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EFFECT OF EDGE-SUBDIVISION ON VERTEX-DOMINATION IN A GRAPH

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Abstract

Let G be a graph with $\Delta(G) > 1$. It can be shown that the domination number of the graph obtained from G by subdividing every edge exactly once is more than that of G. So, let $\xi(G)$ be the least number of edges such that subdividing each of these edges exactly once results in a graph whose domination number is more than that of G. The parameter $\xi(G)$ is called the *subdivision number* of G. This notion has been introduced by S. Arumugam and S. Velammal. They have conjectured that for any graph G with $\Delta(G) > 1$, $\xi(G) \leq 3$. We show that the conjecture is false and construct for any positive integer $n \geq 3$, a graph G of order n with $\xi(G) > \frac{1}{3}\log_2 n$. The main results of this paper are the following: (i) For any connected graph G with at least three vertices, $\xi(G) \leq \gamma(G) + 1$ where $\gamma(G)$ is the domination number of G. (ii) If G is a connected graph of sufficiently large order n, then $\xi(G) \leq 4\sqrt{n}\,\ln n + 5$.

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1. Introduction

All graphs considered in this paper are finite and have neither loops nor multiple edges. For definitions not given here and notations not explained, we refer to [2]. For a graph G, unless otherwise specified, $V(G)$ and $E(G)$ denote respectively the vertex-set and the edge-set of G.

Let $G = (V, E)$ be a graph. For any $a \in V$, its neighbourhood—the set of all vertices which are joined to a—is denoted by $N(a)$. (Sometimes it is denoted by $N_G(a)$ to avoid ambiguity when more graphs are under consideration.) The closed neighbourhood of a—the set $N(a) \cup \{a\}$—is denoted by $N[a]$. Its degree—the number of vertices in $N(a)$—is denoted by $\deg a$. Occasionally we use $I(a)$ or $I_G(a)$ to denote the set of all edges incident with a. By $\delta(G)$ and $\Delta(G)$, we mean $\min_{x\in V} \deg x$ and $\max_{x\in V} \deg x$ respectively. For any $A \subseteq V$, $N(A) = \cup_{x\in A} N(x)$. The induced subgraph defined on A is denoted by $G[A]$.

A *dominating set* of a graph G with vertex-set V, is a subset D of V such that each vertex of $V - D$ has a neighbour in D. The *domination number* of G is the least number that can be the cardinality of a dominating set. The domination number of a graph G is denoted by $\gamma(G)$ or simply γ when there is no ambiguity regarding the graph whose domination number is referred to by γ. (This convention will be adopted for other parameters also.)

Remark 1.1. Let G be a connected graph with at least two vertices. Since any spanning tree is bipartite, $V(G)$ has a bipartition $\{X, Y\}$ such that every vertex of X has a neighbour in Y and vice versa. Therefore both X and Y are dominating sets of G and it follows that $\gamma(G) \leq \min\{|X|, |Y|\} \leq \frac{1}{2}|V(G)|$.

Definition 1.2. Let G be a graph and uv be an edge of G. By *subdividing* the edge uv we mean forming a graph H from G by adding a new vertex w and replacing the edge uv by uw and wv. (Formally, $V(H) = V(G) \cup \{w\}$ and $E(H) = (E(G) - \{uv\}) \cup \{uw, wv\}$.) The graph obtained from G by subdividing each edge exactly once is denoted by $S(G)$.

Remark 1.3. If G is a graph and H is any graph obtained from G by subdividing some edges of G, then $\gamma(H) \geq \gamma(G)$. (From a minimum dominating set D of H, by replacing each vertex x of $D - V(G)$ by a vertex of $V(G)$ which is adjacent to x, we get a dominating set D' of G such that $|D'| \leq |D|$.)

In [6] it has been observed that for a connected graph G with at least 3 vertices, $\gamma(S(G)) > \gamma(G)$. (A lengthy argument has been given to prove this. A simpler proof is the following: Let V and E be respectively the vertex-set and the edge-set of G and n be the number of vertices. Let D be a minimum dominating set of $S(G)$. Let $D_1 = V \cap D$ and $D_2 = D - D_1$. In $S(G)$, since each vertex of D_1 dominates exactly one vertex of V and each vertex of D_2 dominates exactly two vertices of V, it follows that $|D_1| + 2\,|D_2| \geq n$. If $D_1 \neq \emptyset$, then $2\gamma(S(G)) = 2\,|D_1| + 2\,|D_2| \geq n+1$; otherwise, $D = V' - V$ where V' is the vertex-set of $S(G)$ and $\gamma(S(G)) = |V' - V| = |E| \geq n - 1$. In either case, $\gamma(S(G)) > \frac{n}{2}$ and by Remark 1.1, it follows that $\gamma(S(G)) > \gamma(G)$.)

By the above observation, obviously for any graph G with $\Delta > 1$, $\gamma(G) < \gamma(S(G))$. This has prompted S. Arumugam to ask the following question: For a graph G with $\Delta > 1$, what is the minimum number of edges to be subdivided exactly once so that the domination number of the resulting graph exceeds that of G?

Definition 1.4. Let G be a graph with $\Delta > 1$. The least number that can be the cardinality of a set of edges such that subdividing each of them exactly once results in a graph with domination number more than that of G, is called the *subdivision number* of G and is denoted by $\xi(G)$.

In [6], S. Velammal has computed the above parameter for a number of graphs. An interesting result of [6] in this regard is the following.

Proposition 1.5. *For any tree T of order ≥ 3, $\xi \leq 3$.*

Finding that $\xi \leq 3$ holds for each of the graphs considered in this regard in [6], S. Arumugam and S. Velammal have conjectured that for any connected graph G with at least 3 vertices, $\xi(G) \leq 3$.

In [3], an upper bound for the subdivision number of a graph in terms of the minimum degrees of adjacent vertices has been found.

In this paper we show that the above conjecture is false by exhibiting a graph with $\xi > 3$. Using the method for constructing this graph we prove the following result.

Proposition 1.6. *For any integer $n \geq 3$, there exists a graph of order n such that $\xi > \frac{1}{3}\log_2 n$.*

The main results of this paper are the following theorems.

Theorem 1.7. *For a connected graph with at least* 3 *vertices,* $\xi \leq \gamma + 1$.

In [4], a different proof of the above result is given.

Theorem 1.8. *For a connected graph of large order* n, $\xi \leq 4\sqrt{n}\, \ln n + 5$.

We also give a proof of Proposition 1.5, since the argument given in [6] to prove this result is incorrect.

2. Results

First let us prove Proposition 1.5.

If T is a path, then it is easy to verify that the conclusion holds. So, assume that $\Delta(T) \geq 3$. If $P = (v_0, v_1, \ldots, v_n)$ is a path in T such that $\deg v_0 > 2$, $\deg v_i = 2$ for $0 < i < n$ and $\deg v_n = 1$, then P is said to be a *hanging path* and v_0 is called the *support* of P. If any hanging path is of length more than 2, then subdividing three of its edges shows that $\xi(T) \leq 3$. So we assume the following.

(∗∗) Length of any hanging path is at most 2.

Clearly removal of all the hanging paths but retaining their supports yields a tree T'. Let u be a pendant vertex of T'. Then u supports at least two hanging paths. Now by (∗∗) we have two cases.

Case a. u is incident with a pendant edge of T.
Subdivide this pendant edge. If u is incident with one more pendant edge of T, then we find that $\xi(T) = 1$; otherwise subdividing the two edges of any other hanging path supported by u shows that $\xi(T) \leq 3$.

Case b. Every hanging path supported by u is of length 2.
Now subdivide the two edges of one hanging path supported by u. If $V(T') = \{u\}$, then we find that $\xi(T) = 2$. Otherwise, subdividing the edge of T' which is incident with u shows that $\xi(T) \leq 3$.

This completes the proof. ∎

Remark 2.1. Let T be as above and H be any graph. If a graph G is formed by joining a pendant vertex a of T with a vertex b of H (formally $V(G) = V(T) \cup V(H)$, $V(T) \cap V(H) = \emptyset$ and $E(G) = E(T) \cup E(H) \cup$

$\{ab\}$), then $\xi(G) \leq 3$. The above proof works with slight modification just before choosing u: We can assume that $|V(T')| > 1$ for otherwise T is simply a graph obtained from a star by subdividing some of its edges and the conclusion can be easily verified; now let u be a pendant vertex of T' such that a does not lie on any hanging path supported by u. (Note that hanging paths supported by different vertices are vertex-joint.) With this modification, in Case b the possibility that $V(T') = \{u\}$ does not arise.

Disproving the Conjecture. Now let us construct a graph with $\xi > 3$. Let $X = \{1, 2, \ldots, 10\}$. Let $\mathcal{S} = \{A \subset X : |A| = 4\}$. $\mathcal{S}$ has $\binom{10}{4}$ elements. Let G be the bipartite graph with bipartition $\{X, \mathcal{S}\}$ and adjacency defined as follows: For any $x \in X$ and $A \in \mathcal{S}$, x is adjacent to $A \iff x \in A$.

Let D be any dominating set of G. If $|D \cap X| \leq 4$, then $|D| \geq |D \cap \mathcal{S}| \geq \binom{6}{4}$. If $|D \cap X| = 5$, then $|D \cap \mathcal{S}| \geq 5$ implying $|D| \geq 10$. If $|D \cap X| = 6$, then $D \cap \mathcal{S} \neq \emptyset$ implying $|D| \geq 7$. Therefore it can be easily seen that

$$\gamma(G) = 7. \tag{1}$$

Let $\alpha_i A_i$, $1 \leq i \leq 3$ be three edges of G. Let H be the graph obtained from G by subdividing these three edges. For $i = 1, 2, 3$, choose an element $\beta_i \in A_i - \{\alpha_1, \alpha_2, \alpha_3\}$. Let D_1 be a subset of X such that $\{\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3\} \subseteq D_1$ and $|D_1| = 6$. Let $D = D_1 \cup \{X - D_1\}$. It can be verified that D is a dominating set of H. Now by (1) and Remark 1.3, it follows that

$$\gamma(H) = 7. \tag{2}$$

Now (1) and (2) $\Rightarrow \gamma(H) = \gamma(G)$. Therefore $\xi(G) > 3$. ■

Remark 2.2. In the above example taking $X = \{1, 2, \ldots, 9\}$ also works but needs a little more computations.

***Proof of Proposition* 1.6.** Let n be any positive integer. The proposition trivially holds when $n \leq 7$ since $\frac{1}{3}\log_2 n < 1$. When $n = 8$ or 9, we can construct a graph of order n with $\xi = 2$ and the conclusion holds. So, let us assume that $n \geq 10$.

Let k be the positive integer such that

$$3k - 2 + \binom{3k-2}{k} \leq n < 3k + 1 + \binom{3k+1}{k+1}.$$

Note that $k \geq 2$. Now let

$$X = \{1, 2, \ldots, 3k-2\} \text{ and}$$
$$\mathcal{S} = \{A \subset X : |A| = k\} \cup \left\{X \cup \{-i\} : 1 \le i \le n - (3k-2) - \binom{3k-2}{k}\right\}.$$

Let G be as defined in the above example. Then by construction, order of G is n. Let us show that $\gamma(G) = 2k-1$. Let D be a dominating set of G and $\ell = |D \cap X|$. We can assume that $\ell \le 2k-2$. Then

$$\begin{aligned} |D| \ge \ell + \binom{3k-2-\ell}{k} &\ge \ell + 2k - 2 - \ell + 1 \\ &\quad \text{(by using the fact that } \tbinom{k+m}{k} \ge m+1 \text{ when } m \ge 0.) \\ &= 2k-1. \end{aligned}$$

Therefore $\gamma(G) \ge 2k-1$; since a dominating set D of cardinality $2k-1$ can be easily constructed such that $|D \cap X| = 2k-2$, it follows that $\gamma(G) = 2k-1$.

Let $\{\alpha_i A_i : 1 \le i \le k-1\}$ be a set of $k-1$ edges and let H be the graph obtained from G by subdividing these $k-1$ edges. For any $i \le k-1$, choose a positive integer $\beta_i \in A_i - \{\alpha_j : 1 \le j \le k-1\}$. Let D be a subset of X such that $\{\alpha_i : 1 \le i \le k-1\} \cup \{\beta_i : 1 \le i \le k-1\} \subseteq D$ and $|D| = 2k-2$. It can be verified that $D \cup \{X - D\}$ is a dominating set of H. Therefore by Remark 1.3, $\gamma(H) = 2k-1$. Thus we have $\gamma(H) = \gamma(G)$ and it follows that $\xi(G) \ge k$.

Since $n < 3k + 1 + \binom{3k+1}{k+1} = 1 + 3k + \binom{3k}{k} + \binom{3k}{k+1} < 2^{3k}$, we have $3k > \log_2 n$ implying that $\xi > \frac{1}{3}\log_2 n$. This completes the proof. ■

A set M of edges in a graph G is called a *matching* of G (sometimes an *independent set* of edges in G) if no two edges of M have a common end-vertex. The cardinality of a largest matching of G is denoted by $\mu(G)$. The following result is quite well known. (cf. [5, p. 58]; for the sake of completeness, we give a proof of this result.)

Lemma 2.3. *If G is a graph without isolated vertices, then $\gamma \le \mu$.*

Proof. Let M be a maximum matching. Let S be the set of vertices which are not end-vertices of the edges in M. If a is any vertex in S, then a is not joined to any other vertex in S since M is a maximum matching; therefore a is joined to an end-vertex, say x, of an edge in M, since G does not have isolated vertices. Let y be the other end of this edge. If b is any other vertex in S, then b is not joined to y for otherwise $(M - \{xy\}) \cup \{ax, by\}$ would be a matching of size $|M|+1$. Hence it is possible to choose a dominating set D of cardinality μ having exactly one end of each edge in M. Therefore $\gamma \leq \mu$. ■

Remark 2.4. Let G be a graph with vertex set V; suppose A is a subset of V such that $G[V-A]$ has no isolated vertex and $\mu(G[A]) > \gamma(G[A])$. Then because of

$$\begin{aligned} \mu(G) &\geq \mu(G[A]) + \mu(G[V-A]), \\ \gamma(G) &\leq \gamma(G[A]) + \gamma(G[V-A]) \text{ and} \\ \mu(G[V-A]) &\geq \gamma(G[V-A]) \text{ (by Lemma 2.3)} \end{aligned}$$

we have $\mu(G) > \gamma(G)$.

Lemma 2.5. *Suppose G is a graph with vertex-set V which can be partitioned as $\{A_1, B_1, A_2, B_2\}$ such that the following hold:*

- *For $i = 1, 2$, every vertex of A_i is adjacent to every vertex of B_i.*
- *$|A_1|, |A_2| \geq 2$, $|B_1| \geq 3$ and $|B_2| \geq 1$.*
- *A vertex of B_1 is adjacent to a vertex of A_2.*

Then $\mu(G) > \gamma(G)$.

Proof. If $|A_2| = 2$ or $|B_2| = 1$ then $\gamma(G) \leq 3$ and $\mu(G) \geq 4$. So suppose $|A_2| \geq 3$ and $|B_2| \geq 2$. Then $\mu(G) \geq 5$ and $\gamma(G) \leq 4$. Thus it follows that $\mu(G) > \gamma(G)$. ■

Definition 2.6. Let G be a graph with vertex-set V; a subset X of V is said to be *modular* in G, if all the vertices in X have same neighbourhood and $G[V - (X \cup N(X))]$ has no isolated vertices. If in addition X dominates G, then G is called a *module*; G is a *proper module* if $|X| \geq 2$ and $|N(X)| \geq 3$.

Note that any modular set X of a graph is *independent*; i.e., no two vertices of X are adjacent. If $G = (V, E)$ is a module with $E \neq \emptyset$, then $\gamma(G) \leq 2$. (G can be imagined as a graph obtained from the complete bipartite graph with bipartition $\{X, V - X\}$ by adding edges having end-vertices in $V - X$ only.)

Lemma 2.7. *For a graph G without isolated vertices, one of the following holds.*

(i) $\mu(G) > \gamma(G)$.

(ii) *Each connected component is a proper module.*

(iii) *There exists a modular subset A of $V(G)$ such that either $|A| = 1$ or $|N(A)| \leq 2$.*

Proof. By induction; assume that for any graph of order less than that of G, the theorem holds. Let α be any vertex of G such that $\deg \alpha = \delta(G)$. Let $A = \{x \in V(G) : N(x) = N(\alpha)\}$. Let $H = G[V(G) - (A \cup N(A))]$. If $V(H) = \emptyset$ then G is a module and therefore either (ii) or (iii) holds.
When $V(H) \neq \emptyset$, by the construction of A, H has no isolated vertex. We can assume the following for otherwise (iii) holds.

$(**)$ $|A| \geq 2$ and $|N(A)| \geq 3$.

Applying the induction hypothesis for H we have the following cases.

Case 1. $\mu(H) > \gamma(H)$.
Since $G[A \cup N(A)]$ has no isolated vertex, by Remark 2.4, (i) holds.

Case 2. Each component of H is a proper module.
If there is one component J such that $N(V(J)) \cap N(A) \neq \emptyset$, then by Lemma 2.5 and $(**)$, $\mu(G[A \cup N(A) \cup V(J)]) > \gamma(G[A \cup N(A) \cup V(J)])$ and (i) holds by Remark 2.4; otherwise the components of G are those of H and $G[A \cup N(A)]$ and therefore (ii) holds.

Case 3. A subset B of $V(H)$ is modular in H such that either $|B| = 1$ or $|N_H(B)| \leq 2$.
Let $X = A \cup N(A) \cup B \cup N_H(B)$. Note that $G[V(G) - X]$ has no isolated vertex. First suppose $|B| \geq 2$. If there is any edge from $N(A)$ to B then by $(**)$ and Lemma 2.5, $\mu(G[X]) > \gamma(G[X])$ and by Remark 2.4, (i) holds; otherwise $N(B) = N_H(B)$ and (iii) holds with B in place of A.

Now suppose B has only one vertex, say α. If α is not joined to every vertex of $N(A)$, then (iii) holds with B in place of A. So assume that α is joined to every vertex of $N(A)$. Then $\gamma(G[X]) \leq 2$ and $\mu(G[X]) \geq 3$; therefore again we have $\mu(G[X]) > \gamma(G[X])$ and (i) holds. This completes the proof. ∎

***Proof of Theorem* 1.7.** For a graph G, by using induction on its order, let us show the following:

(∗∗) If G is connected and has at least three vertices then there exists a set F of edges of order γ or $\gamma+1$ such that F contains a matching of order γ and subdividing the edges of F results in a graph whose domination number is more than that of G.

When $|V(G)| = 3$, (∗∗) is obvious. So, let $|V(G)| > 3$ and assume that for any graph H with $|V(H)| < |V(G)|$, (∗∗) holds with H in place in G. By Lemma 2.7, we have three cases.

Case 1. G has a matching M of order $\gamma+1$.
Then (∗∗) holds with M in place of F.

Case 2. G is a module.
Then there exists a modular subset A of $V(G)$ such that $V(G) = A \cup N(A)$. If $\gamma(G) = 1$, it is easy to see that (∗∗) holds. So let $\gamma(G) = 2$. Let a, b be two distinct vertices in A.

Suppose $N(A)$ has two adjacent vertices x, y. Since $\gamma(G) = 2$, there must be one more vertex $z \in N(A)$. If $|N(A)| \geq 4$, then $\mu(G) \geq 3$. So, let $N(A) = \{x, y, z\}$. Then neither x nor y is joined to z. If $|A| \geq 3$, then also we have $\mu(G) \geq 3$. So, let $A = \{a, b\}$. Now subdividing the edges of the matching $\{az, xy\}$ shows that $\xi(G) = 2$ and (∗∗) holds.

If $N(A)$ is a set of independent vertices, then subdividing the edges ax, ay, bx where x, y are any two arbitrary vertices in $N(A)$ shows that (∗∗) holds.

Case 3. (iii) of Lemma 2.7 holds.
Let $H_1 = G[A \cup N(A)]$ and $H_2 = G[V(G) - (A \cup N(A))]$. We can assume that $\gamma(H_1) + \gamma(H_2) = \gamma(G)$ for otherwise $\mu(G) \geq \mu(H_1) + \mu(H_2) \geq \gamma(H_1) + \gamma(H_2) > \gamma(G)$ and (∗∗) holds.

Subcase a. $|A| = 1$.
Let $A = \{a\}$ and x be any vertex in $N(A)$.

If H_2 has a component K of order ≥ 3, then by induction hypothesis, there exists a set $F' \subseteq E(K)$ such that (∗∗) holds with K and F' in places of G and F respectively. Since $H_2[V(H_2) - V(K)]$ has no isolated vertex, it has a matching of size $\gamma(H_2) - \gamma(K)$. Now taking $F = \{ax\} \cup F' \cup M$ it can be verified that (∗∗) holds.

So assume that H_2 is a union of copies of K_2. If $\deg a > 1$, then (∗∗) holds with $F = \{ax\} \cup E(H_2)$. So suppose $N(a) = \{x\}$. For any $e \in E(H_2)$, we can assume that both of its end-vertices are not joined to x, for otherwise we would have $\gamma(G) < \mu(G)$. Therefore by connectivity of G, x is joined to exactly one vertex of each edge in $E(H_2)$ and $(**)$ holds with $F = \{ax\} \cup \{xy\} \cup E(H_2)$ where y is a vertex in $V(H_2)$ which is joined to x.

Subcase b. $|A| \geq 2$.
Then $|N(A)| \leq 2$. Let M be any matching in H_2 of size $\gamma(H_2)$. Let a, b be two distinct vertices in A. If $|N(A)| = 1$, then a, b are pendant with the same support, say x; subdividing the edge ax shows that $\xi(G) = 1$ and obviously $(**)$ holds with $F = \{ax\} \cup M$.

So suppose $N(A)$ contains one more vertex, say y. If $\gamma(H_1) = 1$, then $F = \{ax, by\} \cup M$ is a matching of size $\gamma(G) + 1$ and $(**)$ holds; so let $\gamma(H_1) = 2$. Subdividing the edges ax, ay, bx shows that $\xi(G) \leq 3$ and $(**)$ holds with $F = \{ax, ay, bx\} \cup M$. ∎

Now we prove the second main result of this paper. The main tool used in the proof is Alon's result (cf. [1, Page 4]) on domination number of a graph: Any graph G has a dominating set of size $\leq n\frac{1+\ln(\delta+1)}{\delta+1}$ where n is the number of vertices.

***Proof of Theorem* 1.8.** First we settle a few simple cases. (Throughout this proof, we consider a number of cases. Whenever a case is under consideration, it is assumed that the previous cases do not hold.)

Case 1. G has two pendant vertices with same support.

By subdividing one of them, we find $\xi = 1$.

Case 2. There is an edge $e \in E(G)$ such that $G - e$ has two connected components G_1 and G_2 with the property that G_1 is a tree with at least 3 vertices.

Then by Remark 2.1, $\xi \leq 3$.

Case 3. There is a path (u, v, w, x) such that $\deg(u) = \deg(x) = 1$.

Subdividing the three edges of this path shows that $\xi \leq 3$.

So let us assume that none of the above cases holds. Removing all the hanging paths but retaining their supports results in a connected graph G' such that the following hold:

Every pendant vertex in G is connected to a vertex in G' by a path of length at most 2. Any such path of length 1 cannot have a vertex in common with any other path. Any such path of length 2 cannot have an edge in common with any other path.

Let $m = \lceil \sqrt{n}\, \ln n \rceil$. For any pendant vertex u of G let u^* denote its support.

Let $S_1 = \{v \in V(G') \mid \deg(v) \leq m\}$,

$S_2 = \{v \in V(G') \mid \deg(v) > m\}$,

$V_1 = \{v \in V(G) \mid \deg(v) = 1 \text{ and } v^* \in V(G')\}$,

$V_2 = \{v \in V(G) \mid \deg(v) = 1, \deg(v^*) = 2, \text{ and } N(v^*) \cap V(G') \neq \emptyset\}$

and $S = \{v \in S_1 \mid N(v) \cap V_1 \neq \emptyset\}$.

Let ℓ be the number of vertices in V_2 which are joined to vertices in S_2 by paths of length 2. If there is a vertex $v \in V_2$ which is joined by a path of length 2 to a vertex $u \in S_1$; i.e., then by subdividing the edges of this path and all the edges of $E(G')$ which are incident with u, we find that $\xi \leq m+2$.

Let k be the number of vertices in V_1 with supports in S_2. Now let us settle three more cases.

Case 4. There exist $u, v \in S_1$ which are adjacent.

In this case by subdividing all the edges in $I(u) \cup I(v)$ we find that $\xi \leq 2m$.

Case 5. There exist $u \in S_1$ and $v \in S_1 - S$ such that $N(u) \cap N(v) \neq \emptyset$.

Fix a vertex $a \in N(u) \cap N(v)$. Then subdivide all edges in $I_G(u) \cup I_G(v)$. Also for each vertex $x \in (N(u) \cup N(v)) - \{a\}$ we subdivide an edge $xy \in G'$ where $y \notin \{u, v, a\}$. If there is any edge ab with $b \in V_1$, it is also subdivided. If there are paths (a, b, c) with $c \in V_2$ then both the edges of one such path are also subdivided. Hence in this case the subdivision number is at most $4m + 1$.

Case 6. There exist $u, v \in S_1 - S$ and $v_1 v_2 \in E(G')$ with $v_1 \in N(u)$ and $v_2 \in N(v)$.

We subdivide $v_1 v_2$ and the edges in $I_{G'}(u) \cup I_{G'}(v)$. Also for each vertex $x \in N(u) \cup N(v)$ we subdivide an edge $xy \in E(G')$ with $y \notin \{u, v, v_1, v_2\}$. If there is any edge $a_1 a_2 \in E(G')$ with $a_1 \in V_1$ and $a_2 \in \{v_1, v_2\}$ then subdivide it. If there are paths (a_1, a_2, v_1) with $a_1 \in V_2$, then subdivide both the edges of one such path; this is repeated with v_2 in place of v_1. Hence the subdivision number is at most $4m + 5$.

If none of the cases considered so far holds, then by using Alon's result mentioned above we have

$$(*) \qquad \ell + k + |S_1| \le \gamma(G) \le \ell + k + \rho + |S_1|$$

where $\rho = |S_2| \frac{\ln(m+1)+1}{m+1}$.

Case 7. $k > \frac{\sqrt{n}}{\ln n}$.

Then there exist two pendant vertices u, v with supports in S_2 such that $N(N(u)) \cap N(N(v)) \neq \emptyset$. Let x and y be supports of u and v respectively. Let p be a vertex in $N(x) \cap N(y)$. Now we subdivide the edges ux, xp, py and yv. We also subdivide m edges in $I(x)$. If any such edge is incident with a vertex $a \in N(S_1 - S)$ then we also subdivide the edge ab where b is in $S_1 - S$. Now the domination number of the resulting graph is at least $|S_1| + k + \ell + m$ which is more than that of G (if $n \ge 8$). Hence the subdivision number is at most $2m + 4$.

Case 8. $|S_1 - S| \le \frac{m}{4}$.

We take a matching M of size $\min\left(\frac{m}{2}, |V(G') - (S_1 - S)|\right)$ in $G'[V(G') - (S_1 - S)]$ and subdivide all the edges in the matching and all edges uu^* where u^* is an end-vertex of an edge in this matching. For the resulting graph H, the size of the dominating set is at least $\gamma(H) \ge \ell + |S| + \min\left(\frac{m}{2}, |V(G') - (S_1 - S)|\right)$. If $|M| < \frac{m}{2}$ then $\gamma(H) > |S| + \ell = \gamma(G)$; otherwise for $n \ge 235$, by using $(*)$, it can be verified that $\ell + |S| + \frac{m}{2} > \ell + k + \rho + |S_1| \ge \gamma(G)$. Thus, when $n \ge 235$, $\gamma(G) < \gamma(H)$.

Case 9. $|S_1 - S| > \frac{m}{4}$.

First we fix a set $S_1' \subset S_1 - S$ with $|S_1'| = \frac{m}{4}$. Next we get a matching M of size $\frac{m}{4}$ in S_2 such that each edge in M has an end in $N(S_1')$ and for any $a \in S_1'$, there is at most one edge in M having an end-vertex in $N(a)$. Now we subdivide all edges in M and for every vertex $b \in S_1'$ an edge $bb_1 \in E(G')$. We also subdivide all edges of the form $uv \in E(G)$ where u is an end-vertex of an edge in M and $v \in S_1'$. In the resulting graph H the domination number is at least $|S_1| + \frac{m}{4} + \ell$. So as in the last case, if $n \ge 235$ then $\gamma(H) > \gamma(G)$. Therefore the subdivision number in this case is at most $\frac{3m}{4}$.

Thus we conclude that if $n \ge 235$ then $\xi \le 4\sqrt{n}\,\ln n + 5$. ∎

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