F_1 of diameter two and the other F_2 of diameter three. Note that F_2 has to have at least 5 edges, but then F_1 can have at most 5 edges. Also, note that $\delta(F_2) \geqslant 1$, so $\delta(F_1) \leq 3$. The only graph with 5 vertices and at most 5 edges such that its maximum degree is less then 4 and its diameter is 2 is a cycle, but then F_2 is also a cycle with 5 vertices and is not of diameter 3.

The following sketch proves $\phi(1) \leq 6$.



$$\mathrm{diam}(F_0) = 2,\ \mathrm{diam}(F_1) = 3$$

So, the claim is proved. ■

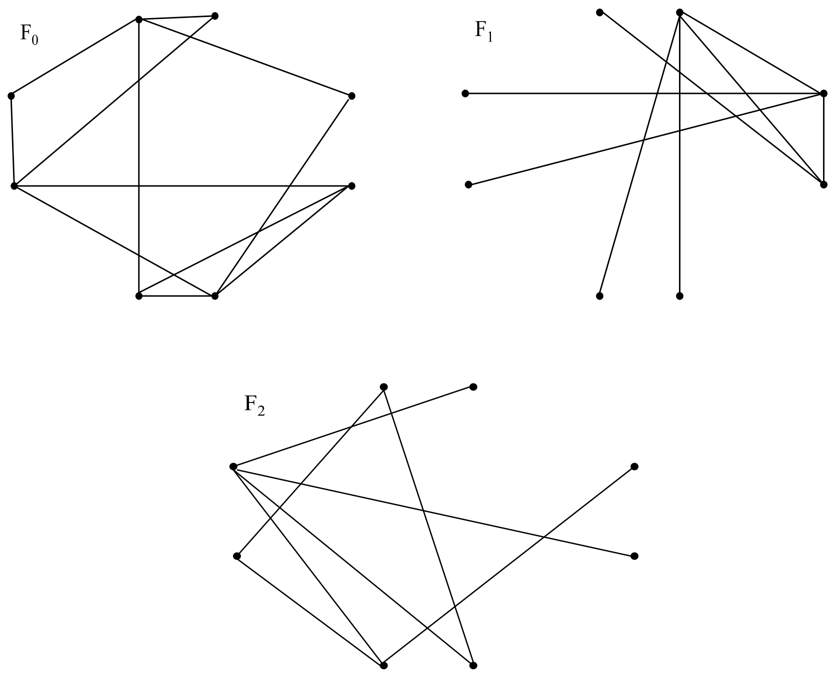
Proposition 4. $\phi(2) = 8$.

Proof. First, we prove that $\phi(2) \geq 8$. Suppose that $\phi(2) < 8$. Than we can decompose K_7 into three factors, one F_1 of diameter two and others of diameter three. By Lemma 2, factors of diameter three have to have at least 7 edges, so $e(F_1) \leq 21 - 2 \cdot 7 = 7$. Each vertex has at least one incident edge in each factor of diameter three, so $\Delta(F_1) \leq 4$. We distinguish two cases.

(1) If each vertex has degree two in F_1, then F_1 is either disconnected or is a cycle of length 7 which is a contradiction.

(2) If there is a vertex x, such that $3 \leq d_{F_1}(x) \leq 4$, then denote by F_1' a graph obtained by deleting this vertex. Let y be an arbitrary vertex of F_1 which is not adjacent to x. Vertex y has to be connected in F_1' to each vertex of F_1' by a path of length at most 2 (otherwise the diameter of F_1 would be greater than 2), so F_1' is connected. But, this is in contradiction to the fact that F_1' has 6 vertices and at most 4 edges.

The following sketch proves that $\phi(2) \leq 8$.



$$\operatorname{diam}(F_0) = 2;\ \operatorname{diam}(F_1) = 3,\ \operatorname{diam}(F_2) = 3$$

So, $\phi(2) = 8$. ■

Proposition 5. $\phi(3) = 10$.

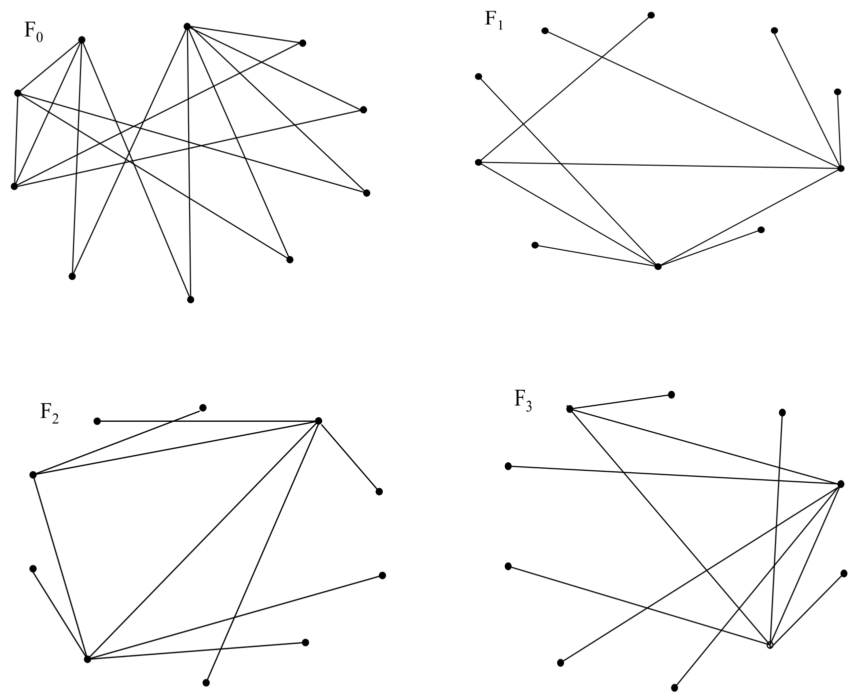
Proof. First, we prove that $\phi(3) \geq 10$. Analogously, as above, suppose that we can decompose K_9 into four factors, one F_1 of diameter two and others of diameter three. By Lemma 2, factors of diameter three have to have at least 9 edges, so $e(F_1) \leq 36 - 3 \cdot 9 = 9$. Each vertex has at least one incident edge in each factor of diameter three, so $\Delta(F_1) \leq 5$. We distinguish two cases.

(1) If each vertex has degree two in F_1, then F_1 is either disconnected or is a cycle of length 9, a contradiction.

(2) If there is a vertex x, such that $3 \leq d_{F_1}(x) \leq 5$, then denote by F_1' a graph obtained by eliminating this vertex. Let y be an arbitrary vertex of F_1 which is not adjacent to x. Vertex y has to be connected in F_1' to each vertex of F_1' by a path of length at most 2 (otherwise the diameter of F_1

would be greater than 2), so F_1' is connected. But, this is in contradiction to the fact that F_1' has 8 vertices and at most 6 edges.

The following sketch proves that $\phi(3) \leq 10$.



$$\operatorname{diam}(F_0) = 2,\ \operatorname{diam}(F_1) = 3,\ \operatorname{diam}(F_2) = 3,\ \operatorname{diam}(F_3) = 3$$

So, the claim is proved. ■

4. The Main Results

First, we give an upper bound for the function ϕ.

Theorem 6. For any $k \in \mathbb{N}$, we have $\phi(k) \leq 2k + 3\lceil\sqrt{k}\rceil + 2t$ where t is the least natural number such that

$$\binom{2t-1}{t-1} \geq k.$$

Proof. We will construct a decomposition of K_n, $n = 2k + 3\lceil\sqrt{k}\rceil + 2t$, in factors $F_0, F_1, F_2, \ldots, F_k$ such that $\mathrm{diam}(F_0) = 2$ and $\mathrm{diam}(F_i) = 3$, $1 \le i \le k$. Let

$$V(K_n) = L \cup D \cup W \cup Z \cup U \cup A \cup B,$$

where

$$L = \{l_1, \ldots, l_k\}, \; D = \{d_1, \ldots, d_k\}, \; W = \left\{w_0, \ldots, w_{\lceil\sqrt{k}\rceil - 1}\right\},$$
$$Z = \left\{z_0, \ldots, z_{\lceil\sqrt{k}\rceil - 1}\right\}, U = \left\{u_1, \ldots, u_{\lceil\sqrt{k}\rceil}\right\}, A = \{a\}, B = \{b_1, \ldots, b_{2t-1}\}.$$

Let $\mathcal{B}$ be the set of all $t-1$ element subsets of the set $\{1, 2, \ldots, 2t-1\}$. Let f be any injection

$$f : \{1, \ldots, k\} \to \mathcal{B}.$$

Let us notice that for each $j \in \{1, \ldots, kt\}$ there are unique numbers q_j and r_j such that

$$j = q_j \cdot \left\lceil\sqrt{k}\right\rceil + r_j, \; 0 \le q_j \le \left\lceil\sqrt{k}\right\rceil - 1, \; 1 \le r_j \le \left\lceil\sqrt{k}\right\rceil.$$

The edges of the factor F_i, $1 \le i \le k$ are

(1) $l_i d_i$,
(2) $l_i l_j, \; 1 \le j < i \le k$,
(3) $d_i l_j, \; 1 \le j < j \le k$,
(4) $d_i d_j, \; 1 \le j < i \le k$,
(5) $l_i d_j, \; 1 \le i < j \le k$,
(6) $l_i a$,
(7) $l_i b_j, \; j \in f(i)$,
(8) $d_i b_j, \; j \in \{1, 2, \ldots, 2t-1\} \setminus f(i)$,
(9) $l_i w_j, \; 1 \le j \le \lceil\sqrt{k}\rceil - 1$,
(10) $d_i z_j, \; 1 \le j \le \lceil\sqrt{k}\rceil - 1$,
(11) $w_{q_i} u_{r_i}$,
(12) $z_{q_i} u_{r_i}$,
(13) $d_i u_j, \; 1 \le j \le k, \; j \ne r_i$.

The other edges are edges of the factor F_0. In each factor F_i, $1 \le i \le k$ all vertices are adjacent to either l_i or d_i, except u_{r_i} which is connected by a path of length 2 to both, l_i and d_i, and also l_i and d_i are adjacent, so we have $\mathrm{diam}(F_i) \le 3$, $1 \le i \le k$. Now, let us prove that $\mathrm{diam}(F_i) \geqslant 3$, $1 \le i \le k$. Let i be an arbitrary number such that $1 \le i \le k$. Let j be an element of the set $\{1, 2, \ldots, 2t-1\} \setminus f(i)$. Note that $d_{F_i}(a, b_j) = 3$, so the claim is proved.

It remains to prove that $\operatorname{diam}(F_0) = 2$. We have to prove that every two vertices of F_0 are adjacent or that they have a common neighbor. We distinguish five cases.

(1) $x \notin L$, $y \notin L$. Then $a \in N_{F_0}(x) \cap N_{F_0}(y)$.

(2) $x, y \in L$. Since

$$|N_{F_0}(x) \cap B| + |N_{F_0}(y) \cap B| = t + t > |B|,$$

by pigeonhole principle we have $b \in B$ such that $b \in N_{F_0}(x) \cap N_{F_0}(y)$.

(3) $x \in L, y \in D$. We distinguish two subcases.

(3a) $x = l_i, y = d_i, 1 \leq i \leq k$. Then $u_{r_i} \in N_{F_0}(l_i) \cap N_{F_0}(d_i)$.

(3b) $x = l_i,\ y = d_j,\ 1 \leq i, j \leq k, i \neq j$. We have

$$|N_{F_0}(l_i) \cap B| + |N_{F_0}(d_j) \cap B| = t - 1 + t = |B|,$$

so either there is a vertex $b \in N_{F_0}(l_i) \cap N_{F_0}(d_j)$ or

$$N_{F_0}(l_i) \cap B = B \setminus N_{F_0}(d_j) = N_{F_0}(l_j) \cap B$$

which is impossible.

(4) $x \in L$, $y \in U \cup Z$. Then x and y are adjacent.

(5) $x \in L$, $y \in W \cup A \cup B$. Then $(\forall z \in Z)(z \in N_{F_0}(x) \cap N_{F_0}(y))$.

So, the claim is proved. ■

From the last theorem, it easily follows

Corollary 7. $\lim_{k\to\infty} \frac{\phi(k)}{k} = 2$.

Proof. Let $k \in \mathbb{N}$ be sufficiently large. Let us find upper and lower bounds for $\phi(k)$.

$$k \cdot (\phi(k) - 1) \leq \binom{\phi(k)}{2} \Rightarrow k \leq \frac{\phi(k)}{2} \Rightarrow \phi(k) \geq 2k.$$

Let us notice that, for sufficiently large k, we have

$$\binom{2\lceil\sqrt{k}\rceil - 1}{\lceil\sqrt{k}\rceil - 1} \geq k,$$

so

$$2k \leq \phi(k) \leq 2k + 5\left(\sqrt{k}+1\right) \Rightarrow 2 \leq \frac{\phi(k)}{k} \leq 2 + \frac{5}{\sqrt{k}} + \frac{5}{k}.$$

$$\Rightarrow \quad 2 \leq \lim_{k\to\infty}\left(\frac{\phi(k)}{k}\right) \leq \lim_{k\to\infty}\left(2 + \frac{5}{\sqrt{k}} + \frac{5}{k}\right).$$

which proves the claim. ■

Now, we give an auxiliary result.

Lemma 8. *Let $k \geq 4$. Then there is a decomposition of K_k into factors F_1' and F_2' such that $\delta(F_1') \geq 1$ and $\delta(F_2') \geq 1$.*

Proof. We prove our claim by induction on k. We denote $W(K_k) = \{1,\ldots,k\}$. For $k = 4$, the claim is trivial. Suppose it is true for j and let us prove it for $j+1$. We decompose the graph induced by vertices $\{1,\ldots,j\}$ as K_j and add to F_1' the edge $\{1, j+1\}$ and add to F_2' the edges $\{i, j+1\}$, $2 \leq i \leq k$. This decomposition proves the lemma. ■

Theroem 9. *Let $k \geq 4$. Then we have $\phi(k) \leq 3k+1$.*

Proof. We shall construct the decomposition of K_n, $n = 3k+1$, into factors $F_0, F_1, F_2, \ldots, F_k$ such that $\mathrm{diam}(F_0) = 2$ and $\mathrm{diam}(F_i) = 3$, $1 \leq i \leq k$. We denote

$$V(K_n) = \{x, y_{ij} : 1 \leq i \leq k,\ 1 \leq j \leq 3\}.$$

Let F_1' and F_2' be the factors of K_k described in previous Lemma. The edges of the factor $F_i, 1 \leq i \leq k$ are

(1) $\{v_{i3}, xt\}$,

(2) $\{v_{i1}, v_{i2}\}$, $\{v_{i2}, v_{i3}\}$, $\{v_{i3}, v_{i1}\}$,

(3) $\{v_{i2}, v_{j2}\}$,, $\{v_{i2}, v_{j3}\}$, $\{v_{i1}, v_{j1}\}$, $1 \leq j < i$, $\{i,j\} \in F_1'$,

(4) $\{v_{i2}, v_{j1}\}$, $\{v_{i2}, v_{j3}\}$, $\{v_{i1}, v_{j2}\}$, $i < j \leq k$, $\{i,j\} \in F_1'$,

(5) $\{v_{i1}, v_{j1}\}$, $\{v_{i1}, v_{j3}\}$, $\{v_{i2}, v_{j2}\}$, $1 \leq j < i$, $\{i,j\} \in F_2'$,

(6) $\{v_{i1}, v_{j2}\}$, $\{v_{i1}, v_{j3}\}$, $\{v_{i2}, v_{j1}\}$, $i < j \leq k$, $\{i,j\} \in F_2'$.

The other edges are edges of the factor F_0. Indeed, $\operatorname{diam}(F_i) = 3$, $1 \le i \le k$, because all its vertices are adjacent to at least one of vertices v_{i1}, v_{i2} and v_{i3}, and these three vertices form a triangle.

It remains to prove that $\operatorname{diam}(F_0) = 2$. We have to prove that each two vertices of F_0 are adjacent or that they have a common neighbor. We distinguish eight cases.

(1) $p = x, q = v_{ij}, 1 \le i \le k, 1 \le j \le 2$. Then x and v_{ij} are adjacent.

(2) $p = x, q = v_{i3}, 1 \le i \le k$. Let us choose j, $j \ne i, 1 \le j \le k$, such that $\{i, j\} \in F_1'$. We have $v_{j1} \in N_{F_0}(x) \cap N_{F_0}(v_{i3})$.

(3) $p = v_{ij},\ q = v_{ab},\ 1 \le i, a \le k,\ 1 \le j, b \le 2$. Then $x \in N_{F_0}(v_{ij}) \cap N_{F_0}(v_{ab})$.

(4) $p = v_{i3}, q = v_{j3},\ 1 \le i, j \le k,\ i \ne j$. Then v_{i3} and v_{j3} are adjacent.

(5) $p = v_{i3},\ q = v_{j1},\ 1 \le i, j \le k,\ \{i, j\} \in F_1'$. Then v_{i3} and v_{j1} are adjacent.

(6) $p = v_{i3},\ q = v_{j1},\ 1 \le i, j \le k,\ \{i, j\} \notin F_1'$. Let us choose m, $m \ne i, m \ne j, 1 \le m \le k$, such that $\{m, j\} \in F_1'$. We have $v_{m3} \in N_{F_0}(v_{i3}) \cap N_{F_0}(v_{j1})$.

(7) $p = v_{i3},\ q = v_{j2},\ 1 \le i, j \le k,\ \{i, j\} \in F_2'$. Then v_{i3} and v_{j2} are adjacent.

(8) $p = v_{i3},\ q = v_{j2},\ 1 \le i, j \le k,\ \{i, j\} \notin F_2'$. Then let us choose m, $m \ne i, m \ne j, 1 \le m \le k$, such that $\{m, j\} \in F_2'$. We have $v_{m3} \in N_{F_0}(v_{i3}) \cap N_{F_0}(v_{j2})$.

So, the claim is proved. ■

Denote by $\mathcal{H}_d'(n, k)$ the set of all graphs with n vertices and with maximal degree at most k and diameter at most d. Put

$$e_d'(n, k) = \min\left\{e(G) : G \in \mathcal{H}_d'(n, k)\right\}.$$

In the proof of Theorem IV. 1.2 in [1], the following statement is proved:

Lemma A. $e_d'(n, n-4) \geqslant 2n - 5$, *if* $n \le 12$.

Corollary 10. $\phi(4) = 13$.

Proof. By the previous Theorem $\phi(4) \le 13$. It remains to prove $\phi(4) \ge 13$. On the contrary, suppose that K_{12} can be decomposed into one factor F_1 of diameter 2 and four factors of diameter 3. From Lemma A it follows that

$e(F_1) \geq 2\cdot 12-5 = 19$. From Lemma 2 it follows that the factors of diameter three have at least 12 edges each, so we have

$$66 = e(K_{12}) \geq 19 + 4 \cdot 12 = 67,$$

which is a contradiction, so our claim is proved. ■

As our last main result, we are going to generalize Corollary 7. First, we give a lemma.

Lemma 11. *There is a function $q : \mathbb{N} \to \mathbb{N}$ such that, for each $p \in \mathbb{N}$, a complete graph $K_{p\cdot q(p)}$ with a set of vertices $\{e_i^\alpha : 1 \leq i \leq q(p), 1 \leq \alpha \leq p\}$ can be decomposed into factors $E_1, E_2, \ldots, E_p$ such that:*

(1) *$e_i^\alpha e_j^\alpha$ is an edge of E_α, $1 \leq i < j \leq q(p)$, $1 \leq \alpha \leq p$,*

(2) $\mathrm{diam}(E_\alpha) \leq 2$, $1 \leq \alpha \leq p$,

(3) $(\forall \alpha, \beta \in \{1, \ldots, p\}, \alpha \neq \beta)(\forall i \in \{1, \ldots, q(p)\})(\exists j \in \{1, \ldots, q(p)\})$ *$(e_i^\alpha e_j^\beta$ is an edge of $E_\beta)$.*

Proof. Let $E'_1, E'_2, \ldots, E'_p$ be a decomposition of a graph $K_{p\cdot q(p)}$, such that:

(a) $e_i^\alpha e_j^\alpha$ is an edge of E'_α, $1 \leq i < j \leq q(p)$, $1 \leq \alpha \leq p$.

(b) The probability that $e_i^\alpha e_j^\beta$, $1 \leq i, j \leq q(p)$, $1 \leq \alpha < \beta \leq p$ is an edge of E'_α is $\frac{1}{2}$ and the probability that it is an edge of E'_β is also $\frac{1}{2}$.

Let us estimate a probability $\mathrm{prob}(\gamma, e_i^\alpha, e_j^\beta)$ that $d_{E'_\gamma}(e_i^\alpha, e_j^\beta) > 2$ for $1 \leq \alpha, \beta, \gamma \leq p$, $1 \leq i, j \leq q(p), e_i^\alpha \neq e_j^\beta$. Distinguish four cases.

(1) $\gamma = \alpha = \beta$. $\mathrm{prob}(\gamma, e_i^\alpha, e_j^\beta) = 0$, because $e_i^\alpha e_j^\alpha$ is an edge of E'_α.

(2) $\gamma = \alpha \neq \beta$. $\mathrm{prob}(\gamma, e_i^\alpha, e_j^\beta)$ is less or equal to the probability that e_j^β is not adjacent to any e_k^α in $E'_\alpha, 1 \leq k \leq q(p)$, so $\mathrm{prob}(\gamma, e_i^\alpha, e_j^\beta) \leq (\frac{1}{2})^{q(p)}$.

(3) $\gamma = \beta \neq \alpha$. Similarly as above $\mathrm{prob}(\gamma, e_i^\alpha, e_j^\beta) \leq (\frac{1}{2})^{q(p)}$.

(4) $\gamma \neq \alpha, \gamma \neq \beta$. Probability that $e_\gamma^k \notin N_{E'_\gamma}(e_i^\alpha) \cap N_{E'_\gamma}(e_j^\beta)$ is $\frac{3}{4}$ for each fixed $k = 1, \ldots, q(p)$, so $\mathrm{prob}(\gamma, e_i^\alpha, e_j^\beta) \leq (\frac{3}{4})^{q(p)}$.

For the sake of simplicity we also define $\mathrm{prob}(\gamma, e_i^\alpha, e_i^\alpha) = 0$. In any case, $\mathrm{prob}(\gamma, e_i^\alpha, e_j^\beta) \leq (\frac{3}{4})^{q(p)}$. Let us find a probability $\mathrm{prob}(\beta, e_i^\alpha)$ that for e_i^α, $1 \leq i \leq q(p)$, $1 \leq \alpha \leq p$ and $\beta \neq \alpha$, $1 \leq \beta \leq p$ there is no j, $1 \leq j \leq q(p)$

such that $e_i^\alpha e_j^\beta$ is an edge of E'_β. The probability that $e_i^\alpha e_j^\beta$ is not an edge of E'_β for a fixed j, $1 \le j \le q(p)$ is $\frac{1}{2}$, so $\mathrm{prob}(\beta, e_i^\alpha) \le (\frac{1}{2})^{q(p)}$.

Now, we can find a lower bound for the probability $X^p_{q(p)}$ that the random decomposition $E'_1, E'_2 \ldots, E'_p$ of $K_{p \cdot q(p)}$, described above, has properties required in Lemma. It holds that

$$\begin{aligned} X^p_{q(p)} &\ge 1 - \left(\sum_{\substack{1 \le i \le q(p) \\ 1 \le \alpha,\beta \le p \\ \alpha \ne \beta}} \mathrm{prob}\,(\beta, e_i^\alpha) + \sum_{\substack{1 \le i,j \le q(p) \\ 1 \le \alpha,\beta,\gamma \le p}} \mathrm{prob}\left(\beta, e_i^\alpha, e_j^\beta\right) \right) \\ &\ge 1 - \left(q\,(p) \cdot p^2 \cdot \left(\frac{1}{2}\right)^{q(p)} + p^3 \cdot (q\,(p))^2 \left(\frac{3}{4}\right)^{q(p)} \right). \end{aligned}$$

Since

$$\lim_{q(p)\to\infty} \left(1 - \left(q\,(p) \cdot p^2 \cdot \left(\frac{1}{2}\right)^{q(p)} + p^2 \cdot (q\,(p))^2 \left(\frac{3}{4}\right)^{q(p)} \right) \right) = 1 > 0,$$

for any p and sufficiently large $q(p)$ we have

$$X^p_{q(p)} > 0,$$

so there is a decomposition $E_1, \ldots, E_p$ with the required properties. ∎

Theorem 12. $\lim_{k\to\infty} \frac{f(\underbrace{2,2,\ldots,2}_{p\text{-times}},\underbrace{3,3\ldots,3}_{k\text{-times}})}{k} = 2$, *where p is a fixed natural number.*

Proof. Analogously, as in the proof of Corollary 7, we have

$$f\left(\underbrace{2,2,\ldots,2}_{p\text{-times}},\underbrace{3,3,\ldots,3}_{k\text{-times}}\right) \ge 2k. \tag{1}$$

Now, we are going to prove that for sufficiently large k,

$$(2)\ f\left(\underbrace{2,2,\ldots,2}_{p\text{-times}},\underbrace{3,3,\ldots,3}_{k\text{-times}}\right) \leq 2k+5p\cdot\left\lceil\sqrt{k}\right\rceil+\binom{p}{2}\left\lceil\sqrt{k}\right\rceil+2\cdot p\cdot q\left(p\right),$$

where q is the function from the previous Lemma.

Denote $n = 2k + 5p \cdot \lceil\sqrt{k}\rceil + \binom{p}{2}\lceil\sqrt{k}\rceil + 2 \cdot p \cdot q(p)$. Let $E_1, E_2, \ldots, E_p$ be a decomposition of $K_{p\cdot q(p)}$ from Lemma 11. We describe a decomposition of K_n into factors $F_1, F_2, \ldots, F_p$ of diameter 2 and factors $G_1, G_2, \ldots, G_k$ of diameter 3. Let

$$V\left(K_n\right) = L \cup D \cup \bigcup_{\alpha=1}^{p}\left(W_\alpha \cup Z_\alpha \cup U_\alpha \cup A_\alpha \cup B_\alpha \cup C_\alpha\right) \cup \bigcup_{1\leq\alpha<\beta\leq p} S_{\alpha\beta},$$

where

$$\begin{aligned}
L &= \left\{l_1,\ldots,l_k\right\},\\
D &= \left\{d_1,\ldots,d_k\right\},\\
W_\alpha &= \left\{w_0^\alpha,\ldots,w^\alpha_{\lceil\sqrt{k}\rceil-1}\right\}, 1\leq\alpha\leq p,\\
Z_\alpha &= \left\{z_0^\alpha,\ldots,z^\alpha_{\lceil\sqrt{k}\rceil-1}\right\}, 1\leq\alpha\leq p,\\
U_\alpha &= \left\{u_1^\alpha,\ldots,u^\alpha_{\lceil\sqrt{k}\rceil}\right\}, 1\leq\alpha\leq p,\\
A_\alpha &= \left\{a_1^\alpha,\ldots,a^\alpha_{q(p)}\right\},\ 1\leq\alpha\leq p,\\
B_\alpha &= \left\{b_1^\alpha,\ldots,b^\alpha_{2\lceil\sqrt{k}\rceil}\right\}, 1\leq\alpha\leq p,\\
C_\alpha &= \left\{c_1^\alpha,c_2^\alpha,\ldots,c^\alpha_{q(p)}\right\}, 1\leq\alpha\leq p,\\
S_{\alpha\beta} &= \left\{s_1^{\alpha\beta},\ldots,s^{\alpha\beta}_{\lceil\sqrt{k}\rceil}\right\},\ 1\leq\alpha<\beta\leq p.
\end{aligned}$$

Let $\mathcal{B}$ be the set of all $\lceil\sqrt{k}\rceil$ element subsets of the set $\{1, 2, \ldots, 2\lceil\sqrt{k}\rceil\}$. Let f be any injection

$$f : \{1, \ldots, k\} \to \mathcal{B}.$$

f exists, because

$$\binom{2 \cdot \lceil \sqrt{k} \rceil}{\lceil \sqrt{k} \rceil} \geq k$$

for a sufficiently large k. Let us notice that for each $j \in \{1, \ldots, k\}$ there are unique numbers q_j and r_j such that

$$j = q_j \cdot \left\lceil \sqrt{k} \right\rceil + r_j,\ 0 \leq q_j \leq \left\lceil \sqrt{k} \right\rceil - 1,\ 1 \leq r_j \leq \left\lceil \sqrt{k} \right\rceil.$$

The edges of a factor $G_i, 1 \leq i \leq k$ are

(1) $l_i d_i$,
(2) $l_i l_j,\ 1 \leq j < i \leq k$,
(3) $d_i l_j,\ 1 \leq i < j \leq k$,
(4) $d_i d_j,\ 1 \leq j < i \leq k$,
(5) $l_i d_j,\ 1 \leq i < j \leq k$,
(6) $l_i a_j^\alpha,\ 1 \leq \alpha \leq p,\ 1 \leq j \leq q(p)$,
(7) $l_i b_j^\alpha,\ j \in f(i),\ 1 \leq \alpha \leq p$,
(8) $d_i b_j^\alpha,\ j \in \{1, 2, \ldots, 2\lceil k \rceil\} \setminus f(i),\ 1 \leq \alpha \leq p$,
(9) $d_i c_j^\alpha,\ 1 \leq \alpha \leq p,\ 1 \leq j \leq q(p)$,
(10) $l_i w_j^\alpha,\ 0 \leq j \leq \lceil \sqrt{k} \rceil - 1,\ 1 \leq \alpha \leq p$,
(11) $d_i z_j^\alpha,\ 0 \leq j \leq \lceil \sqrt{k} \rceil - 1,\ 1 \leq \alpha \leq p$,
(12) $w_{q_i}^\alpha u_{r_i}^\alpha,\ 1 \leq \alpha \leq p$,
(13) $z_{q_i}^\alpha u_{r_i}^\alpha,\ 1 \leq \alpha \leq p$,
(14) $d_i u_j^\alpha,\ 1 \leq j \leq k,\ j \neq r_i,\ 1 \leq \alpha \leq p$,
(15) $s_{q_i}^{\alpha\beta} u_{r_i}^\alpha,\ 1 \leq \alpha < \beta \leq p$,
(16) $s_{q_i}^{\alpha\beta} u_{r_i}^\beta,\ 1 \leq \alpha < \beta \leq p$,
(17) $l_i s_j^{\alpha\beta}, 1 \leq \alpha < \beta \leq p,\ 1 \leq j \leq \lceil \sqrt{k} \rceil$.

The edges of a factor $F_\alpha, 1 \leq \alpha \leq p$ are

(1) xy such that $x, y \in L \cup D \cup W_\alpha \cup Z_\alpha \cup U_\alpha \cup A_\alpha \cup B_\alpha \cup C_\alpha$ and xy is not an edge of any graph $G_i, 1 \leq i \leq k$.
(2) xy such that $x \in A_\alpha \cup C_\alpha$ and
$y \in \bigcup_{\substack{1 \leq \beta \leq p \\ \beta \neq \alpha}} (W_\beta \cup Z_\beta \cup U_\beta \cup B_\beta) \cup \bigcup_{1 \leq \beta < \gamma \leq p} S_{\beta\gamma}$.

(3) $a_i^\alpha c_j^\beta$, so that $e_i^\alpha e_j^\beta \in E_\alpha,\ 1 \le i,j \le q(p),\ 1 \le \beta \le p$.

(4) $a_j^\alpha c_i^\beta$, so that $e_i^\alpha e_j^\beta \in E_\alpha,\ 1 \le i,j \le q(p),\ 1 \le \beta \le p$.

(5) $a_i^\alpha a_j^\beta$, so that $e_i^\alpha e_j^\beta \in E_\alpha,\ 1 \le i,j \le q(p),\ 1 \le \beta \le p$.

(6) $c_i^\alpha c_j^\beta$, so that $e_i^\alpha e_j^\beta \in E_\alpha,\ 1 \le i,j \le q(p),\ 1 \le \beta \le p$.

Now, we shall prove that the diameter of $G_i,\ 1 \le i \le k$, is 3. First, we prove that for each $x, y \in G_i$ is $d_{G_i}(x,y) \le 3$. Distinguish 4 cases.

(1) $x, y \in \{l_i, d_i\} \cup N_{G_i}(l_i) \cup N_{G_i}(d_i)$.

(2) $x = \{u_{r_i}^\alpha : 1 \le \alpha \le p\},\ y \in N_{G_i}(l_i) \cup \{l_i\}$.

(3) $x = \{u_{r_i}^\alpha : 1 \le \alpha \le p\},\ y \in N_{G_i}(d_i) \cup \{d_i\}$.

(4) $x, y \in \{u_{r_i}^\alpha : 1 \le \alpha \le p\}$.

In each case a simple analysis shows that there is a path of length ≤ 3.

Let us prove that the diameter of $G_i,\ 1 \le i \le k$, is $\geqslant 3$. Let j be an arbitrary number such that $\{1, 2, \ldots, 2\lceil k \rceil\} \setminus f(i)$. Then $d_{G_i}(a_1^1, b_j^1) = 3$.

It remains to prove that the diameter of each $F_\alpha,\ 1 \le \alpha \le p$, is 2. So, we have to prove that each $x, y \in F_\alpha$ are adjacent or have a common neighbor. Distinguish eight cases.

$$x, y \in L \cup D \cup W_\alpha \cup Z_\alpha \cup U_\alpha \cup A_\alpha \cup B_\alpha \cup C_\alpha. \tag{1}$$

This case can be proved by complete analogy with the proof of Theorem 6.

$$\begin{gathered} x \in L \cup D \cup W_\alpha \cup Z_\alpha \cup U_\alpha \cup A_\alpha \cup B_\alpha \cup C_\alpha, \\ y \in \bigcup_{\substack{1 \le \beta \le p \\ \beta \ne \alpha}} (W_\beta \cup Z_\beta \cup U_\beta \cup B_\beta) \cup \bigcup_{1 \le \beta < \gamma \le p} S_{\beta\gamma}. \end{gathered} \tag{2}$$

We have $A_\alpha \cup C_\alpha \subseteq N_{F_\alpha}(y)$ and $N_{F_\alpha}(x) \cap (A_\alpha \cup C_\alpha) \ne \emptyset$, so $N_{F_\alpha}(x) \cap N_{F_\alpha}(y) \ne \emptyset$.

$$\begin{gathered} x \in L \cup D \cup W_\alpha \cup Z_\alpha \cup U_\alpha \cup A_\alpha \cup B_\alpha \cup C_\alpha, y = a_i^\beta, \\ 1 \le \beta \le p,\ 1 \le i \le q\left(p\right). \end{gathered} \tag{3}$$

There is an edge $e_i^\beta e_j^\alpha$ in E_α, for some $j,\ 1 \le j \le q(p)$, so $\{a_j^\alpha, c_j^\alpha\} \subseteq N_{F_\alpha}(y)$. Also we have $\{a_j^a, c_j^a\} \cap N_{F_\alpha}(x) \ne \emptyset$, so $N_{F_\alpha}(x) \cap N_{F_\alpha}(y) \ne \emptyset$.

(4)
$$x \in L \cup D \cup W_\alpha \cup Z_\alpha \cup U_\alpha \cup A_\alpha \cup B_\alpha \cup C_\alpha, y = c_i^\beta,$$
$$1 \le \beta \le p,\ 1 \le i \le q(p).$$

There is an edge $e_i^\beta e_j^\alpha$ in E_α, for some $j,\ 1 \le j \le q(p)$, so $\{a_j^\alpha, c_j^\alpha\} \subseteq N_{F_\alpha}(y)$. Also we have $\{a_j^\alpha, c_j^\alpha\} \cap N_{F_\alpha}(x) \neq \emptyset$, so $N_{F_\alpha}(x) \cap N_{F_\alpha}(y) \neq \emptyset$.

(5)
$$x, y \in \bigcup_{\substack{1 \le \beta \le p \\ \beta \neq \alpha}} (W_\beta \cup Z_\beta \cup U_\beta \cup B_\beta) \cup \bigcup_{1 \le \beta < \gamma \le p} S_{\alpha_\beta}.$$

We have $a_1^\alpha \in N_{F_\alpha}(x) \cap N_{F_\alpha}(y)$

(6)
$$\begin{aligned} x &\in \bigcup_{\substack{1 \le \beta \le p \\ \beta \neq \alpha}} (W_\beta \cup Z_\beta \cup U_\beta \cup B_\beta) \cup \bigcup_{1 \le \beta < \gamma \le p} S_{\alpha\beta}, \\ y &= a_i^\gamma, 1 \le \gamma \le p, \alpha \neq \gamma, 1 \le i \le q(p). \end{aligned}$$

There is an edge $e_i^\gamma e_j^\alpha$ in E_α, for some $j,\ 1 \le j \le q(p)$.
So $a_j^\alpha \in N_{F_\alpha}(x) \cap N_{F_\alpha}(y)$

(7)
$$\begin{aligned} x &\in \bigcup_{\substack{1 \le \beta \le p \\ \beta \neq \alpha}} (W_\beta \cup Z_\beta \cup U_\beta \cup B_\beta) \cup \bigcup_{1 \le \beta < \gamma \le p} S_{\alpha\beta}, \\ y &= c_i^\gamma, 1 \le \gamma \le p, \gamma \neq \alpha, 1 \le i \le q(p). \end{aligned}$$

There is an edge $e_i^\gamma e_j^\alpha$ in E_α, for some $j,\ 1 \le j \le q(p)$.
So $a_j^\alpha \in N_{F_\alpha}(x) \cap N_{F_\alpha}(y)$.

(8) $\quad x \in A_\beta \cup C_\beta, y \in A_\gamma \cup C_\gamma,\ 1 \le \beta, \gamma \le p,\ \alpha \neq \beta,\ \alpha \neq \gamma, x \neq y.$

We distinguish four subcases

(8a) $x = a_i^\beta,\ y = a_j^\gamma,$

(8b) $x = a_i^\beta,\ y = c_j^\gamma,$

(8c) $x = c_i^\beta,\ y = a_j^\gamma,$

(8d) $x = c_i^\beta,\ y = c_j^\gamma.$

As proofs of this subcases are completely analogous, we prove only (8a). Since $d(e_i^\beta, e_j^\gamma) \leq 2$, either e_i^β and e_j^γ are adjacent in E_α or there is a vertex $e_k^\alpha \in N_{E_\alpha}(e_i^\beta) \cap N_{E_\alpha}(e_j^\alpha)$. In the first case a_i^β and a_j^γ are adjacent in F_α, and in the second case $a_k^\alpha \in N_{F_\alpha}(a_i^\beta) \cap N_{F_\alpha}(a_j^\gamma)$.

So, the inequality (2) is proved.

From (1) and (2) we get

$$\begin{aligned} 2k &\leq f\left(\underbrace{2,2,\ldots,2}_{p\text{-times}},\underbrace{3,3,\ldots,3}_{k\text{-times}}\right) \\ &\leq 2k + 5p\cdot\left\lceil\sqrt{k}\right\rceil + \binom{p}{2}\left\lceil\sqrt{k}\right\rceil + 2\cdot p\cdot q\,(p)\,. \end{aligned}$$

$$\begin{aligned} 2k &\leq f\left(\underbrace{2,2,\ldots,2}_{p\text{-times}},\underbrace{3,3,\ldots,3}_{k\text{-times}}\right) \\ &\leq 2k + \left(5p+\binom{p}{2}\right)\sqrt{k} + \left(5p+\binom{p}{2}\right) + 2\cdot p\cdot q\,(p)\,. \end{aligned}$$

Dividing by k and passing to the limit, we get

$$\begin{aligned} 2 &\leq \lim_{k\to\infty} \frac{f\left(\underbrace{2,2,\ldots,2}_{p\text{-times}},\underbrace{3,3,\ldots,3}_{k\text{-times}}\right)}{k} \\ &\leq \lim_{k\to\infty} 2 + \frac{\left(5p+\binom{p}{2}\right)}{\sqrt{k}} + \frac{\left(5p+\binom{p}{2}\right)}{k} + \frac{2\cdot p\cdot q(p)}{k} \end{aligned}$$

which proves the theorem. ■

Acknowledgement

The author thanks D. Veljan for useful advise and help.

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Received 25 May 2001
Revised 5 September 2002