# EFFECT OF EDGE-SUBDIVISION ON VERTEX-DOMINATION IN A GRAPH 

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#### Abstract

Let $G$ be a graph with $\Delta(G)>1$. It can be shown that the domination number of the graph obtained from $G$ by subdividing every edge exactly once is more than that of $G$. So, let $\xi(G)$ be the least number of edges such that subdividing each of these edges exactly once results in a graph whose domination number is more than that of $G$. The parameter $\xi(G)$ is called the subdivision number of $G$. This notion has been introduced by S. Arumugam and S. Velammal. They have conjectured that for any graph $G$ with $\Delta(G)>1, \xi(G) \leq 3$. We show that the conjecture is false and construct for any positive integer $n \geq 3$, a graph $G$ of order $n$ with $\xi(G)>\frac{1}{3} \log _{2} n$. The main results of this paper are the following: (i) For any connected graph $G$ with at least three vertices, $\xi(G) \leq \gamma(G)+1$ where $\gamma(G)$ is the domination number of $G$. (ii) If $G$ is a connected graph of sufficiently large order $n$, then $\xi(G) \leq 4 \sqrt{n} \ln n+5$.


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## 1. Introduction

All graphs considered in this paper are finite and have neither loops nor multiple edges. For definitions not given here and notations not explained, we refer to [2]. For a graph $G$, unless otherwise specified, $V(G)$ and $E(G)$ denote respectively the vertex-set and the edge-set of $G$.

Let $G=(V, E)$ be a graph. For any $a \in V$, its neighbourhood-the set of all vertices which are joined to $a$-is denoted by $N(a)$. (Sometimes it is denoted by $N_{G}(a)$ to avoid ambiguity when more graphs are under consideration.) The closed neighbourhood of $a$-the set $N(a) \cup\{a\}$-is denoted by $N[a]$. Its degree - the number of vertices in $N(a)$-is denoted by $\operatorname{deg} a$. Occasionally we use $I(a)$ or $I_{G}(a)$ to denote the set of all edges incident with $a$. By $\delta(G)$ and $\Delta(G)$, we mean $\min _{x \in V} \operatorname{deg} x$ and $\max _{x \in V} \operatorname{deg} x$ respectively. For any $A \subseteq V, N(A)=\cup_{x \in A} N(x)$. The induced subgraph defined on $A$ is denoted by $G[A]$.

A dominating set of a graph $G$ with vertex-set $V$, is a subset $D$ of $V$ such that each vertex of $V-D$ has a neighbour in $D$. The domination number of $G$ is the least number that can be the cardinality of a dominating set. The domination number of a graph $G$ is denoted by $\gamma(G)$ or simply $\gamma$ when there is no ambiguity regarding the graph whose domination number is referred to by $\gamma$. (This convention will be adopted for other parameters also.)

Remark 1.1. Let $G$ be a connected graph with at least two vertices. Since any spanning tree is bipartite, $V(G)$ has a bipartition $\{X, Y\}$ such that every vertex of $X$ has a neighbour in $Y$ and vice versa. Therefore both $X$ and $Y$ are dominating sets of $G$ and it follows that $\gamma(G) \leq \min \{|X|,|Y|\} \leq \frac{1}{2}|V(G)|$.

Definition 1.2. Let $G$ be a graph and $u v$ be an edge of $G$. By subdividing the edge $u v$ we mean forming a graph $H$ from $G$ by adding a new vertex $w$ and replacing the edge $u v$ by $u w$ and $w v$. (Formally, $V(H)=V(G) \cup\{w\}$ and $E(H)=(E(G)-\{u v\}) \cup\{u w, w v\}$.) The graph obtained from $G$ by subdividing each edge exactly once is denoted by $S(G)$.

Remark 1.3. If $G$ is a graph and $H$ is any graph obtained from $G$ by subdividing some edges of $G$, then $\gamma(H) \geq \gamma(G)$. (From a minimum dominating set $D$ of $H$, by replacing each vertex $x$ of $D-V(G)$ by a vertex of $V(G)$ which is adjacent to $x$, we get a dominating set $D^{\prime}$ of $G$ such that $\left|D^{\prime}\right| \leq|D|$.)

In [6] it has been observed that for a connected graph $G$ with at least 3 vertices, $\gamma(S(G))>\gamma(G)$. (A lengthy argument has been given to prove this. A simpler proof is the following: Let $V$ and $E$ be respectively the vertex-set and the edge-set of $G$ and $n$ be the number of vertices. Let $D$ be a minimum dominating set of $S(G)$. Let $D_{1}=V \cap D$ and $D_{2}=D-D_{1}$. In $S(G)$, since each vertex of $D_{1}$ dominates exactly one vertex of $V$ and each vertex of $D_{2}$ dominates exactly two vertices of $V$, it follows that $\left|D_{1}\right|+2\left|D_{2}\right| \geq n$. If $D_{1} \neq \emptyset$, then $2 \gamma(S(G))=2\left|D_{1}\right|+2\left|D_{2}\right| \geq n+1$; otherwise, $D=V^{\prime}-V$ where $V^{\prime}$ is the vertex-set of $S(G)$ and $\gamma(S(G))=\left|V^{\prime}-V\right|=|E| \geq n-1$. In either case, $\gamma(S(G))>\frac{n}{2}$ and by Remark 1.1, it follows that $\gamma(S(G))>$ $\gamma(G)$.)

By the above observation, obviously for any graph $G$ with $\Delta>1$, $\gamma(G)<\gamma(S(G))$. This has prompted S. Arumugam to ask the following question: For a graph $G$ with $\Delta>1$, what is the minimum number of edges to be subdivided exactly once so that the domination number of the resulting graph exceeds that of $G$ ?

Definition 1.4. Let $G$ be a graph with $\Delta>1$. The least number that can be the cardinality of a set of edges such that subdividing each of them exactly once results in a graph with domination number more than that of $G$, is called the subdivision number of $G$ and is denoted by $\xi(G)$.

In [6], S. Velammal has computed the above parameter for a number of graphs. An interesting result of [6] in this regard is the following.

Proposition 1.5. For any tree $T$ of order $\geq 3, \xi \leq 3$.
Finding that $\xi \leq 3$ holds for each of the graphs considered in this regard in [6], S. Arumugam and S. Velammal have conjectured that for any connected graph $G$ with at least 3 vertices, $\xi(G) \leq 3$.

In [3], an upper bound for the subdivision number of a graph in terms of the minimum degrees of adjacent vertices has been found.

In this paper we show that the above conjecture is false by exhibiting a graph with $\xi>3$. Using the method for constructing this graph we prove the following result.

Proposition 1.6. For any integer $n \geq 3$, there exists a graph of order $n$ such that $\xi>\frac{1}{3} \log _{2} n$.

The main results of this paper are the following theorems.

Theorem 1.7. For a connected graph with at least 3 vertices, $\xi \leq \gamma+1$.
In [4], a different proof of the above result is given.
Theorem 1.8. For a connected graph of large order $n, \xi \leq 4 \sqrt{n} \ln n+5$.
We also give a proof of Proposition 1.5, since the argument given in [6] to prove this result is incorrect.

## 2. Results

First let us prove Proposition 1.5.
If $T$ is a path, then it is easy to verify that the conclusion holds. So, assume that $\Delta(T) \geq 3$. If $P=\left(v_{0}, v_{1}, \ldots, v_{n}\right)$ is a path in $T$ such that $\operatorname{deg} v_{0}>2, \operatorname{deg} v_{i}=2$ for $0<i<n$ and $\operatorname{deg} v_{n}=1$, then $P$ is said to be a hanging path and $v_{0}$ is called the support of $P$. If any hanging path is of length more than 2, then subdividing three of its edges shows that $\xi(T) \leq 3$. So we assume the following.
(**) Length of any hanging path is at most 2.
Clearly removal of all the hanging paths but retaining their supports yields a tree $T^{\prime}$. Let $u$ be a pendant vertex of $T^{\prime}$. Then $u$ supports at least two hanging paths. Now by $(* *)$ we have two cases.

Case a. $u$ is incident with a pendant edge of $T$.
Subdivide this pendant edge. If $u$ is incident with one more pendant edge of $T$, then we find that $\xi(T)=1$; otherwise subdividing the two edges of any other hanging path supported by $u$ shows that $\xi(T) \leq 3$.

Case b. Every hanging path supported by $u$ is of length 2 . Now subdivide the two edges of one hanging path supported by $u$. If $V\left(T^{\prime}\right)=$ $\{u\}$, then we find that $\xi(T)=2$. Otherwise, subdividing the edge of $T^{\prime}$ which is incident with $u$ shows that $\xi(T) \leq 3$.

This completes the proof.
Remark 2.1. Let $T$ be as above and $H$ be any graph. If a graph $G$ is formed by joining a pendant vertex $a$ of $T$ with a vertex $b$ of $H$ (formally $V(G)=V(T) \cup V(H), V(T) \cap V(H)=\emptyset$ and $E(G)=E(T) \cup E(H) \cup$
$\{a b\})$, then $\xi(G) \leq 3$. The above proof works with slight modification just before choosing $u$ : We can assume that $\left|V\left(T^{\prime}\right)\right|>1$ for otherwise $T$ is simply a graph obtained from a star by subdividing some of its edges and the conclusion can be easily verified; now let $u$ be a pendant vertex of $T^{\prime}$ such that $a$ does not lie on any hanging path supported by $u$. (Note that hanging paths supported by different vertices are vertex-joint.) With this modification, in Case b the possibility that $V\left(T^{\prime}\right)=\{u\}$ does not arise.

Disproving the Conjecture. Now let us construct a graph with $\xi>3$. Let $X=\{1,2, \ldots, 10\}$. Let $\mathcal{S}=\{A \subset X:|A|=4\}$. $\mathcal{S}$ has $\binom{10}{4}$ elements. Let $G$ be the bipartite graph with bipartition $\{X, \mathcal{S}\}$ and adjacency defined as follows: For any $x \in X$ and $A \in \mathcal{S}, x$ is adjacent to $A \Longleftrightarrow x \in A$.

Let $D$ be any dominating set of $G$. If $|D \cap X| \leq 4$, then $|D| \geq|D \cap \mathcal{S}| \geq$ $\binom{6}{4}$. If $|D \cap X|=5$, then $|D \cap \mathcal{S}| \geq 5$ implying $|D| \geq 10$. If $|D \cap X|=6$, then $D \cap \mathcal{S} \neq \emptyset$ implying $|D| \geq 7$. Therefore it can be easily seen that

$$
\begin{equation*}
\gamma(G)=7 \tag{1}
\end{equation*}
$$

Let $\alpha_{i} A_{i}, 1 \leq i \leq 3$ be three edges of $G$. Let $H$ be the graph obtained from $G$ by subdividing these three edges. For $i=1,2,3$, choose an element $\beta_{i} \in A_{i}-\left\{\alpha_{1}, \alpha_{2}, \alpha_{3}\right\}$. Let $D_{1}$ be a subset of $X$ such that $\left\{\alpha_{1}, \alpha_{2}, \alpha_{3}\right.$, $\left.\beta_{1}, \beta_{2}, \beta_{3}\right\} \subseteq D_{1}$ and $\left|D_{1}\right|=6$. Let $D=D_{1} \cup\left\{X-D_{1}\right\}$. It can be verified that $D$ is a dominating set of $H$. Now by (1) and Remark 1.3, it follows that

$$
\begin{equation*}
\gamma(H)=7 \tag{2}
\end{equation*}
$$

Now (1) and $(2) \Rightarrow \gamma(H)=\gamma(G)$. Therefore $\xi(G)>3$.
Remark 2.2. In the above example taking $X=\{1,2, \ldots, 9\}$ also works but needs a little more computations.

Proof of Proposition 1.6. Let $n$ be any positive integer. The proposition trivially holds when $n \leq 7$ since $\frac{1}{3} \log _{2} n<1$. When $n=8$ or 9 , we can construct a graph of order $n$ with $\xi=2$ and the conclusion holds. So, let us assume that $n \geq 10$.

Let $k$ be the positive integer such that

$$
3 k-2+\binom{3 k-2}{k} \leq n<3 k+1+\binom{3 k+1}{k+1}
$$

Note that $k \geq 2$. Now let

$$
\begin{aligned}
X & =\{1,2, \ldots, 3 k-2\} \text { and } \\
\mathcal{S} & =\{A \subset X:|A|=k\} \cup\left\{X \cup\{-i\}: 1 \leq i \leq n-(3 k-2)-\binom{3 k-2}{k}\right\} .
\end{aligned}
$$

Let $G$ be as defined in the above example. Then by construction, order of $G$ is $n$. Let us show that $\gamma(G)=2 k-1$. Let $D$ be a dominating set of $G$ and $\ell=|D \cap X|$. We can assume that $\ell \leq 2 k-2$. Then

$$
\begin{aligned}
|D| \geq \ell+\binom{3 k-2-\ell}{k} \geq & \ell+2 k-2-\ell+1 \\
& \left.\quad \text { by using the fact that }\binom{k+m}{k} \geq m+1 \text { when } m \geq 0 .\right) \\
= & 2 k-1 .
\end{aligned}
$$

Therefore $\gamma(G) \geq 2 k-1$; since a dominating set $D$ of cardinality $2 k-1$ can be easily constructed such that $|D \cap X|=2 k-2$, it follows that $\gamma(G)=2 k-1$.

Let $\left\{\alpha_{i} A_{i}: 1 \leq i \leq k-1\right\}$ be a set of $k-1$ edges and let $H$ be the graph obtained from $G$ by subdividing these $k-1$ edges. For any $i \leq k-1$, choose a positive integer $\beta_{i} \in A_{i}-\left\{\alpha_{j}: 1 \leq j \leq k-1\right\}$. Let $D$ be a subset of $X$ such that $\left\{\alpha_{i}: 1 \leq i \leq k-1\right\} \cup\left\{\beta_{i}: 1 \leq i \leq k-1\right\} \subseteq D$ and $|D|=2 k-2$. It can be verified that $D \cup\{X-D\}$ is a dominating set of $H$. Therefore by Remark 1.3, $\gamma(H)=2 k-1$. Thus we have $\gamma(H)=\gamma(G)$ and it follows that $\xi(G) \geq k$.

Since $n<3 k+1+\binom{3 k+1}{k+1}=1+3 k+\binom{3 k}{k}+\binom{3 k}{k+1}<2^{3 k}$, we have $3 k>\log _{2} n$ implying that $\xi>\frac{1}{3} \log _{2} n$. This completes the proof.
A set $M$ of edges in a graph $G$ is called a matching of $G$ (sometimes an independent set of edges in $G$ ) if no two edges of $M$ have a common endvertex. The cardinality of a largest matching of $G$ is denoted by $\mu(G)$. The following result is quite well known. (cf. [5, p. 58]; for the sake of completeness, we give a proof of this result.)

Lemma 2.3. If $G$ is a graph without isolated vertices, then $\gamma \leq \mu$.
Proof. Let $M$ be a maximum matching. Let $S$ be the set of vertices which are not end-vertices of the edges in $M$. If $a$ is any vertex in $S$, then $a$ is not joined to any other vertex in $S$ since $M$ is a maximum matching; therefore $a$ is joined to an end-vertex, say $x$, of an edge in $M$, since $G$ does not have isolated vertices. Let $y$ be the other end of this edge. If $b$ is any other vertex in $S$, then $b$ is not joined to $y$ for otherwise $(M-\{x y\}) \cup\{a x, b y\}$ would
be a matching of size $|M|+1$. Hence it is possible to choose a dominating set $D$ of cardinality $\mu$ having exactly one end of each edge in $M$. Therefore $\gamma \leq \mu$.

Remark 2.4. Let $G$ be a graph with vertex set $V$; suppose $A$ is a subset of $V$ such that $G[V-A]$ has no isolated vertex and $\mu(G[A])>\gamma(G[A])$. Then because of

$$
\begin{aligned}
\mu(G) & \geq \mu(G[A])+\mu(G[V-A]), \\
\gamma(G) & \leq \gamma(G[A])+\gamma(G[V-A]) \text { and } \\
\mu(G[V-A]) & \geq \gamma(G[V-A])(\text { by Lemma } 2.3)
\end{aligned}
$$

we have $\mu(G)>\gamma(G)$.
Lemma 2.5. Suppose $G$ is a graph with vertex-set $V$ which can be partitioned as $\left\{A_{1}, B_{1}, A_{2}, B_{2}\right\}$ such that the following hold:

- For $i=1,2$, every vertex of $A_{i}$ is adjacent to every vertex of $B_{i}$.
- $\left|A_{1}\right|,\left|A_{2}\right| \geq 2,\left|B_{1}\right| \geq 3$ and $\left|B_{2}\right| \geq 1$.
- A vertex of $B_{1}$ is adjacent to a vertex of $A_{2}$.

Then $\mu(G)>\gamma(G)$.
Proof. If $\left|A_{2}\right|=2$ or $\left|B_{2}\right|=1$ then $\gamma(G) \leq 3$ and $\mu(G) \geq 4$. So suppose $\left|A_{2}\right| \geq 3$ and $\left|B_{2}\right| \geq 2$. Then $\mu(G) \geq 5$ and $\gamma(G) \leq 4$. Thus it follows that $\mu(G)>\gamma(G)$.

Definition 2.6. Let $G$ be a graph with vertex-set $V$; a subset $X$ of $V$ is said to be modular in $G$, if all the vertices in $X$ have same neighbourhood and $G[V-(X \cup N(X))]$ has no isolated vertices. If in addition $X$ dominates $G$, then $G$ is called a module; $G$ is a proper module if $|X| \geq 2$ and $|N(X)| \geq 3$.

Note that any modular set $X$ of a graph is independent; i.e., no two vertices of $X$ are adjacent. If $G=(V, E)$ is a module with $E \neq \emptyset$, then $\gamma(G) \leq 2$. ( $G$ can be imagined as a graph obtained from the complete bipartite graph with bipartition $\{X, V-X\}$ by adding edges having end-vertices in $V-X$ only.)

Lemma 2.7. For a graph $G$ without isolated vertices, one of the following holds.
(i) $\mu(G)>\gamma(G)$.
(ii) Each connected component is a proper module.
(iii) There exists a modular subset $A$ of $V(G)$ such that either $|A|=1$ or $|N(A)| \leq 2$.

Proof. By induction; assume that for any graph of order less than that of $G$, the theorem holds. Let $\alpha$ be any vertex of $G$ such that $\operatorname{deg} \alpha=\delta(G)$. Let $A=\{x \in V(G): N(x)=N(\alpha)\}$. Let $H=G[V(G)-(A \cup N(A))]$. If $V(H)=\emptyset$ then $G$ is a module and therefore either (ii) or (iii) holds.
When $V(H) \neq \emptyset$, by the construction of $A, H$ has no isolated vertex. We can assume the following for otherwise (iii) holds.
$(* *)|A| \geq 2$ and $|N(A)| \geq 3$.
Applying the induction hypothesis for $H$ we have the following cases.
Case 1. $\mu(H)>\gamma(H)$.
Since $G[A \cup N(A)]$ has no isolated vertex, by Remark 2.4, (i) holds.
Case 2. Each component of $H$ is a proper module.
If there is one component $J$ such that $N(V(J)) \cap N(A) \neq \emptyset$, then by Lemma 2.5 and $(* *), \mu(G[A \cup N(A) \cup V(J)])>\gamma(G[A \cup N(A) \cup V(J)])$ and (i) holds by Remark 2.4; otherwise the components of $G$ are those of $H$ and $G[A \cup N(A)]$ and therefore (ii) holds.

Case 3. A subset $B$ of $V(H)$ is modular in $H$ such that either $|B|=1$ or $\left|N_{H}(B)\right| \leq 2$.
Let $X=A \cup N(A) \cup B \cup N_{H}(B)$. Note that $G[V(G)-X]$ has no isolated vertex. First suppose $|B| \geq 2$. If there is any edge from $N(A)$ to $B$ then by $(* *)$ and Lemma 2.5, $\mu(G[X])>\gamma(G[X])$ and by Remark 2.4, (i) holds; otherwise $N(B)=N_{H}(B)$ and (iii) holds with $B$ in place of $A$.

Now suppose $B$ has only one vertex, say $\alpha$. If $\alpha$ is not joined to every vertex of $N(A)$, then (iii) holds with $B$ in place of $A$. So assume that $\alpha$ is joined to every vertex of $N(A)$. Then $\gamma(G[X]) \leq 2$ and $\mu(G[X]) \geq 3$; therefore again we have $\mu(G[X])>\gamma(G[X])$ and (i) holds. This completes the proof.

Proof of Theorem 1.7. For a graph $G$, by using induction on its order, let us show the following:
$(* *)$ If $G$ is connected and has at least three vertices then there exists a set $F$ of edges of order $\gamma$ or $\gamma+1$ such that $F$ contains a matching of order $\gamma$ and subdividing the edges of $F$ results in a graph whose domination number is more than that of $G$.

When $|V(G)|=3,(* *)$ is obvious. So, let $|V(G)|>3$ and assume that for any graph $H$ with $|V(H)|<|V(G)|,(* *)$ holds with $H$ in place in $G$. By Lemma 2.7, we have three cases.

Case 1. $G$ has a matching $M$ of order $\gamma+1$. Then $(* *)$ holds with $M$ in place of $F$.

Case 2. $G$ is a module.
Then there exists a modular subset $A$ of $V(G)$ such that $V(G)=A \cup N(A)$. If $\gamma(G)=1$, it is easy to see that $(* *)$ holds. So let $\gamma(G)=2$. Let $a, b$ be two distinct vertices in $A$.

Suppose $N(A)$ has two adjacent vertices $x, y$. Since $\gamma(G)=2$, there must be one more vertex $z \in N(A)$. If $|N(A)| \geq 4$, then $\mu(G) \geq 3$. So, let $N(A)=\{x, y, z\}$. Then neither $x$ nor $y$ is joined to $z$. If $|A| \geq 3$, then also we have $\mu(G) \geq 3$. So, let $A=\{a, b\}$. Now subdividing the edges of the matching $\{a z, x y\}$ shows that $\xi(G)=2$ and $(* *)$ holds.

If $N(A)$ is a set of independent vertices, then subdividing the edges $a x, a y, b x$ where $x, y$ are any two arbitrary vertices in $N(A)$ shows that $(* *)$ holds.

Case 3. (iii) of Lemma 2.7 holds.
Let $H_{1}=G[A \cup N(A)]$ and $H_{2}=G[V(G)-(A \cup N(A))]$. We can assume that $\gamma\left(H_{1}\right)+\gamma\left(H_{2}\right)=\gamma(G)$ for otherwise $\mu(G) \geq \mu\left(H_{1}\right)+\mu\left(H_{2}\right) \geq \gamma\left(H_{1}\right)+$ $\gamma\left(H_{2}\right)>\gamma(G)$ and $(* *)$ holds.

Subcase a. $|A|=1$.
Let $A=\{a\}$ and $x$ be any vertex in $N(A)$.
If $H_{2}$ has a component $K$ of order $\geq 3$, then by induction hypothesis, there exists a set $F^{\prime} \subseteq E(K)$ such that $(* *)$ holds with $K$ and $F^{\prime}$ in places of $G$ and $F$ respectively. Since $H_{2}\left[V\left(H_{2}\right)-V(K)\right]$ has no isolated vertex, it has a matching of size $\gamma\left(H_{2}\right)-\gamma(K)$. Now taking $F=\{a x\} \cup F^{\prime} \cup M$ it can be verified that $(* *)$ holds.

So assume that $H_{2}$ is a union of copies of $K_{2}$. If $\operatorname{deg} a>1$, then $(* *)$ holds with $F=\{a x\} \cup E\left(H_{2}\right)$. So suppose $N(a)=\{x\}$. For any $e \in E\left(H_{2}\right)$, we can assume that both of its end-vertices are not joined to $x$,
for otherwise we would have $\gamma(G)<\mu(G)$. Therefore by connectivity of $G$, $x$ is joined to exactly one vertex of each edge in $E\left(H_{2}\right)$ and ( $* *$ ) holds with $F=\{a x\} \cup\{x y\} \cup E\left(H_{2}\right)$ where $y$ is a vertex in $V\left(H_{2}\right)$ which is joined to $x$.

Subcase b. $|A| \geq 2$.
Then $|N(A)| \leq 2$. Let $M$ be any matching in $H_{2}$ of size $\gamma\left(H_{2}\right)$. Let $a, b$ be two distinct vertices in $A$. If $|N(A)|=1$, then $a, b$ are pendant with the same support, say $x$; subdividing the edge $a x$ shows that $\xi(G)=1$ and obviously ( $* *$ ) holds with $F=\{a x\} \cup M$.

So suppose $N(A)$ contains one more vertex, say $y$. If $\gamma\left(H_{1}\right)=1$, then $F=\{a x, b y\} \cup M$ is a matching of size $\gamma(G)+1$ and $(* *)$ holds; so let $\gamma\left(H_{1}\right)=2$. Subdividing the edges $a x, a y, b x$ shows that $\xi(G) \leq 3$ and $(* *)$ holds with $F=\{a x, a y, b x\} \cup M$.
Now we prove the second main result of this paper. The main tool used in the proof is Alon's result (cf. [1, Page 4]) on domination number of a graph: Any graph $G$ has a dominating set of size $\leq n \frac{1+\ln (\delta+1)}{\delta+1}$ where $n$ is the number of vertices.

Proof of Theorem 1.8. First we settle a few simple cases. (Throughout this proof, we consider a number of cases. Whenever a case is under consideration, it is assumed that the previous cases do not hold.)

Case 1. $G$ has two pendant vertices with same support.
By subdividing one of them, we find $\xi=1$.
Case 2. There is an edge $e \in E(G)$ such that $G-e$ has two connected components $G_{1}$ and $G_{2}$ with the property that $G_{1}$ is a tree with at least 3 vertices.

Then by Remark 2.1, $\xi \leq 3$.
Case 3. There is a path $(u, v, w, x)$ such that $\operatorname{deg}(u)=\operatorname{deg}(x)=1$.
Subdividing the three edges of this path shows that $\xi \leq 3$.
So let us assume that none of the above cases holds. Removing all the hanging paths but retaining their supports results in a connected graph $G^{\prime}$ such that the following hold:

Every pendant vertex in $G$ is connected to a vertex in $G^{\prime}$ by a path of length at most 2. Any such path of length 1 cannot have a vertex in
common with any other path. Any such path of length 2 cannot have an edge in common with any other path.

Let $m=\lceil\sqrt{n} \ln n\rceil$. For any pendant vertex $u$ of $G$ let $u^{*}$ denote its support.
Let $S_{1}=\left\{v \in V\left(G^{\prime}\right) \mid \operatorname{deg}(v) \leq m\right\}$,
$S_{2}=\left\{v \in V\left(G^{\prime}\right) \mid \operatorname{deg}(v)>m\right\}$,
$V_{1}=\left\{v \in V(G) \mid \operatorname{deg}(v)=1\right.$ and $\left.v^{*} \in V\left(G^{\prime}\right)\right\}$,
$V_{2}=\left\{v \in V(G) \mid \operatorname{deg}(v)=1, \operatorname{deg}\left(v^{*}\right)=2\right.$, and $\left.N\left(v^{*}\right) \cap V\left(G^{\prime}\right) \neq \emptyset\right\}$ and $S=\left\{v \in S_{1} \mid N(v) \cap V_{1} \neq \emptyset\right\}$.

Let $\ell$ be the number of vertices in $V_{2}$ which are joined to vertices in $S_{2}$ by paths of length 2 . If there is a vertex $v \in V_{2}$ which is joined by a path of length 2 to a vertex $u \in S_{1}$; i.e., then by subdividing the edges of this path and all the edges of $E\left(G^{\prime}\right)$ which are incident with $u$, we find that $\xi \leq m+2$.

Let $k$ be the number of vertices in $V_{1}$ with supports in $S_{2}$. Now let us settle three more cases.

Case 4. There exist $u, v \in S_{1}$ which are adjacent.
In this case by subdividing all the edges in $I(u) \cup I(v)$ we find that $\xi \leq 2 m$.
Case 5. There exist $u \in S_{1}$ and $v \in S_{1}-S$ such that $N(u) \cap N(v) \neq \emptyset$.
Fix a vertex $a \in N(u) \cap N(v)$. Then subdivide all edges in $I_{G}(u) \cup I_{G}(v)$. Also for each vertex $x \in(N(u) \cup N(v))-\{a\}$ we subdivide an edge $x y \in G^{\prime}$ where $y \notin\{u, v, a\}$. If there is any edge $a b$ with $b \in V_{1}$, it is also subdivided. If there are paths ( $a, b, c$ ) with $c \in V_{2}$ then both the edges of one such path are also subdivided. Hence in this case the subdivision number is at most $4 m+1$.

Case 6. There exist $u, v \in S_{1}-S$ and $v_{1} v_{2} \in E\left(G^{\prime}\right)$ with $v_{1} \in N(u)$ and $v_{2} \in N(v)$.

We subdivide $v_{1} v_{2}$ and the edges in $I_{G^{\prime}}(u) \cup I_{G^{\prime}}(v)$. Also for each vertex $x \in N(u) \cup N(v)$ we subdivide an edge $x y \in E\left(G^{\prime}\right)$ with $y \notin\left\{u, v, v_{1}, v_{2}\right\}$. If there is any edge $a_{1} a_{2} \in E\left(G^{\prime}\right)$ with $a_{1} \in V_{1}$ and $a_{2} \in\left\{v_{1}, v_{2}\right\}$ then subdivide it. If there are paths $\left(a_{1}, a_{2}, v_{1}\right)$ with $a_{1} \in V_{2}$, then subdivide both the edges of one such path; this is repeated with $v_{2}$ in place of $v_{1}$. Hence the subdivision number is at most $4 m+5$.

If none of the cases considered so far holds, then by using Alon's result mentioned above we have

$$
\begin{equation*}
\ell+k+\left|S_{1}\right| \leq \gamma(G) \leq \ell+k+\rho+\left|S_{1}\right| \tag{*}
\end{equation*}
$$

where $\rho=\left|S_{2}\right| \frac{\ln (m+1)+1}{m+1}$.
Case 7. $k>\frac{\sqrt{n}}{\ln n}$.
Then there exist two pendant vertices $u, v$ with supports in $S_{2}$ such that $N(N(u)) \cap N(N(v)) \neq \emptyset$. Let $x$ and $y$ be supports of $u$ and $v$ respectively. Let $p$ be a vertex in $N(x) \cap N(y)$. Now we subdivide the edges $u x, x p, p y$ and $y v$. We also subdivide $m$ edges in $I(x)$. If any such edge is incident with a vertex $a \in N\left(S_{1}-S\right)$ then we also subdivide the edge $a b$ where $b$ is in $S_{1}-S$. Now the domination number of the resulting graph is at least $\left|S_{1}\right|+k+\ell+m$ which is more than that of $G$ (if $n \geq 8$ ). Hence the subdivision number is at most $2 m+4$.

Case 8. $\left|S_{1}-S\right| \leq \frac{m}{4}$.
We take a matching $M$ of size $\min \left(\frac{m}{2},\left|V\left(G^{\prime}\right)-\left(S_{1}-S\right)\right|\right)$ in $G^{\prime}\left[V\left(G^{\prime}\right)-\right.$ $\left(S_{1}-S\right)$ ] and subdivide all the edges in the matching and all edges $u u^{*}$ where $u^{*}$ is an end-vertex of an edge in this matching. For the resulting graph $H$, the size of the dominating set is at least $\gamma(H) \geq \ell+|S|+$ $\min \left(\frac{m}{2},\left|V\left(G^{\prime}\right)-\left(S_{1}-S\right)\right|\right)$. If $|M|<\frac{m}{2}$ then $\gamma(H)>|S|+\ell=\gamma(G)$; otherwise for $n \geq 235$, by using $(*)$, it can be verified that $\ell+|S|+\frac{m}{2}>$ $\ell+k+\rho+\left|S_{1}\right| \geq \gamma(G)$. Thus, when $n \geq 235, \gamma(G)<\gamma(H)$.

Case 9. $\left|S_{1}-S\right|>\frac{m}{4}$.
First we fix a set $S_{1}^{\prime} \subset S_{1}-S$ with $\left|S_{1}^{\prime}\right|=\frac{m}{4}$. Next we get a matching $M$ of size $\frac{m}{4}$ in $S_{2}$ such that each edge in $M$ has an end in $N\left(S_{1}^{\prime}\right)$ and for any $a \in S_{1}^{\prime}$, there is at most one edge in $M$ having an end-vertex in $N(a)$. Now we subdivide all edges in $M$ and for every vertex $b \in S_{1}^{\prime}$ an edge $b b_{1} \in E\left(G^{\prime}\right)$. We also subdivide all edges of the form $u v \in E(G)$ where $u$ is an end-vertex of an edge in $M$ and $v \in S_{1}^{\prime}$. In the resulting graph $H$ the domination number is at least $\left|S_{1}\right|+\frac{m}{4}+\ell$. So as in the last case, if $n \geq 235$ then $\gamma(H)>\gamma(G)$. Therefore the subdivision number in this case is at most $\frac{3 m}{4}$.

Thus we conclude that if $n \geq 235$ then $\xi \leq 4 \sqrt{n} \ln n+5$.

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