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2-PLACEMENT OF (p, q)-TREES

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Abstract

Let $G = (L, R; E)$ be a bipartite graph such that $V(G) = L \cup R$, $|L| = p$ and $|R| = q$. G is called (p, q)-*tree* if G is connected and $|E(G)| = p + q - 1$.

Let $G = (L, R; E)$ and $H = (L', R'; E')$ be two (p, q)-tree. A bijection $f : L \cup R \to L' \cup R'$ is said to be a biplacement of G and H if $f(L) = L'$ and $f(x)f(y) \notin E'$ for every edge xy of G. A biplacement of G and its copy is called 2-placement of G. A bipartite graph G is 2-placeable if G has a 2-placement. In this paper we give all (p, q)-trees which are not 2-placeable.

Keywords: tree, bipartite graph, packing graph.

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1. Definitions

We shall use standard graph theory notation. All graphs will be assumed to have neither loops nor multiple edges. Let $G = (L, R; E)$ be a bipartite graph with a vertex set $V(G) = L \cup R$, where $L \cap R = \varnothing$ $L(G) = L$, $R(G) = R$ are *left* and *right set of bipartition* of the vertex set, an edge set $E(G) = E$ and size $e(G)$. For a vertex $x \in V(G)$ by $N(x, G)$ and $d(x, G)$ we denote the set of its neighbors in G and the degree of the vertex x in G, respectively. $\Delta_L(G)$ and $\Delta_R(G)$ are the maximum vertex degree in the set $L(G)$ and $R(G)$, respectively. By P_n we denote the path of length $n - 1$. Bipartite graph $G = (L, R; E)$ is said (p, q)-*bipartite* if $|L| = p$ and $|R| = q$. $K_{p,q}$ is the complete (p, q)-bipartite graph. $\bar{G}$ is the complement of

G in $K_{p,q}$. A bipartite graph $G = (L, R; E)$ is a *subgraph* of bipartite graph $H = (L', R'; E')$ if $L \subseteq L'$, $R \subseteq R'$ and $E \subseteq E'$.

Let $G = (L, R; E)$ and $H = (L', R'; E')$ be two (p, q)-bipartite graphs. We say that G and H are *mutually placeable* (for short *mp*) if there is a bijection $f : L \cup R \to L' \cup R'$ such that $f(L) = L'$ and $f(x)f(y)$ is not edge in H whenever xy is an edge of G. The function f is called the *biplacement* of G and H. Thus G and H are *mp* if and only if G is contained in the graph $\bar{H}$, i.e., G is subgraph of $\bar{H}$. 2-*placement* of G is a biplacement of G and its copy. If such a 2-placement of G exists then we say that G is 2-placeable.

In the proof of the main theorem of this paper we use the *adjacency matrices* defined as follows.

Let $G = (L, R; E)$ be a (p, q)-bipartite graph, $L = \{x_1, \dots, x_p\}$ and $R = \{y_1, \dots, y_q\}$. The matrix $M_G = (a_{ij})_{\substack{i=1,\dots,p \\ j=1,\dots,q}}$ where:

$$a_{ij} = \begin{cases} 1, & x_i x_j \in E(G), \\ 0, & x_i x_j \notin E(G) \end{cases}$$

is called *adjacency matrix* of the graph G. Let G and H be mutually placeable (p, q)-bipartite graphs and let f be a biplacement of G and H. We may define the new $p \times q$ matrix $M_{G,H} = (b_{i,j})$ by the formula

$$b_{ij} = \begin{cases} 1, & \text{when } x_i x_j \in E(H), \\ 2, & \text{when } x_i x_j \in E(f(G)), \\ 0, & \text{when } x_i x_j \notin E(H) \text{ and } x_i x_j \notin E(f(G)). \end{cases}$$

The matrix $M_{G,H}$ is said to be *the matrix of biplacement of G and H.* Next, instead of looking for biplacement of G and H we shall look for a matrix $M_{G,H}$.

A (p, q)-bipartite graph G is called (p, q)-*tree* if G is connected and $|E(G)| = p + q - 1$. Thus each (p, q)-tree is a tree and for each tree T there exist integers p and q such that T is (p, q)-tree.

Let T be a (p, q)-tree and $y \in V(T)$. Let us denote by U_y the set of all $z \in N(y, T)$ such that $d(z, T) = 1$. We shall call U_y *the bough* with *the center* y. We say that $\{x, y\} \subset L$ (or $\{x, y\} \subset R$) is a *good pair of vertices* (for short *good pair*) if there exist vertices w and z such, that $x \in U_w$, $y \in U_z$ and $w \neq z$.

2. Results

Let G be a general graph of order n. The following theorem has been proved in [2].

Theorem 1. *If $e(G) \leq n-1$ and $n \geq 8$ then either G is contained in $\bar{G}$ or G is isomorphic to one of the following graphs: $K_{1,n-1}$, $K_{1,n-4} \cup K_3$.*

Wang and Saver proved the following result in [6].

Theorem 2. *A tree of order $n \geq 7$ is not 3-placeable if and only if it is isomorphic to the star S_n or the graph obtained from S_{n-1} by inserting a new vertex into an edge of S_{n-1}.*

Makheo, Saclé and Woźniak in [4] characterized all triples of trees $\{T_n, T'_n, T''_n\}$ which are not mutually placeable in K_n.

For bipartite graphs, J.L. Fouquet and A.P. Wojda in [3] characterized those (p, q)-bipartite graphs of size $p+q-2$ which are not 2-placeable in $K_{p,q}$.

All pairs of (p, q)-bipartite graphs G, H which are not placeable, $e(G) \leq p+q-1$, $e(H) \leq p$ and $p \leq q$ are given in [5].

The main result to be presented in this paper is that any (p, q)-tree T such that $\Delta_R(T) < p$, $\Delta_L(T) < q$, $p \geq 3$, $q \geq 3$ and $p+q \geq 7$ is either 2-placeable or T is in the family $\mathcal{T}(p, q)$ of graphs which are defined below:

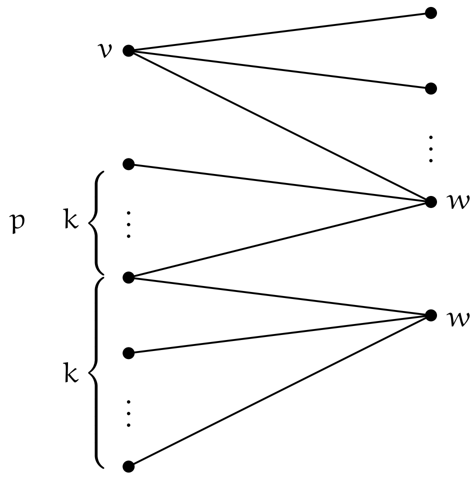
$T'L(p, q, k)$ is the (p, q)-tree T such that, there are three vertices v, w, w' such that $v \in L$ and $d(v, T) = q-1$, $w' \in R \setminus N(v, T)$, $d(w', T) = k$, $w \in N(v, T)$ and $d(w, T) = p-k+1$ (see Figure 1). We shall called the vertex v the left center of T.

It is not difficult to see that $T'L(p, q, k)$ is 2-placeable if and only if $1 < k \leq \frac{p}{2}$. Let $\mathcal{T}L(p, q) = \bigcup\{T'L(p, q, k); k > \frac{p}{2}\}$. Analogically we define the tree $T'R(p, q, k)$ and the set $\mathcal{T}R(p, q) = \{T'R(p, q, k); k > \frac{q}{2}\}$. The tree $T'R(p, q, k)$ is shown in Figure 2.

By $\mathcal{T}(p, q)$ we denote the set $\mathcal{T}R(p, q) \cup \mathcal{T}L(p, q)$.

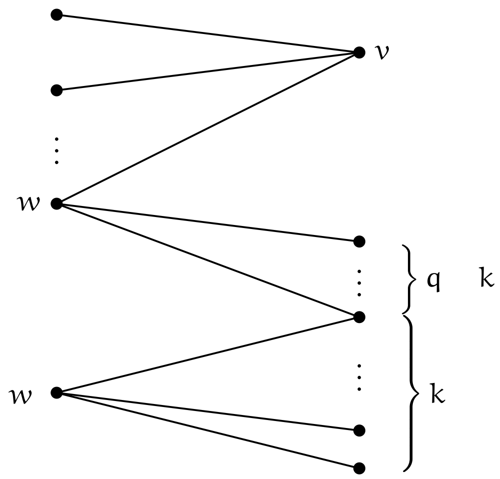
Now, we can formulate our main result.

Theorem A. *Let $T = (L, R; E)$ be a (p, q)-tree such that $\Delta_L(T) < q$, $\Delta_R(T) < p$, $p \geq 3$, $q \geq 3$ and $p+q \geq 7$. Then either T is 2-placeable or $T \in \mathcal{T}(p, q)$.*



$\mathcal{T}'L(p,q,k)$

Figure 1



$\mathcal{T}'R(p,q,k)$

Figure 2

3. Proof of Theorem A

To prove Theorem A we shall need two lemmas and some observations.

Lemma 3.1. *Let $T = (L, R; E)$ be a (p,q)-tree such that there are two different vertices y and y' such that either $y, y' \in L$ or $y, y' \in R$, $U_y \neq \varnothing$ and $U_{y'} \neq \varnothing$. Let $|U_y| = k$, $U_y = \{x_1, \ldots, x_k\}$, $|U_{y'}| = k', U_{y'} = \{x'_1, \ldots, x'_{k'}\}$, and $k \leq k'$. Denote by $U^*_{y'}$ the set $\{x'_1, \ldots, x'_k\}$.*

If $T \setminus (U_y \cup U^*_{y'})$ is 2-placeable, then T is 2-placeable, too.

Proof. Let $T' = T \setminus (U_y \cup U^*_{y'})$ and let f be a 2-placement of T'. We may define a 2-placement f^* of T in the following way:

- $f^*(v) = f(v)$, for each vertex v of T',
- if $f(y') = y'$ or $f(y) = y$ then $f^*(U_y) = U^*_{y'}$, $f^*(U^*_{y'}) = U_y$,
- if $f(y') \neq y'$ and $f(y) \neq y$ then $f^*(U_y) = U_y$, $f^*(U^*_{y'}) = U^*_{y'}$. ■

Lemma 3.2. *Let $T = (L, R; E)$ be $(3,q)$-tree, $\Delta_L(T) < q$, $\Delta_R(T) < 3$ and $q \geq 4$. Then T is 2-placeable unless $T \in \mathcal{T}(3,q)$.*

Proof. Let $T = (L, R; E)$ be a $(3,q)$-tree, $\Delta_L(T) < q$ and $\Delta_R(T) < 3$. Let $L = \{a, b, c\}$, $d(a,T) = k_1$, $d(b,T) = k_2$ and $d(c,T) = k_3$. Note that two of sets $N(a,T) \cap N(b,T)$, $N(b,T) \cap N(c,T)$, $N(c,T) \cap N(a,T)$ are 1-sets, while the third is empty. We assume that $N(a,T) \cap N(b,T) \neq N(b,T) \cap N(c,T)$, otherwise $\Delta_R(T) = 3$. Let z be a common neighbor of vertices a and b, and let y be a common neighbor of vertices b and c. Let $N(a,T) = \{x_1, \ldots, x_{k_1}\}$, $x_{k_1} = z$, $N(b,T) = \{x_{k_1}, \ldots, x_{k_1+k_2-1}\}$, $x_{k_1+k_2-1} = y$ and $N(c,T) = \{x_{k_1+k_2-1}, \ldots, x_q\}$. The tree T and the matrix M_T is shown in Figure 3.

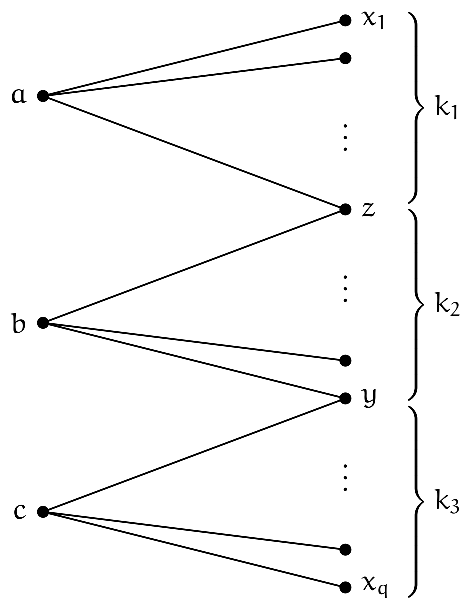
Observe that $k_1 \geq 1$, $k_3 \geq 1$, $k_2 \geq 2$ and $k_1 + k_2 + k_3 - 2 = q$. If $k_1 = 1$ and $k_3 > \frac{q}{2}$ or $k_3 = 1$ and $k_1 > \frac{q}{2}$ then $T \in \mathcal{T}(3,q)$. If $k_1 = 1$ and $k_3 \leq \frac{q}{2}$ then any function $f : L \cup R \to L \cup R$ such that $f(N(b,T)) = \{x_{q-k_2+1}, \ldots, x_q\}$ and $f(N(c,T)) = \{x_1, \ldots, x_{q-k_2+1}\}$, $f(b) = a$, $f(a) = b$, $f(c) = c$ is 2-placement of T. For $k_3 = 1$ and $k_1 \leq \frac{q}{2}$ we define a 2-placement of T analogically.

So, we assume that for each $i \in \{1,2,3\}$ $k_i \geq 2$. Let $k = \max\{k_1, k_2, k_3\}$. We consider two cases.

Case 1. $k \neq k_2$

We may assume that $k = k_3$. The function f such that $f(c) = a$, $f(b) = b$, $f(c) = a$, $f(N(a,T)) = \{x_1, \ldots, x_k\}$, $f(N(b,T)) = \{x_1, x_{k_1+k_3}, \ldots, x_q\}$ and

$f(N(c,T)) = \{x_{k_1+1}, \dots, x_{k_1+k_3-1}, x_q\}$ is a 2-placement of T. For $k_1 = 4$, $k_2 = 4$ and $k_3 = 6$ the matrix $M_{T,T}$ is shown in the Figure 4.



$$\begin{array}{c} \\ a \\ b \\ c \end{array} \begin{array}{c} \begin{array}{ccccccccccc} x_1 & \cdots & & z & & \cdots & & y & & \cdots & x_q \end{array} \\ \begin{bmatrix} 1 & \cdots & 1 & 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 1 & 1 & \cdots & 1 & 1 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 1 & 1 & \cdots & 1 \end{bmatrix} \end{array}$$

Figure 3

$$\begin{array}{c} \\ a \\ b \\ c \end{array} \begin{array}{c} \begin{array}{cccccccccccc} x_1 & \cdots & & x_4 & \cdots & & x_7 & & \cdots & & & x_{12} \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 1 & 2 & 2 & 2 & 2 & 2 & 0 & 0 & 2 \\ 2 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 2 & 2 & 2 \\ 2 & 2 & 2 & 2 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} \end{array}$$

Figure 4

Case 2. $k = k_2$
Without loss of the generality, we may suppose that $k_1 \leq k_3 < k_2$. The 2-placement of T we may define as follows: $f(a) = b$, $f(b) = a$, $f(c) = c$, $f(N(b,T)) = \{x_{q-k_2+1}, \dots, x_q\}$, $f(N(a,T)) = \{x_1, \dots, x_{k_1-1}, x_q\}$, $f(N(c,T)) = \{x_{k_1}, \dots, x_{q-k_2+1}\}$. The matrix of $M_{T,T}$ when $k_1 = 4$, $k_2 = 6$ and $k_3 = 5$ is shown in Figure 5. ∎

$$\begin{array}{c@{\,}c}
 & \begin{array}{ccccccccccccc} x_1 & \cdots & & x_4 & & \cdots & & & x_9 & & \cdots & & x_{13} \end{array} \\
\begin{array}{c} a \\ b \\ c \end{array} & \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 2 & 2 & 2 & 2 & 2 & 2 \\ 2 & 2 & 2 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 2 & 2 & 2 & 2 & 2 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}
\end{array}$$

Figure 5

Let T be (p,q)-tree, such that $\Delta_R(T) < p$ $\Delta_L(T) < q$, $5 \leq p \leq q$ and $6 \leq q$. Let $\{x,y\}$ be a good pair of vertices. We say that $\{x,y\}$ is a *very good pair* if either $\Delta_L(T \setminus \{x,y\}) < q-2$ and $T \setminus \{x,y\} \notin \mathcal{T}(p,q-2)$ when $\{x,y\} \subset R$ or $\Delta_R(T \setminus \{x,y\}) < p-2$ and $T \setminus \{x,y\} \notin \mathcal{T}(p-2,q)$ when $\{x,y\} \subset L$.

Observations.

1. If $T \in \mathcal{T}(p,q)$ then if v is the left (or right) center of T, then there is exactly one vertex which is not pendent in $N(v,T)$.
2. If $T \in \mathcal{T}(p,q)$ and z is the common neighbor of the vertices w and w' then $d(z,T) = 2$.

***Proof of Theorem* A.** We shall give the main idea of the proof, leaving to reader long but easy verification of some details. The proof is by induction on $p+q$.

Without the loss of the generality we may assume that $p \leq q$. By Lemma 3.2 the theorem holds if $p = 3$ and $q \geq 4$. So, we assume that $p \geq 4$, $q \geq p$ and the theorem is true for every (p',q')-tree if $p'+q' < p+q$.

Let T be a (p,q)-tree verifying assumptions of the theorem. Then there is a pendent vertex in R.

To prove that T is 2-placeable unless $T \in \mathcal{T}(p,q)$ we shall distinguish two cases.

Case 1. There are two pendent vertices in R, say x and y, having different neighbors — $\{x,y\}$ is a good pair in R. When $q = 4$ then the theorem is easy to check. So, we may assume that $q \geq 5$.

Let $T' = T \setminus \{x,y\}$. If $\{x,y\}$ is a very good pair, then by the induction hypothesis T' is 2-placeable. The 2-placement of T we have by the Lemma 3.1. Now, we suppose that $\{x,y\}$ is not a very good pair. We consider three subcases.

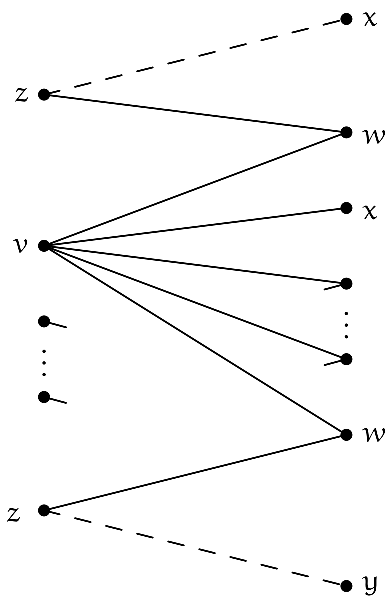


Figure 6

Subcase 1.1. $\Delta_L(T') = q - 2$
Let v be a vertex in L such that $d(v, T') = q - 2$. First, we assume that $d(v, T) = q - 2$. Let $N(x) = \{z\}$ and $N(y) = \{z'\}$ (see Figure 6). Observe that if $p \leq q - 2$ then there is a pendent vertex, say x', in the set $N(v, T)$ and $\{x, x'\}$ is a very good pair in R. In fact, if $T'' = T \setminus \{x, x'\}$ then $\Delta_L(T'') = q - 3 < q - 2$ and $\Delta_R(T'') = \Delta_R(T) < p$. Suppose that $T'' \in \mathcal{T}L(p, q - 2)$. Then the only possible center is the vertex v. But then $R(T'') \setminus N(v, T'') = \{y\}$ and $d(y, T'') = 1$, a contradiction.

Now, we suppose that $p = q \geq 6$ or $p = q - 1 \geq 5$ and each neighbor of the vertex v has the degree at least two. In this case either $T = T_1$ or $T = T_2$ else $T = T_3$ where T_1, T_2 and T_3 are the graphs defined in the Figure 7.

Note that there is a very good pair of vertices in L. Let $\{x', y'\}$ be a very good pair in L. By induction hypothesis $T \setminus \{x', y'\}$ has 2-placement. T is 2-placeable by the Lemma 3.1.

When $p = q = 5$ and there are no very good pairs in L and each neighbour of the vertex v has the degree at least two or if $p = 4$ the proof may be completed by checking all possible cases.

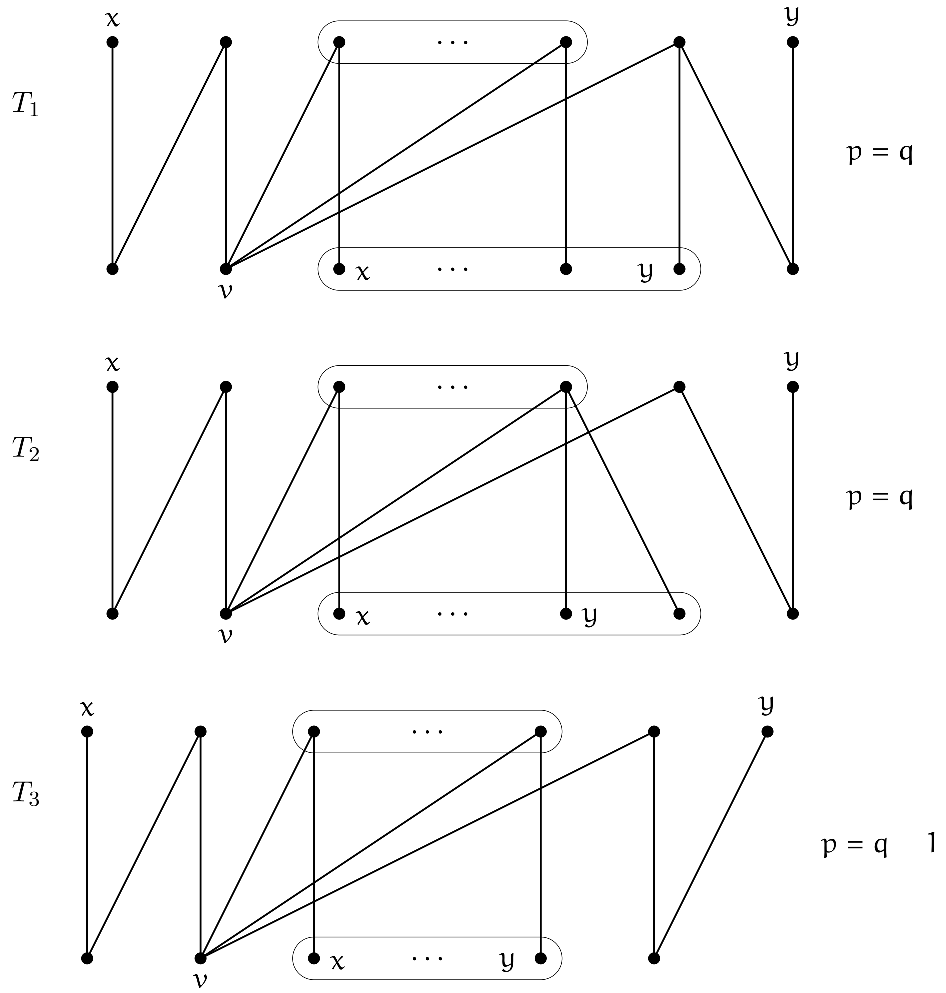


Figure 7

Let us suppose now, that $d(v,T) = q-1$ and $y \notin N(v,T)$ (see Figure 8).

If there is a 2-placement f of $T \setminus \{x\}$ then $f(v) \neq \{v\}$ and the map defined by $f^*(z') = f(z')$, for $z' \neq x$, $f^*(x) = x$ is 2-placement of T.

Observe that $T \setminus \{x\}$ is $(p, q-1)$-tree, $\Delta_L(T \setminus \{x\}) = q-2 < q-1$ and $\Delta_R(T \setminus \{x\}) = \Delta_R(T) < p$. There are at least two vertices of the degree at least two in the set $N(v,T)$. In the other case $\Delta_R(T) = p$. Therefore, by Observation 1, $T \setminus \{x\} \notin \mathcal{T}L(p, q-1)$. If there is a vertex of degree $p-1$ in $N(v,T) \setminus \{y_1\}$, where $\{y_1\} = N(v,T) \cap N(z,T)$, then $T \setminus \{x\} \in \mathcal{T}R(p, q-1)$.

But the degree of the vertex z, which is not adjacent to the right center of T, is two. Hence we conclude that $T \setminus \{x\} \notin \mathcal{T}R(p, q-1)$ and, by the induction hypothesis, there is a 2-placement f of $T \setminus \{x\}$.

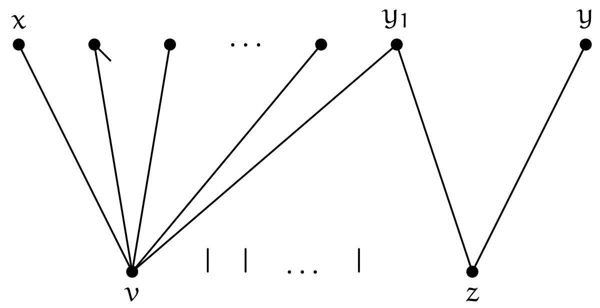


Figure 8

Subcase 1.2. $T' \in \mathcal{T}R(p, q-2)$
First we assume that $d(w, T') \geq 3$. Then either $T = T_1$, or $T = T_2$, or $T = T_3$, else $T = T_4$ (see Figure 9).

Let $T = T_1$ and let x' be a pendent neighbor of the vertex w'. The tree $T \setminus \{x', y\}$ has two neighbors of vertex v of degree at least two. Hence, by Observation 1, $T \setminus \{x', y\} \notin \mathcal{T}(p, q-2)$ and $\{x', y\}$ is very good pair.

Analogically, we may show that $\{x', y\}$ is a very good pair if $T = T_2$ and x' is pendent in $N(w')$ or if $T = T_3$, $x' \in N(w)$ and $d(x', T) = 1$. When $T = T_4$ then $T \in \mathcal{T}R(p, q)$.

If $d(w, T') = 2$ and $T = T_3$ then there is no very good pair in $V(T)$. Let then the tree $T = T_{3'}$. The matrix $M_{T_3', T_3'}$ is shown in Figure 10.

Subcase 1.3. $T' \in \mathcal{T}L(p, q-2)$
At the beginning we assume that $d(w', T') = p-1$. In this case either there are very good pair in R or $T \in \mathcal{T}R(p, q)$ else $T = T_3'$ (See Figure 10).

For $d(w', T') = p-2$, unless $T = T_5$ or $T = T_6$ (See Figure 11), there is a very good pair of vertices in T'. The matrices M_{T_5, T_5} and M_{T_6, T_6} are not difficult to find.

If $d(w', T) \leq p-3$ then there is very good pair of vertices $V(T)$.

Case 2. There is a vertex in L, say z_0, such that each pendent vertex in R is its neighbor.

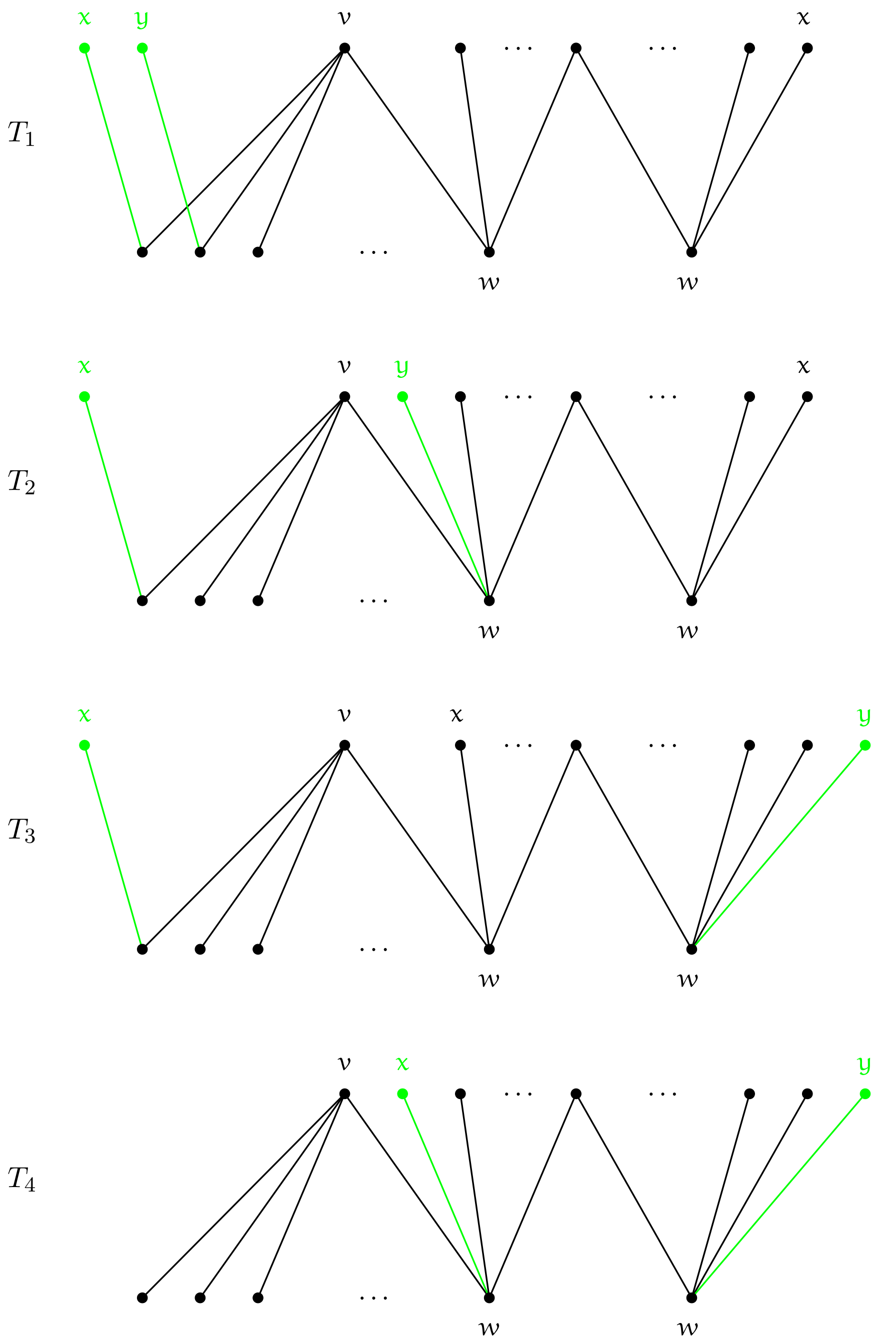
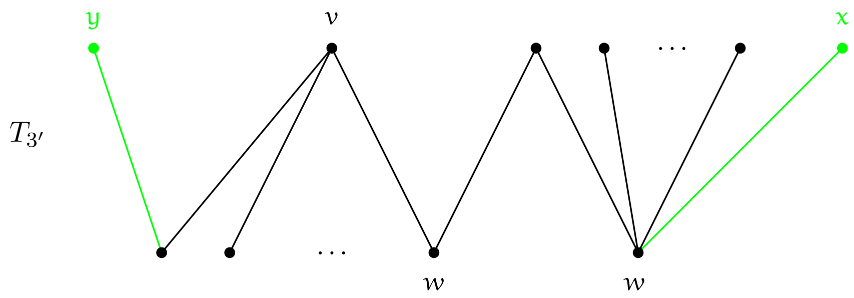


Figure 9



$$
\begin{array}{c}
\begin{array}{c} \\ 1 \\ 2 \\ \\ \vdots \\ \\ p-1 \\ 0 \end{array}
\begin{array}{c}
\begin{array}{cccccc} 1 & 2 & & & \cdots & q \end{array} \\
\begin{bmatrix}
1 & 1 & 2 & 2 & \cdots & 2 \\
2 & 1 & 2 & 0 & \cdots & 0 \\
2 & 1 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & & & \vdots \\
 & & 0 & & & \\
2 & 1 & 1 & 0 & \cdots & 0 \\
2 & 2 & 1 & 1 & \cdots & 1
\end{bmatrix}
\end{array}
\end{array}
$$

Figure 10

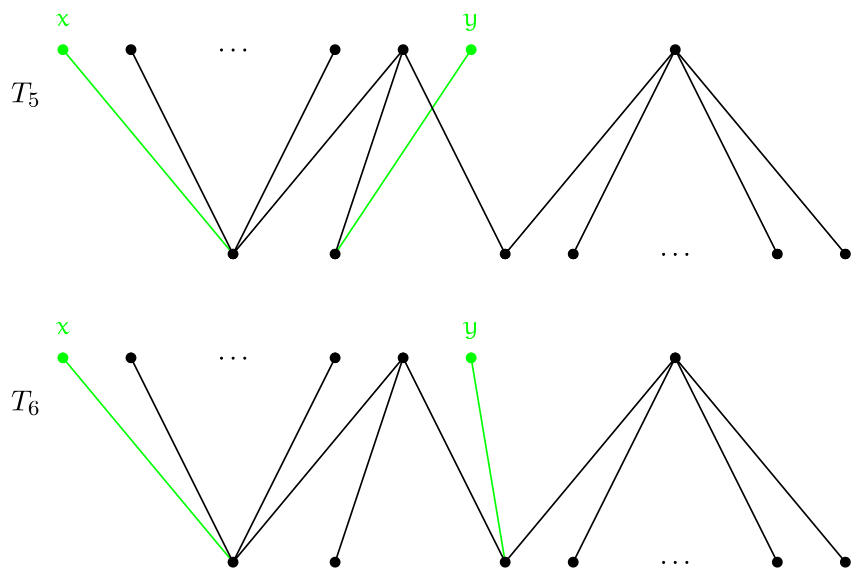


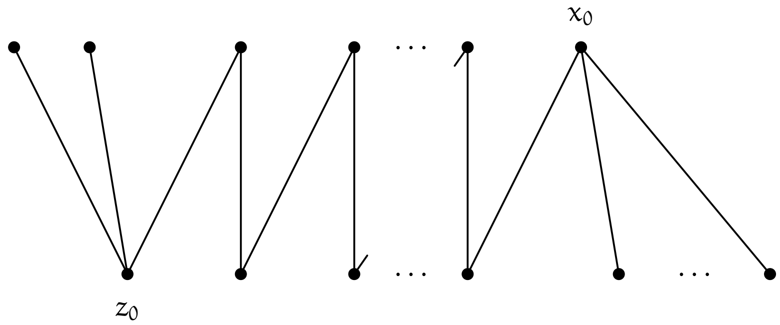
Figure 11

Let us denote by U_{z_0} the bough with center z_0 and let $|U_{z_0}| = m$. Note that $d(z_0, T) \geq m$. If $d(z_0, T) = m$ then $m = q$ and $T = K_{1,q}$. So, we suppose now, that $d(z_0, T) \geq m + 1$. Observe, that there is at least one pendent vertex in L. In the other case there is a good pair of the vertices in R.

First, we assume that there is a good pair, say x' and y', in L. When $p = 4$ then $m = q - 2$ or $m = q - 3$ and is easy to check the theorem.

For $p \geq 5$ $T'' = T \setminus \{x', y'\}$ is $(p-2, q)$-tree, $(p - 2 \geq 3)$ and if $\{x', y'\}$ is very good pair then T'' is 2-placeable by the induction hypothesis. T has 2-placement by Lemma 3.1.

Now, we suppose that there is no very good pair in L — i.e., $\{x', y'\}$ is a good pair but either $\Delta_R(T'') = p - 2$ or $T'' \in \mathcal{T}(p-2, q)$. Observe that $\Delta_R(T'') < p - 2$. In the other case either $\Delta_L(T) = q$ or there is a cycle C_4 in T.



$$\begin{bmatrix}
1 & 1 & \cdots & 1 & 1 & 1 & 0 & 0 & 2 & 0 \\
2 & 2 & \cdots & 2 & 2 & 1 & 1 & 0 & 2 & 2 \\
0 & 0 & \cdots & 0 & 2 & 2 & 1 & 1 & 0 & 0 \\
 & & & & 0 & 2 & 2 & \ddots & \ddots & 0 \\
\vdots & \vdots & & & & & \ddots & \ddots & 1 & 1 \\
 & & & & & & 0 & 2 & 2 & 1 \\
 & & & & & & & 0 & 2 & 1 \\
\vdots & \vdots & & & & & & \vdots & \vdots & \vdots \\
 & & & & & & & 0 & 2 & 1 \\
0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 2 & 1
\end{bmatrix}$$

Figure 12

If $T'' \in \mathcal{T}R(p-2,q)$ or $T'' \in \mathcal{T}L(p-2,q)$, then either $T \in \mathcal{T}L(p,q)$ or $T = T_{3'}$.

Finally, we assume that all pendent vertices in L have a common neighbor. Let x_0 be a vertex in R such that if $v' \in L$ and $d(v',T) = 1$ then $v' \in N(x_0)$ and let $|U_{x_0}| = l$. Observe, that $T''' = T \setminus U_{z_0} \setminus U_{x_0} = P_{2n}$, where $n = q - m = p - l$. When $n = 1$ then $\Delta_L(T) = q$. If $n = 2$ then $T \in \mathcal{T}L(p,q)$. For $n \geq 3$ the tree $T = T_{10}$ and the matrix $M_{T_{10},T_{10}}$ shown in Figure 12.

This completes the proof of the theorem. ∎

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