# GENERALIZED EDGE-CHROMATIC NUMBERS AND ADDITIVE HEREDITARY PROPERTIES OF GRAPHS 

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#### Abstract

An additive hereditary property of graphs is a class of simple graphs which is closed under unions, subgraphs and isomorphisms. Let $\mathcal{P}$ and $\mathcal{Q}$ be hereditary properties of graphs. The generalized edge-chromatic number $\rho_{\mathcal{Q}}^{\prime}(\mathcal{P})$ is defined as the least integer $n$ such that $\mathcal{P} \subseteq n \mathcal{Q}$. We investigate the generalized edge-chromatic numbers of the properties $\rightarrow H, \mathcal{I}_{k}, \mathcal{O}_{k}, \mathcal{W}_{k}^{*}, \mathcal{S}_{k}$ and $\mathcal{D}_{k}$.


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## 1. Introduction

Following [1] we denote the class of all finite simple graphs by $\mathcal{I}$.
A property of graphs is a non-empty isomorphism-closed subclass of $\mathcal{I}$. We say that a graph $G$ has the property $\mathcal{P}$ if $G \in \mathcal{P}$. A property $\mathcal{P}$ is called hereditary if $G \in \mathcal{P}$ and $H \subseteq G$ implies $H \in \mathcal{P}$. $\mathcal{P}$ is called additive if $G \cup H \in \mathcal{P}$ whenever $G \in \mathcal{P}$ and $H \in \mathcal{P}$. A homomorphism of a graph $G$ to a graph $H$ is a mapping of the vertex set $V(G)$ into $V(H)$ such that if $e=\{u, v\} \in E(G)$, then $f(e)=\{f(u), f(v)\} \in E(H)$. Given a graph $G$ and a positive integer $k$ we define $G[k]$ to be the graph with $V(G[k])=$ $V(G) \times\{1,2, \ldots, k\}$ and $E(G[k])=\left\{\left(u, l_{1}\right)\left(v, l_{2}\right): u v \in E(G)\right\} ; G[k]$ is called a multiplication of $G$. The clique number $\omega(G)$ of a graph $G$ is the maximum order of a complete subgraph of $G$. A trail in a graph is a sequence $u_{1} u_{2}, u_{2} u_{3}, \ldots, u_{k-1} u_{k}$ of edges, with no edge repeating. If $u_{1} \neq u_{k}$ then the trail is open. Since we will only be interested in the length of a trail, we associate a trail $T$ with the set of edges in $T$.

Example 1. For a positive integer $k$ and a given graph $H$ we define the following well-known properties:
$\mathcal{O}=\{G \in \mathcal{I}: E(G)=\emptyset\}$,
$\mathcal{I}_{k}=\left\{G \in \mathcal{I}: G\right.$ does not contain $\left.K_{k+2}\right\}$,
$\mathcal{O}_{k}=\{G \in \mathcal{I}:$ each component of $G$ has at most $k+1$ vertices $\}$,
$\mathcal{W}_{k}=\{G \in \mathcal{I}:$ each path in $G$ has at most $k$ edges $\}$,
$\mathcal{W}_{k}^{*}=\{G \in \mathcal{I}:$ each open trail in $G$ has at most $k$ edges $\}$,
$\mathcal{S}_{k}=\{G \in \mathcal{I}:$ the maximum degree of $G$ is at most $k\}$,
$\mathcal{D}_{k}=\{G \in \mathcal{I}: G$ is $k$-degenerate, i.e., every subgraph of $G$ has a vertex of degree at most $k\}$,
$\rightarrow H=\{G \in \mathcal{I}:$ there is a homomorphism from $G$ to $H\}$,
$\mathcal{O}^{k}=\{G \in \mathcal{I}: G$ is $k$-colourable $\}=\rightarrow K_{k}$.
Note that for a graph $G$ we have that $G \in H$ iff $G$ is a subgraph of a multiplication of $H$. A property of the form $\rightarrow H$ is called a hom-property.

Every hereditary property $\mathcal{P}$ is determined by the set of minimal forbidden subgraphs $\mathbf{F}(\mathcal{P})=\{G \in \overline{\mathcal{P}}$ : every proper subgraph of $G$ is in $\mathcal{P}\}$.

If $G=(V, E)$ is a graph and $E^{\prime} \subseteq E$ then the subgraph of $G$ induced by $E^{\prime}$ is the graph $\left(V, E^{\prime}\right)$ and is denoted by $G\left[E^{\prime}\right]$.

Let $\mathcal{Q}_{1}, \mathcal{Q}_{2}, \ldots, \mathcal{Q}_{n}$ be arbitrary hereditary properties of graphs. An edge $\left(\mathcal{Q}_{1}, \mathcal{Q}_{2}, \ldots, \mathcal{Q}_{n}\right)$-decomposition of a graph $G$ is a decomposition
$\left\{E_{1}, E_{2}, \ldots, E_{n}\right\}$ of $E(G)$ such that for each $i=1,2, \ldots, n$ the induced subgraph $G\left[E_{i}\right]$ has the property $\mathcal{Q}_{i}$. The property $\mathcal{R}=\mathcal{Q}_{1} \oplus \mathcal{Q}_{2} \oplus \cdots \oplus \mathcal{Q}_{n}$ is defined as the set of all graphs having an edge $\left(\mathcal{Q}_{1}, \mathcal{Q}_{2}, \ldots, \mathcal{Q}_{n}\right)$-decomposition. It is easy to see that if $\mathcal{Q}_{1}, \mathcal{Q}_{2}, \ldots, \mathcal{Q}_{n}$ are additive and hereditary, then $\mathcal{R}=\mathcal{Q}_{1} \oplus \mathcal{Q}_{2} \oplus \cdots \oplus \mathcal{Q}_{n}$ is additive and hereditary too. If $\mathcal{Q}_{1}=\mathcal{Q}_{2}=\cdots=$ $\mathcal{Q}_{n}=\mathcal{Q}$, then we write $n \mathcal{Q}=\mathcal{Q}_{1} \oplus \mathcal{Q}_{2} \oplus \cdots \oplus \mathcal{Q}_{n}$.

The generalized edge-chromatic number $\rho_{\mathcal{Q}}^{\prime}(G)$ of a graph $G$ is defined as the least integer $n$ such that $G \in n \mathcal{Q}$. For a property $\mathcal{P}, \rho_{\mathcal{Q}}^{\prime}(\mathcal{P})$ is then the least $n$ such that $\mathcal{P} \subseteq n \mathcal{Q}$.

As an example of the non-existence of $\rho_{\mathcal{Q}}^{\prime}(\mathcal{P})$ we have $\rho_{\mathcal{S}_{1}}^{\prime}\left(\mathcal{D}_{1}\right)$ since there exist graphs in $\mathcal{D}_{1}$ of arbitrary maximum degree. Theorem 1.1 by J. Nešetřil and V. Rödl (see [6]) implies that for some properties $\mathcal{P}$, $\rho_{\mathcal{Q}}^{\prime}(\mathcal{P})$ exists iff $\rho_{\mathcal{Q}}^{\prime}(\mathcal{P})=1$. Here a graph $G$ is called 3-chromatic connected if there is no $S \subseteq V(G)$ such that $G-S$ is disconnected and $G[S]$ is bipartite.

Theorem 1.1 [6]. Let $\mathbf{F}(\mathcal{P})$ be a set of 3-chromatic connected graphs. Then for every positive integer $k$ and graph $G \in \mathcal{P}$ there exists a graph $H \in \mathcal{P}$ such that for any decomposition $\left\{E_{1}, E_{2}, \ldots, E_{k}\right\}$ of $E(H)$ there is an $i \in\{1,2, \ldots, k\}$, for which $G \subseteq H\left[E_{i}\right]$.

Corollary 1.2. If $\mathbf{F}(\mathcal{P})$ is a set of 3 -chromatic connected graphs, then for any hereditary property $\mathcal{Q}, \rho_{\mathcal{Q}}^{\prime}(\mathcal{P})$ exists if and only if $\mathcal{P} \subseteq \mathcal{Q}$.

Proof. Suppose that $\mathcal{P} \nsubseteq \mathcal{Q}$ but $\mathcal{P} \in n \mathcal{Q}$ for some $n$. Let $G \in \mathcal{P}$ and $G \notin \mathcal{Q}$. By Theorem 1.1 there is an $H \in \mathcal{P}$ such that for every decomposition $\left\{E_{1}, E_{2}, \ldots, E_{n}\right\}$ of $E(H)$ there is an $i \in\{1,2, \ldots, n\}$ for which $G \subseteq H\left[E_{i}\right]$. Let $\left\{E_{1}, E_{2}, \ldots, E_{n}\right\}$ be an $n \mathcal{Q}$-decomposition of $E(H)$. Then $G \subseteq H\left[E_{i}\right]$ $\in \mathcal{Q}$ for some $i$, a contradiction. The converse is trivial.
In particular, for every $k$ and any hereditary property $\mathcal{Q}$ we have that $\rho_{\mathcal{Q}}^{\prime}\left(\mathcal{I}_{k}\right)$ exists iff $\mathcal{I}_{k} \subseteq \mathcal{Q}$.

Lemma 1.3. Let $\mathcal{P}_{1}, \mathcal{P}_{2}$ and $\mathcal{Q}$ be any properties. If $\mathcal{P}_{1} \subseteq \mathcal{P}_{2}$, then $\rho_{\mathcal{Q}}^{\prime}\left(\mathcal{P}_{1}\right) \leq \rho_{\mathcal{Q}}^{\prime}\left(\mathcal{P}_{2}\right)$.

Lemma 1.4. Let $\mathcal{Q}_{1}, \mathcal{Q}_{2}$ and $\mathcal{P}$ be any properties. If $\mathcal{Q}_{1} \subseteq \mathcal{Q}_{2}$, then $\rho_{\mathcal{Q}_{2}}^{\prime}(\mathcal{P}) \leq \rho_{\mathcal{Q}_{1}}^{\prime}(\mathcal{P})$.

The lattice of (additive) hereditary properties is discussed in [1] - we use the supremum and infimum of properties in our next result without further discussion. A similar result is proved in [5].

Theorem 1.5. Let $\mathcal{P}_{1}$ and $\mathcal{P}_{2}$ be hereditary properties and $\mathcal{Q}$ an additive hereditary property such that $\rho_{\mathcal{Q}}^{\prime}\left(\mathcal{P}_{1}\right)$ and $\rho_{\mathcal{Q}}^{\prime}\left(\mathcal{P}_{2}\right)$ are finite. The following hold:
(i) $\rho_{\mathcal{Q}}^{\prime}\left(\mathcal{P}_{1} \cup \mathcal{P}_{2}\right)=\rho_{\mathcal{Q}}^{\prime}\left(\mathcal{P}_{1} \vee \mathcal{P}_{2}\right)=\max \left\{\rho_{\mathcal{Q}}^{\prime}\left(\mathcal{P}_{1}\right), \rho_{\mathcal{Q}}^{\prime}\left(\mathcal{P}_{2}\right)\right\}$.
(ii) $\rho_{\mathcal{Q}}^{\prime}\left(\mathcal{P}_{1} \cap \mathcal{P}_{2}\right) \leq \min \left\{\rho_{\mathcal{Q}}^{\prime}\left(\mathcal{P}_{1}\right), \rho_{\mathcal{Q}}^{\prime}\left(\mathcal{P}_{2}\right)\right\}$.
(iii) $\max \left\{\rho_{\mathcal{Q}}^{\prime}\left(\mathcal{P}_{1}\right), \rho_{\mathcal{Q}}^{\prime}\left(\mathcal{P}_{2}\right)\right\} \leq \rho_{\mathcal{Q}}^{\prime}\left(\mathcal{P}_{1} \oplus \mathcal{P}_{2}\right) \leq \rho_{\mathcal{Q}}^{\prime}\left(\mathcal{P}_{1}\right)+\rho_{\mathcal{Q}}^{\prime}\left(\mathcal{P}_{2}\right)$.

In the rest of this paper we aim to study the generalized edge-chromatic number $\rho_{\mathcal{Q}}^{\prime}(\mathcal{P})$ with $\mathcal{Q}$ and $\mathcal{P}$ amongst the properties listed in Example 1.

## 2. Some Values of $\rho_{\mathcal{Q}}^{\prime}(\mathcal{P})$

The well-known results of Vizing and Petersen on edge-colourings of graphs imply the following result - see [3] for details.

Theorem 2.1. Let $p$ and $q$ be any positive integers. Then

1. $\mathcal{S}_{p} \oplus \mathcal{S}_{q} \subseteq \mathcal{S}_{p+q}$.
2. $\mathcal{S}_{p} \subseteq(p+1) \mathcal{S}_{1}$.
3. If $p$ and $q$ are even then $\mathcal{S}_{p+q}=\mathcal{S}_{p} \oplus \mathcal{S}_{q}$.
4. If $q$ is odd then $\mathcal{S}_{p+q} \nsubseteq \mathcal{S}_{p} \oplus \mathcal{S}_{q}$.

Corollary 2.2. For all positive integers $k$ and $n$,

$$
\rho_{\mathcal{S}_{n}}^{\prime}\left(\mathcal{S}_{k}\right)=\left\{\begin{aligned}
\left\lceil\frac{k}{n}\right\rceil, & n \text { even or } k \leq n \\
\left\lceil\frac{k+1}{n}\right\rceil, & \text { otherwise }
\end{aligned}\right.
$$

Proof. The result is clearly true if $k \leq n$. If $n$ is even then it follows from Part 3 of Theorem 2.1 that $\mathcal{S}_{k} \subseteq\left\lceil\frac{k}{n}\right\rceil \mathcal{S}_{n}$ while the lower bound follows by observing that $k>n\left(\left\lceil\frac{k}{n}\right\rceil-1\right)$ so that $\mathcal{S}_{k} \nsubseteq \mathcal{S}_{n\left(\left\lceil\frac{k}{n}\right\rceil-1\right)}=\left(\left\lceil\frac{k}{n}\right\rceil-1\right) \mathcal{S}_{n}$.

Now let $n$ be odd and $k>n$. By Theorem 2.1 we have that $\mathcal{S}_{k} \subseteq(k+1)$ $\mathcal{S}_{1} \subseteq n\left\lceil\frac{k+1}{n}\right\rceil \mathcal{S}_{1} \subseteq\left\lceil\frac{k+1}{n}\right\rceil \mathcal{S}_{n}$. Let $c=\left\lceil\frac{k+1}{n}\right\rceil-1$. Since $\left\lceil\frac{k+1}{n}\right\rceil \leq \frac{k+1}{n}+\frac{n-1}{n}$ it follows that $k \geq n c$. If $c=1$ then, since $k>n, \rho_{\mathcal{S}_{n}}^{\prime}\left(\mathcal{S}_{k}\right) \geq 2=c+1=\left\lceil\frac{k+1}{n}\right\rceil$, so assume that $c \geq 2$. Now $\mathcal{S}_{k} \supseteq \mathcal{S}_{c n}=\mathcal{S}_{(c-1) n+n} \nsubseteq \mathcal{S}_{(c-1) n} \oplus \mathcal{S}_{n} \supseteq$ $(c-1) \mathcal{S}_{n} \oplus \mathcal{S}_{n} \supseteq c \mathcal{S}_{n}$ so that $\rho_{\mathcal{S}_{n}}^{\prime}\left(\mathcal{S}_{k}\right) \geq c+1$.

Our next result states that, in some cases, the determination of the generalized edge-chromatic number $\rho_{\mathcal{Q}}^{\prime}(\rightarrow H)$ reduces to the determination of $\rho_{\mathcal{Q}}^{\prime}(H)$.

Theorem 2.3. For any additive hereditary property $\mathcal{Q}$ which is closed under multiplications and any graph $H, \rho_{\mathcal{Q}}^{\prime}(\rightarrow H)=\rho_{\mathcal{Q}}^{\prime}(H)$.

Proof. Since $H \in H$ we have $\rho_{\mathcal{Q}}^{\prime}(\rightarrow H) \geq \rho_{\mathcal{Q}}^{\prime}(H)$. Now suppose that $H \in m \mathcal{Q}$ and let $\left(E_{1}, E_{2}, \ldots, E_{m}\right)$ be an $m \mathcal{Q}$-decomposition of $E(H)$. If $G \in \rightarrow H$ then $G$ is a subgraph of a multiplication of $H$. Let, for every $i \in\{1,2, \ldots, m\}, E_{i}^{\prime}=\left\{\left(u, l_{1}\right)\left(v, l_{2}\right): u v \in E_{i}\right\}$. Then $G\left[E_{i}^{\prime}\right]$ is a subgraph of a multiplication of $H\left[E_{i}\right]$ for every $i$ and, since $\mathcal{Q}$ is closed under multiplications and hereditary, $G\left[E_{i}^{\prime}\right] \in \mathcal{Q}$. Therefore $\left(E_{1}^{\prime}, E_{2}^{\prime}, \ldots, E_{m}^{\prime}\right)$ is an $m \mathcal{Q}$-decomposition of $E(G)$, hence $\rho_{\mathcal{Q}}^{\prime}(\rightarrow H) \leq \rho_{\mathcal{Q}}^{\prime}(H)$.

Theorem 2.4. For all positive integers $n \geq 2$ and $k$, if $\mathcal{P}$ satisfies $\mathcal{O}_{k-1} \subseteq$ $\mathcal{P} \subseteq \mathcal{O}^{k}$, then $\rho_{\mathcal{O}^{n}}^{\prime}(\mathcal{P})=\left\lceil\log _{n} k\right\rceil$.

Proof. It is well known that $\mathcal{O}^{a b}=\mathcal{O}^{a} \oplus \mathcal{O}^{b}$ (see e.g. [3]). This implies that $\mathcal{O}^{k} \subseteq \mathcal{O}^{n^{\left\lceil\log _{n} k\right\rceil}}=\left\lceil\log _{n} k\right\rceil \mathcal{O}^{n}$ hence $\rho_{\mathcal{O}^{n}}^{\prime}\left(\mathcal{O}^{k}\right) \leq\left\lceil\log _{n} k\right\rceil$.

Since $n^{\left\lceil\log _{n} k\right\rceil-1}<n^{\log _{n} k}=k$ it follows that $K_{k} \notin \mathcal{O}^{n^{\left[\log _{n} k\right\rceil-1}}=$ $\left(\left\lceil\log _{n} k\right\rceil-1\right) \mathcal{O}^{n}$. Therefore $\mathcal{O}_{k-1} \nsubseteq\left(\left\lceil\log _{n} k\right\rceil-1\right) \mathcal{O}^{n}$ and thus $\rho_{\mathcal{O}^{n}}^{\prime}\left(\mathcal{O}_{k-1}\right) \geq$ $\left\lceil\log _{n} k\right\rceil$. Therefore, by Lemma 1.3 it follows that $\rho_{\mathcal{O}^{n}}^{\prime}(\mathcal{P})=\left\lceil\log _{n} k\right\rceil$.
For our next result we define $\rho_{\chi}(\mathcal{P})$ to be the least $k$ such that $\mathcal{P} \subseteq \mathcal{O}^{k}$ and $\chi^{*}(\mathcal{P})$ to be the greatest $k$ such that $\mathcal{O}^{k} \subseteq \mathcal{P}$.

Corollary 2.5. For any additive hereditary properties $\mathcal{Q}, \mathcal{P} \neq \mathcal{I}$ for which $\rho_{\chi}(\mathcal{P})$ and $\rho_{\chi}(\mathcal{Q})$ exist, $\left\lceil\log _{\rho_{\chi}(\mathcal{Q})} \chi^{*}(\mathcal{P})\right\rceil \leq \rho_{\mathcal{Q}}^{\prime}(\mathcal{P}) \leq\left\lceil\log _{\chi^{*}(\mathcal{Q})} \rho_{\chi}(\mathcal{P})\right\rceil$.

Proof. Since $\mathcal{O}^{*}(\mathcal{Q}) \subseteq \mathcal{Q}$ and $\mathcal{P} \subseteq \mathcal{O}^{\rho_{\chi}(\mathcal{P})}$ we have by Lemma 1.3, Lemma 1.4 and Theorem 2.4 that $\left\lceil\log _{\chi^{*}(\mathcal{Q})} \rho_{\chi}(\mathcal{P})\right\rceil \geq \rho_{\mathcal{Q}}^{\prime}(\mathcal{P})$. Similarly, since $\mathcal{Q} \subseteq$ $\mathcal{O}^{\rho_{\chi}(\mathcal{Q})}$ and $\mathcal{O} \chi^{*}(\mathcal{P}) \subseteq \mathcal{P}$ we have that $\rho_{\mathcal{Q}}^{\prime}(\mathcal{P}) \geq\left\lceil\log _{\rho_{\chi}(\mathcal{Q})} \chi^{*}(\mathcal{P})\right\rceil$.
Since, for any graph $H, \rho_{\chi}(\rightarrow H)=\chi(H)$ and $\chi^{*}(\rightarrow H)=\omega(H)$ we have the following corollary.

Corollary 2.6. For all graphs $G$ and $H$,

$$
\left\lceil\log _{\chi(G)} \omega(H)\right\rceil \leq \rho_{\rightarrow G}^{\prime}(\rightarrow H) \leq\left\lceil\log _{\omega(G)} \chi(H)\right\rceil .
$$

## 3. Some Results on $\mathcal{D}_{k}$

The next result is stated in [2].
Theorem 3.1. For all positive integers $a$ and $b$, we have $\mathcal{D}_{a+b} \subseteq \mathcal{D}_{a} \oplus \mathcal{D}_{b}$.
From this theorem it follows that, for all positive integers $c$ and $n, \mathcal{D}_{c n} \subseteq$ $c \mathcal{D}_{n}$. We now show that this cannot be improved, even if we restrict the graphs to be bipartite.

Theorem 3.2. For all positive integers $c$ and $n, \mathcal{D}_{c n+1} \cap \mathcal{O}^{2} \nsubseteq c \mathcal{D}_{n}$.
Proof. Let $t=(n+1) c^{c n+1}$. Clearly, $G=K_{c n+1, t} \in \mathcal{D}_{c n+1} \cap \mathcal{O}^{2}$. We show that $G \notin c \mathcal{D}_{n}$ : Suppose, to the contrary, that $\left\{E_{1}, E_{2}, \ldots, E_{c}\right\}$ is a ${ }^{\mathcal{D}_{n}}$-decomposition of $E(G)$. Let $V_{1}=\left\{v_{1}, v_{2}, \ldots, v_{c n+1}\right\}$ be the partite set of order $c n+1$ and $V_{2}$ the partite set of order $t$. Consider the edges incident with $v_{1}$. At least $t / c$ of them must be in the same colour class, hence there is a subset $U_{1}$ of $V_{2}$ with $\left|U_{1}\right|=t / c$ such that all edges in $G\left[U_{1} \cup V_{1}\right]$ incident with $v_{1}$ have the same colour. Similarly, there is a subset $U_{2}$ of $U_{1}$ with $\left|U_{2}\right|=t / c^{2}$ such that all edges in $G\left[U_{2} \cup V_{1}\right]$ incident with $v_{2}$ have the same colour (not necessarily the same as for $v_{1}$ ). Continuing in this way we obtain a subset $U$ of $V_{2}$ with $|U|=n+1$ such that, for every $v \in V_{1}$, all edges of $G\left[U \cup V_{1}\right]$ incident with $v$ have the same colour.

Since there are $c$ colours it follows that for some $i \in\{1,2, \ldots, c\}$ we have that $K_{n+1, n+1} \subseteq G\left[E_{i}\right]$. This is a contradiction, since $K_{n+1, n+1} \notin \mathcal{D}_{n}$. Thus $K_{c n+1, t} \notin c \mathcal{D}_{n}$.

Theorem 3.3. For all positive integers $k$ and $n$, we have that

$$
\rho_{\mathcal{D}_{n}}^{\prime}\left(\mathcal{D}_{k}\right)=\left\lceil\frac{k}{n}\right\rceil .
$$

Proof. It follows from Theorem 3.1, by induction on $c$, that $\mathcal{D}_{c n} \subseteq c \mathcal{D}_{n}$ for all $c$ and $n$. Now let $k$ and $n$ be positive integers and let $c=\left\lceil\frac{k}{n}\right\rceil$. Then $k \leq c n$ hence $\mathcal{D}_{k} \subseteq \mathcal{D}_{c n} \subseteq c \mathcal{D}_{n}$ and the upper bound follows.

For the lower bound, since $k \geq(c-1) n+1$ we have that $\mathcal{D}_{k} \supseteq$ $\mathcal{D}_{(c-1) n+1} \nsubseteq(c-1) \mathcal{D}_{n}$ by Theorem 3.2.
We know that if $p q>a+b$, then $\mathcal{D}_{a+b} \subseteq \mathcal{O}^{a+b+1} \subseteq \mathcal{O}^{p q}=\mathcal{O}^{p} \oplus \mathcal{O}^{q}$ and $\mathcal{D}_{a+b} \subseteq \mathcal{D}_{a} \oplus \mathcal{D}_{b}$. Our next result shows that for graphs in $\mathcal{D}_{a+b}$ we can find simultaneous $\left(\mathcal{O}^{p}, \mathcal{O}^{q}\right)$ - and $\left(\mathcal{D}_{a}, \mathcal{D}_{b}\right)$-partitions. First a set-theoretic lemma.

Lemma 3.4. Let $a, b, p$ and $q$ be positive integers such that $a \geq b, 2 \leq q \leq$ $b+1$ and $p q>a+b$. If $X$ is $a$ set with $a+b$ elements and $\left\{U_{1}, U_{2}, \ldots, U_{p}\right\}$ and $\left\{V_{1}, V_{2}, \ldots, V_{q}\right\}$ are partitions of $X$ then there exists a partition $\{A, B\}$ of $X$ and $i$ and $j$ such that $|A|=a, A \cap U_{i}=\emptyset$ and $B \cap V_{j}=\emptyset$.

Proof. It is sufficient (and necessary) to find $i$ and $j$ such that $U_{i} \cap V_{j}=\emptyset$, $\left|U_{i}\right| \leq b$ and $\left|V_{j}\right| \leq a$. Let $k$ be the number of $U_{i}$ 's such that $\left|U_{i}\right|>b$ and $m$ the number of $V_{j}$ 's such that $\left|V_{j}\right|>a$. We will show that $(p-k)(q-m)>$ $c=\left|X \backslash\left(\bigcup\left\{U_{i}:\left|U_{i}\right|>b\right\} \cup \bigcup\left\{V_{j}:\left|V_{j}\right|>a\right\}\right)\right|$. It then follows that among the sets of the required size there is a disjoint pair (there are $(p-k)(q-m)$ ways to choose a pair $\left(U_{i}, V_{j}\right)$ of sets of the required size. Since the $U_{i}$ 's are pairwise disjoint and the $V_{j}$ 's are pairwise disjoint it would follow that $c \geq(p-k)(q-m)$ if all such pairs have nonempty intersection). Note that $m \leq 1$ since $a \geq b$ and that $c \leq \min \{a+b-k(b+1), a+b-m(a+1)\}$. Also, $k<p$, for otherwise we get $a+b=|X| \geq p(b+1) \geq p q$. We have three cases to consider.
(1) $m=0$ : In this case we have $(p-k) q=p q-k q \geq a+b+1-k(b+1)>c$.
(2) $m=1$ and $k \leq \frac{a+1}{b+1}$ : We want to show that $(p-k)(q-1)>b-1$ since $c \leq b-1$. If $q=b+1$ this is clearly true, hence we assume that $q \leq b$. We have

$$
\begin{aligned}
\frac{b-1}{q-1}+k q-a & \leq \frac{a+1}{b+1} q-a+\frac{b-1}{q-1} \\
& =a\left(\frac{q}{b+1}-1\right)+\frac{b-1}{q-1}+\frac{q}{b+1} \\
& \leq b\left(\frac{q}{b+1}-1\right)+\frac{b-1}{q-1}+\frac{q}{b+1} \quad \text { since } a \geq b \text { and } q \leq b \\
& =b\left(\frac{1}{q-1}-1\right)+q-\frac{1}{q-1} \\
& \leq q\left(\frac{1}{q-1}-1\right)+q-\frac{1}{q-1} \\
& =1
\end{aligned}
$$

Suppose now that $(p-k)(q-1) \leq b-1$. Then we have $p q \leq \frac{b-1}{q-1} q+k q=$ $b-1+\frac{b-1}{q-1}+k q-a+a \leq a+b$, a contradiction.
(3) $m=1$ and $k>\frac{a+1}{b+1}$ : Again we may assume that $q \leq b$. We show that $(p-k)(q-1)>a+b-k(b+1) \geq c$. We have

$$
\begin{aligned}
-k(b+1)+ & \frac{a+b-k(b+1)}{q-1}+k q \\
& =\frac{a+b}{q-1}+k\left(q-(b+1)-\frac{b+1}{q-1}\right) \\
& \leq \frac{a+b}{q-1}+\frac{a+1}{b+1}\left(q-(b+1)-\frac{b+1}{q-1}\right) \quad \text { since } q \leq b \\
& =a\left(\frac{q}{b+1}-1\right)+\frac{q}{b+1}+\frac{b-q}{q-1} \\
& \leq b\left(\frac{q}{b+1}-1\right)+\frac{q}{b+1}+\frac{b-q}{q-1} \\
& =(q-b)\left(1-\frac{1}{q-1}\right) \\
& \leq 0
\end{aligned}
$$

Suppose now that $(p-k)(q-1) \leq a+b-k(b+1)$. Then we have $p q \leq$ $q \frac{a+b-k(b+1)}{q-1}+k q=a+b-k(b+1)+\frac{a+b-k(b+1)}{q-1}+k q \leq a+b$.
Theorem 3.5. Let $a, b, p$ and $q$ be positive integers such that $a \geq b$, $2 \leq q \leq b+1$ and $p q>a+b$. Then $\mathcal{D}_{a+b} \subseteq\left(\mathcal{D}_{a} \cap \mathcal{O}^{p}\right) \oplus\left(\mathcal{D}_{b} \cap \mathcal{O}^{q}\right)$.

Proof. Let $G$ be a counterexample of minimum order and let $v$ be a vertex of $G$ of degree at most $a+b$. Then $G-v$ has a $\left(\mathcal{D}_{a} \cap \mathcal{O}^{p}, \mathcal{D}_{b} \cap \mathcal{O}^{q}\right)$ decomposition and Lemma 3.4 is exactly what we need to extend this decomposition to $G$ for a contradiction.
These results now put us in a position to refine Theorem 3.3.
Theorem 3.6. For all positive integers $k$, $n$ and $p \geq 2$ we have that:

$$
\begin{aligned}
\rho_{\mathcal{D}_{n} \cap \mathcal{O}^{p}}^{\prime}\left(\mathcal{D}_{k}\right) & =\left\lceil\log _{p}(k+1)\right\rceil, \text { if } k \leq n \\
& =\left\lceil\frac{k}{n}\right\rceil, \text { if } k>n \text { and } p^{2}>2 n, \\
& \leq\left\lceil\log _{p}(n+1)\right\rceil+\left\lceil\frac{k}{n}\right\rceil-1, \text { otherwise. }
\end{aligned}
$$

Proof. Firstly we note that from Theorem 3.5 it follows that $\mathcal{D}_{c n} \subseteq$ $\mathcal{D}_{(c-1) n} \oplus\left(\mathcal{D}_{n} \cap \mathcal{O}^{2}\right) \subseteq \mathcal{D}_{(c-1) n} \oplus\left(\mathcal{D}_{n} \cap \mathcal{O}^{p}\right)$ for all $c \geq 2$ and therefore $\mathcal{D}_{c n} \subseteq \mathcal{D}_{2 n} \oplus(c-2)\left(\mathcal{D}_{n} \cap \mathcal{O}^{p}\right)$.

Suppose that $k \leq n$. Then $\rho_{\mathcal{D}_{n} \cap \mathcal{O}^{p}}^{\prime}\left(\mathcal{D}_{k}\right)=\rho_{\mathcal{O}^{p}}^{\prime}\left(\mathcal{D}_{k}\right)=\left\lceil\log _{p}(k+1)\right]$ by Theorem 2.4.

Now suppose that $k>n$ and $p^{2}>2 n$. Then $\mathcal{D}_{c n} \subseteq \mathcal{D}_{2 n} \oplus(c-2)\left(\mathcal{D}_{n} \cap\right.$ $\left.\mathcal{O}^{p}\right) \subseteq c\left(\mathcal{D}_{n} \cap \mathcal{O}^{p}\right)$, using Theorem 3.5 and the fact that $p^{2}>2 n$. Now $\mathcal{D}_{k} \subseteq \mathcal{D}_{\left\lceil\frac{k}{n}\right\rceil n} \subseteq\left\lceil\frac{k}{n}\right\rceil\left(\mathcal{D}_{n} \cap \mathcal{O}^{p}\right)$ giving the upper bound. The lower bound follows from Theorem 3.3 and Lemma 1.4.

Suppose that $p^{2} \leq 2 n$. From $\mathcal{D}_{c n} \subseteq \mathcal{D}_{2 n} \oplus(c-2)\left(\mathcal{D}_{n} \cap \mathcal{O}^{p}\right)$ we get that $\mathcal{D}_{c n} \subseteq \mathcal{D}_{n} \oplus(c-1)\left(\mathcal{D}_{n} \cap \mathcal{O}^{p}\right)$. Moreover, by Theorem 2.4 we have that $\mathcal{D}_{n} \subseteq \mathcal{O}^{n+1} \subseteq\left\lceil\log _{p}(n+1)\right\rceil\left(\mathcal{D}_{n} \cap \mathcal{O}^{p}\right)$. Therefore $\mathcal{D}_{k} \subseteq \mathcal{D}_{\left\lceil\frac{k}{n}\right\rceil n} \subseteq$ $\mathcal{D}_{n} \oplus\left(\left\lceil\frac{k}{n}\right\rceil-1\right)\left(\mathcal{D}_{n} \cap \mathcal{O}^{p}\right) \subseteq\left(\left\lceil\log _{p}(n+1)\right\rceil+\left\lceil\frac{k}{n}\right\rceil-1\right)\left(\mathcal{D}_{n} \cap \mathcal{O}^{p}\right)$ giving the desired bound.

## 4. Results on $\mathcal{W}_{k}^{*}$ and $\mathcal{W}_{k}$

It has been conjectured (see e.g. [4]) that the generalized vertex-chromatic number $\rho_{\mathcal{W}_{n}}\left(\mathcal{W}_{k}\right)$ equals $\left\lceil\frac{k+1}{n+1}\right\rceil$. We now consider the similar problems of determining $\rho_{\mathcal{W}_{n}^{*}}^{\prime}\left(\mathcal{W}_{k}^{*}\right)$ and $\rho_{\mathcal{W}_{n}}^{\prime}\left(\mathcal{W}_{k}\right)$.

We will say that two trails in a graph intersect if they have a common edge.

Theorem 4.1. For $a \geq 9$ and $b \geq 1$ we have $\mathcal{W}_{\left\lceil\frac{2 a-6}{*}\right\rceil+b}^{*} \subseteq \mathcal{W}_{a}^{*} \oplus \mathcal{W}_{b}^{*}$.
Proof. Consider any graph $G$ in $\mathcal{W}_{\left\lceil\frac{2 a-6}{3}\right\rceil+b}^{*}$. Take $E_{1}$ to be a maximal subset of $E(G)$ such that $G\left[E_{1}\right]$ is in $\mathcal{W}_{a}^{*}$. Let $E_{2}=E(G)-E_{1}$. Suppose that there is an open trail $T$ in $G\left[E_{2}\right]$ of length $b+1$ and let $e_{1}$ and $e_{2}$ denote the end-edges of $T$. Since $E_{1}$ is maximal in $\mathcal{W}_{a}^{*}$ it follows that there is an open trail $T_{1}$ of length $a+1$ in $G\left[E_{1} \cup\left\{e_{1}\right\}\right]$ and an open trail $T_{2}$ of length $a+1$ in $G\left[E_{1} \cup\left\{e_{2}\right\}\right]$. Let $T_{11}$ and $T_{12}$ denote the trails on either side of $e_{1}$ such that $T_{11} \cup\left\{e_{1}\right\} \cup T_{12}=T_{1}$. Similarly, let $T_{21} \cup\left\{e_{2}\right\} \cup T_{22}=T_{2}$. Now suppose, without loss of generality, that $x=\left|E\left(T_{11}\right)\right| \leq y=\left|E\left(T_{12}\right)\right|$, so that $x+y=a$.

It is easily seen that if $y \geq\left\lfloor\frac{2 a}{3}\right\rfloor+1$, then by taking the trail $T_{12} \cup T$ or $T_{12} \cup\left(T-e_{1}\right)$, as the case may be, we get a trail of length at least $\left\lfloor\frac{2 a}{3}\right\rfloor+1+b$ and therefore an open trail of length at least $\left\lfloor\frac{2 a}{3}\right\rfloor+1+b-$ $1 \geq \frac{2 a-2}{3}+b>\frac{2 a-4}{3}+b \geq\left\lceil\frac{2 a-6}{3}\right\rceil+b$ in $G$, a contradiction. Therefore
$\left\lceil\frac{a}{2}\right\rceil \leq y \leq\left\lfloor\frac{2 a}{3}\right\rfloor$. Moreover, each $T_{i j}, i, j \in\{1,2\}$ has length at least $\left\lfloor\frac{a}{3}\right\rfloor$, since $x=a-y \geq a-\left\lfloor\frac{2 a}{3}\right\rfloor \geq a-\frac{2 a}{3}=\frac{a}{3} \geq\left\lfloor\frac{a}{3}\right\rfloor$.

Note that $T_{11}$ and $T_{12}$ are neccessarily edge disjoint as are $T_{21}$ and $T_{22}$. $T_{12}$ must intersect $T_{21}$ and $T_{22}$, otherwise we get an open trail of length at least $\left\lceil\frac{a}{2}\right\rceil+b-2+\left\lfloor\frac{a}{3}\right\rfloor \geq \frac{a}{2}+\frac{a-2}{3}+b-2=\frac{5 a-16}{6}+b>\left\lceil\frac{2 a-6}{3}\right\rceil+b$ in $G$; containing $T_{12}, T-e_{1}-e_{2}$ and $T_{21}$ or $T_{22}$.

In the following, when we say that $T_{21}$ intersects $T_{12}$ first we mean that there is a trail starting from an end-vertex of $e_{2}$, following $T_{21}$ and ending with an edge of $T_{12}$, containing no edge of $T_{11}$. Similarly for $T_{22}$ intersecting $T_{12}$ first or $T_{2 i}$ intersecting $T_{11}$ first. Note that since $T_{11}$ and $T_{12}$ are disjoint and $T_{12}$ intersects $T_{21}$ and $T_{22}$, we must have that $T_{2 i}, i \in\{1,2\}$ intersects one of $T_{11}$ and $T_{12}$ first.

Suppose that both $T_{21}$ and $T_{22}$ intersect $T_{12}$ first. Then we obtain an open trail of length at least $x+b-1+\left\lceil\frac{y}{2}\right\rceil \geq a-y+\frac{y}{2}+b-1 \geq a-\frac{1}{2} y-1+b \geq$ $a-\frac{1}{2}\left\lfloor\frac{2 a}{3}\right\rfloor-1+b \geq a-\frac{1}{2}\left(\frac{2 a}{3}\right)-1+b=\frac{2 a-3}{3}+b>\left\lceil\frac{2 a-6}{3}\right\rceil+b$ in $G$; containing $T_{11}, T-e_{1}$ and at least a half of $T_{12}$.

Now, suppose that $T_{21}$ or $T_{22}$ intersects $T_{11}$ first, say $T_{21}$. Then we obtain an open trail of length at least $y+\left\lceil\frac{x}{2}\right\rceil+b-2=y+\left\lceil\frac{1}{2}(a-y)\right\rceil+b-2 \geq$ $y+\frac{a-y}{2}+b-2 \geq \frac{a}{2}+\frac{1}{2}\left\lceil\frac{a}{2}\right\rceil+b-2 \geq \frac{3 a}{4}+b-2>\left\lceil\frac{2 a-6}{3}\right\rceil+b$ in $G$; containing $T_{12}, T-e_{1}-e_{2}$ and at least a half of $T_{11}$.

We remark that a similar result has been proved for vertex partitions and $\mathcal{W}_{k}$ in [5].

Theorem 4.2. For all positive integers $k$ and $n \geq 9$, $\rho_{\mathcal{W}_{n}^{*}}^{\prime}\left(\mathcal{W}_{k}^{*}\right) \leq\left\lceil\frac{3 k}{2 n-6}\right\rceil$.
$\boldsymbol{P r o o f}$. From Theorem 4.1 it follows by induction on $c$ that $\mathcal{W}_{c\left\lceil\frac{2 n-6}{3}\right\rceil} \subseteq$ $c \mathcal{W}_{n}^{*}$ for all positive integers $c$ and $n$. Now, with $c=\left\lceil\frac{3 k}{2 n-6}\right\rceil$ we have that $\mathcal{W}_{k}^{*} \subseteq \mathcal{W}_{c\left\lceil\frac{2 n-6}{3}\right\rceil}^{*} \subseteq c \mathcal{W}_{n}^{*}$.

Theorem 4.3. For all positive integers $k$ and $n \geq 2,\left\lfloor\frac{k-2}{n-1}\right\rfloor+1 \leq \rho_{\mathcal{W}_{n}}^{\prime}\left(\mathcal{W}_{k}\right)$ $\leq 2 k$.

Proof. We first show that $\mathcal{W}_{2 a c+2} \nsubseteq c \mathcal{W}_{2 a+1}$ for every positive integer $c$ : Clearly, $G=K_{a c+1, t} \in \mathcal{W}_{2 a c+2}$ for every $t$. Let $t$ be large and suppose that $G \in c \mathcal{W}_{2 a+1}$. Let $\left\{E_{1}, E_{2}, \ldots, E_{c}\right\}$ be a corresponding decomposition
of $E(G)$. As in the proof of Theorem 3.2 we get, if $t$ is large enough, for some $i \in\{1,2, \ldots, c\}$ that $K_{a+1, a+2} \subseteq G\left[E_{i}\right]$, a contradiction.

Now let $a=\frac{n-1}{2}$ and $c=\left\lfloor\frac{k-2}{n-1}\right\rfloor$. Since $k \geq 2 a c+2$ we have $\mathcal{W}_{k} \supseteq$ $\mathcal{W}_{2 a c+2} \nsubseteq c \mathcal{W}_{n}$. Therefore $\rho_{\mathcal{W}_{n}}^{\prime}\left(\mathcal{W}_{k}\right) \geq c+1$.

For the upper bound we have $\mathcal{W}_{k} \subseteq \mathcal{D}_{k} \subseteq k \mathcal{D}_{1} \subseteq 2 k \mathcal{W}_{2} \subseteq 2 k \mathcal{W}_{n}$ from Theorem 3.3 and the well-known fact that every tree has a $2\left(\mathcal{W}_{2} \cap \mathcal{D}_{1}\right)$ edge decomposition.

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