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A CHARACTERIZATION OF ROMAN TREES

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Abstract

A Roman dominating function (RDF) on a graph $G = (V, E)$ is a function $f: V \to \{0, 1, 2\}$ satisfying the condition that every vertex u for which $f(u) = 0$ is adjacent to at least one vertex v for which $f(v) = 2$. The weight of f is $w(f) = \sum_{v \in V} f(v)$. The Roman domination number is the minimum weight of an RDF in G. It is known that for every graph G, the Roman domination number of G is bounded above by twice its domination number. Graphs which have Roman domination number equal to twice their domination number are called Roman graphs. At the Ninth Quadrennial International Conference on Graph Theory, Combinatorics, Algorithms, and Applications held at Western Michigan University in June 2000, Stephen T. Hedetniemi in his principal talk entitled "Defending the Roman Empire" posed the open problem of characterizing the Roman trees. In this paper, we give a characterization of Roman trees.

Keywords: dominating set, Roman dominating function.

2000 Mathematics Subject Classification: 05C069.

1. Introduction

Cockayne, Dreyer, Hedetniemi, and Hedetniemi [1] defined a *Roman dominating function* (RDF) on a graph $G = (V, E)$ to be a function $f: V \to$

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$\{0, 1, 2\}$ satisfying the condition that every vertex u for which $f(u) = 0$ is adjacent to at least one vertex v for which $f(v) = 2$. For a real-valued function $f: V \to R$ the *weight* of f is $w(f) = \sum_{v \in V} f(v)$, and for $S \subseteq V$ we define $f(S) = \sum_{v \in S} f(v)$, so $w(f) = f(V)$. The *Roman domination number*, denote $\gamma_R(G)$, is the minimum weight of an RDF in G; that is, $\gamma_R(G) = \min\{w(f) \mid f \text{ is an RDF in } G\}$. An RDF of weight $\gamma_R(G)$ we call a $\gamma_R(G)$-*function*.

This definition of a Roman dominating function was motivated by an article in *Scientific American* by Ian Stewart entitled "Defend the Roman Empire" [6]. Each vertex in our graph represents a location in the Roman Empire. A location (vertex v) is considered *unsecured* if no legions are stationed there (i.e., $f(v) = 0$) and *secured* otherwise (i.e., if $f(v) \in \{1, 2\}$). An unsecured location (vertex v) can be secured by sending a legion to v from an adjacent location (an adjacent vertex u). But Emperor Constantine the Great, in the fourth century A.D., decreed that a legion cannot be sent from a secured location to an unsecured location if doing so leaves that location unsecured. Thus, two legions must be stationed at a location ($f(v) = 2$) before one of the legions can be sent to an adjacent location. In this way, Emperor Constantine the Great can defend the Roman Empire. Since it is expensive to maintain a legion at a location, the Emperor would like to station as few legions as possible, while still defending the Roman Empire. A Roman dominating function of weight $\gamma_R(G)$ corresponds to such an optimal assignment of legions to locations.

It is shown in [1] that for every graph G, the Roman domination number of G is bounded above by twice its domination number. Graphs which have Roman domination number equal to twice their domination number are called *Roman graphs*. At the Ninth Quadrennial International Conference on Graph Theory, Combinatorics, Algorithms, and Applications held at Western Michigan University in June 2000, Stephen T. Hedetniemi in his principal talk entitled "Defending the Roman Empire" posed the open problem of characterizing the Roman trees (see [1, 2, 5]).

Our aim in this paper is to give a characterization of Roman trees.

2. Notation

For notation and graph theory terminology we in general follow [3]. Specifically, let $G = (V, E)$ be a graph with vertex set V of order n and edge set E, and let v be a vertex in V. The open neighborhood of v is $N(v) =$

$\{u \in V \mid uv \in E\}$ and the closed neighborhood of v is $N[v] = \{v\} \cup N(v)$. For a set $S \subseteq V$, its *open neighborhood* $N(S) = \cup_{v \in S} N(v)$ and its *closed neighborhood* $N[S] = N(S) \cup S$. A vertex u is called a *private neighbor of* v *with respect to* S, or simply an S-pn of v, if $N[u] \cap S = \{v\}$. The set $\mathrm{pn}(v, S) = N[v] - N[S - \{v\}]$ of all S-pns of v is called the *private neighbor set of* v *with respect to* S. We define the *external private neighbor set of* v *with respect to* S by $\mathrm{epn}(v, S) = \mathrm{pn}(v, S) - \{v\}$. Hence, the set $\mathrm{epn}(v, S)$ consists of all S-pns of v that belong to $V - S$.

For ease of presentation, we mostly consider *rooted trees*. For a vertex v in a (rooted) tree T, we let $C(v)$ and $D(v)$ denote the set of children and descendants, respectively, of v, and we define $D[v] = D(v) \cup \{v\}$. The maximal subtree at v is the subtree of T induced by $D[v]$, and is denoted by T_v. A *leaf* of T is a vertex of degree 1, while a *support vertex* of T is a vertex adjacent to a leaf. We denote the set of support vertices of T by $S(T)$. A *strong support vertex* is adjacent to at least two leaves.

Let $G = (V, E)$ be a graph and let $S \subseteq V$. A set S dominates a set U, denoted $S \succ U$, if every vertex in U is adjacent to a vertex of S. If $S \succ V - S$, then S is called a *dominating set* of G. The *domination number* $\gamma(G)$ is the minimum cardinality of a dominating set of G. A dominating set of cardinality $\gamma(G)$ we call a $\gamma(G)$-*set*. Domination and its variations in graphs are now well studied. The literature on this subject has been surveyed and detailed in the two books by Haynes, Hedetniemi, and Slater [3, 4].

3. The Family $\mathcal{T}$

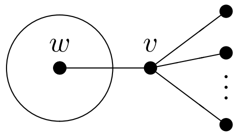
We describe a procedure to build trees. For this purpose, we define two families of trees as follows. Let $\mathcal{F}_1^*$ denote the family of all rooted trees such that every leaf different from the root is at distance 2 from the root and all, except possibly one, child of the root is a strong support vertex. Let $\mathcal{F}_2^*$ denote the family of all rooted trees such that every leaf is at distance 2 from the root and all but two children of the root are strong support vertices.

For a tree T, we let $V_S(T) = \{v \in V(T) \mid v \in S(T) \text{ and } \gamma_R(T - v) \geq \gamma_R(T)\}$. Note that every strong support vertex of T belongs to $V_S(T)$.

Let $\mathcal{T}$ be the family of unlabelled trees T that can be obtained from a sequence $T_1, \ldots, T_j$ ($j \geq 1$) of trees such that T_1 is a star $K_{1,r}$ for $r \geq 1$, and, if $j \geq 2$, T_{i+1} can be obtained recursively from T_i by one of the three operations $\mathcal{T}_1$, $\mathcal{T}_2$ and $\mathcal{T}_3$.

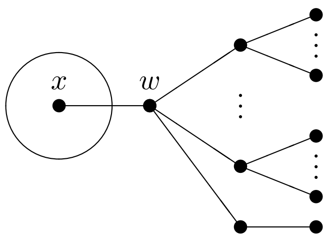
Operation $\mathcal{T}_1$. Assume $w \in V_S(T_i)$. Then the tree T_{i+1} is obtained from T_i by adding a star $K_{1,s}$ for $s \geq 2$ with central vertex v and adding the edge vw.

$\mathcal{T}_1$:



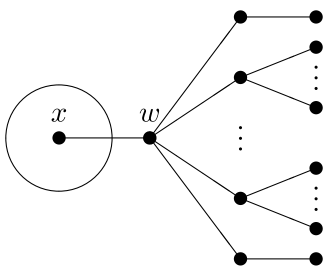
Operation $\mathcal{T}_2$. Assume $x \in V(T_i)$. Then the tree T_{i+1} is obtained from T_i by adding a tree T from the family $\mathcal{F}_1^*$ by adding the edge xw, where w is a leaf of T if $T = P_3$ or w is the central vertex of T if $T \neq P_3$.

$\mathcal{T}_2$:



Operation $\mathcal{T}_3$. Assume $x \in V_S(T_i)$. Then the tree T_{i+1} is obtained from T_i by adding a tree T from the family $\mathcal{F}_2^*$ and adding the edge xw, where w denotes the central vertex of T.

$\mathcal{T}_3$:



4. Preliminary Results

In this section, we use the notation from the definition of the three operations in Section 3. In the proofs of Lemmas 1, 2, and 3, we let f_{i+1} be a $\gamma_R(T_{i+1})$-

function and we let f_i be the restriction of f_{i+1} to T_i; that is, $f_i(u) = f_{i+1}(u)$ for each $u \in V(T_i)$. We may assume that no adjacent vertices of f_{i+1} are both assigned 1, for otherwise we can assign to one vertex the weight 2 and to the other vertex the weight 0. Further, we may assume that f_{i+1} assigns to every strong support vertex the weight 2 and to every leaf adjacent to a strong support vertex the weight 0, and that no leaf is assigned the weight 2 (for otherwise, this weight can simply be shifted up to its parent).

Lemma 1. *If T_i is a Roman tree and if T_{i+1} is obtained from T_i by operation $\mathcal{T}_1$, then T_{i+1} is also a Roman tree.*

Proof. Suppose, to the contrary, that T_{i+1} is not a Roman tree. Then, $w(f_{i+1}) \le 2\gamma(T_{i+1}) - 1$. Since v is a strong support vertex, $f_{i+1}(v) = 2$ and $f_{i+1}(u) = 0$ for each leaf u adjacent to v. Thus, $w(f_i) = w(f_{i+1}) - 2 = \gamma_R(T_{i+1}) - 2 \le 2\gamma(T_{i+1}) - 3$. Any $\gamma(T_i)$-set can be extended to a dominating set of T_{i+1} by adding the vertex v, and so $\gamma(T_{i+1}) \le \gamma(T_i) + 1$. Thus, $w(f_i) \le 2\gamma(T_i) - 1$.

If f_i is an RDF of T_i, then $\gamma_R(T_i) \le w(f_i) < 2\gamma(T_i)$, contradicting the assumption that T_i is a Roman tree. Hence, f_i cannot be an RDF of T_i. Thus it must be the case that $f_i(w) = 0$. Since f_{i+1} is an RDF of T_{i+1}, it follows that f_i must be an RDF of $T_i - w$. Thus, $\gamma_R(T_i - w) \le w(f_i) \le 2\gamma(T_i) - 1$. However, since $w \in V_S(T_i)$ and since T_i is a Roman tree, $\gamma_R(T_i - w) \ge \gamma_R(T_i) = 2\gamma(T_i)$, producing a contradiction. Hence, T_{i+1} must be a Roman tree. ■

Lemma 2. *If T_i is a Roman tree and if T_{i+1} is obtained from T_i by operation $\mathcal{T}_2$, then T_{i+1} is also a Roman tree.*

Proof. Suppose, to the contrary, that T_{i+1} is not a Roman tree. Then, $w(f_{i+1}) \le 2\gamma(T_{i+1}) - 1$.

Suppose that T_{i+1} is obtained from T_i by adding the rooted tree T with root w such that every leaf different from w is at distance 2 from w and all, except possibly one, child of w is a strong support vertex. If $T \neq P_3$, then let $v_1, \ldots, v_k$, where $k \ge 1$, denote the children of w that are strong support vertices.

If $k \ge 1$, then for $j = 1, \ldots, k$, each v_j is a strong support vertex, and so $f_{i+1}(v_j) = 2$ and $f_{i+1}(z) = 0$ for each leaf z adjacent to v_j. Suppose w has a child v of degree 2. Let u be the child of v. Suppose $f_{i+1}(u) = 1$. Then, $f_{i+1}(v) = 0$ and $f_{i+1}(w) = 2$. If $f_{i+1}(x) \ge 1$, then changing the weights assigned to u, v and w to be 0, 2 and 0, respectively, and leaving all other weights unchanged, produces an RDF of T_{i+1} of weight $w(f_{i+1})-1 = \gamma_R(T_{i+1})-1$, which is impossible. Hence, $f_{i+1}(x)=0$. Then, $f\colon V(T_{i+1}) \to \{0,1,2\}$ defined by $f(u)=0$, $f(v)=2$, $f(w)=0$, $f(x)=1$ and $f(z) = f_{i+1}(z)$ for all remaining vertices of T_{i+1} is an RDF of T_{i+1} with $w(f) = w(f_{i+1})$. Hence, we may assume that $f_{i+1}(u)=0$ and that $f_{i+1}(v)=2$.

By our assumptions, every child of w has weight 2, and so we may assume that $f_{i+1}(w)=0$ (if w is assigned a positive weight, then this weight can simply be shifted up to its parent x). Thus, f_i is an RDF of T_i. Hence, $\gamma_R(T_i) \le w(f_i) = w(f_{i+1}) - 2|C(w)| = \gamma_R(T_{i+1}) - 2|C(w)| \le 2\gamma(T_{i+1}) - 2|C(w)| - 1$. Any $\gamma(T_i)$-set can be extended to a dominating set of T_{i+1} by adding the children of w, and so $\gamma(T_{i+1}) \le \gamma(T_i) + |C(w)|$. It follows that $\gamma_R(T_i) \le 2\gamma(T_i)-1$, contradicting the assumption that T_i is a Roman tree. Hence, T_{i+1} must be a Roman tree. ■

Lemma 3. *If T_i is a Roman tree and if T_{i+1} is obtained from T_i by operation $\mathcal{T}_3$, then T_{i+1} is also a Roman tree.*

Proof. Suppose, to the contrary, that T_{i+1} is not a Roman tree. Then, $w(f_{i+1}) \le 2\gamma(T_{i+1})-1$.

Suppose that T_{i+1} is obtained from T_i by adding the rooted tree T with root w such that every leaf different from w is at distance 2 from w and all but two children of w are strong support vertices. Let v_1 and v_2 be the two children of w of degree 2 and let u_1 and u_2 be their respective children. If $|C(w)| = k \ge 3$, then let $C(w) - \{v_1, v_2\} = \{v_3, \ldots, v_k\}$.

We may assume that $f_{i+1}(w)=2$ and, for $i=1,2$, $f_{i+1}(v_i)=0$ and $f_{i+1}(u_i)=1$. If $|C(w)|=k\ge 3$, then for $j=3,\ldots,k$, each v_j is a strong support vertex, and so $f_{i+1}(v_j)=2$ and $f_{i+1}(z)=0$ for each leaf z adjacent to v_j. Then, $w(f_i) = w(f_{i+1}) - 2|C(w)| = \gamma_R(T_{i+1}) - 2|C(w)| \le 2\gamma(T_{i+1}) - 2|C(w)| - 1$. Any $\gamma(T_i)$-set can be extended to a dominating set of T_{i+1} by adding the children of w, and so $\gamma(T_{i+1}) \le \gamma(T_i) + |C(w)|$. It follows that $w(f_i) \le 2\gamma(T_i)-1$.

If f_i is an RDF of T_i, then $\gamma_R(T_i) \le w(f_i) < 2\gamma(T_i)$, contradicting the assumption that T_i is a Roman tree. Hence, f_i cannot be an RDF of T_i. Thus it must be the case that $f_i(x)=0$. Now let f_i' be the restriction of f_i to $T_i - x$. Since f_{i+1} is an RDF of T_{i+1} and $f_i(x)=0$, it follows that f_i' must be an RDF of $T_i - x$. Thus, $\gamma_R(T_i - x) \le w(f_i') = w(f_i) \le 2\gamma(T_i)-1$. However, since $x \in V_S(T_i)$ and since T_i is a Roman tree, $\gamma_R(T_i - x) \ge \gamma_R(T_i) = 2\gamma(T_i)$, producing a contradiction. Hence, T_{i+1} must be a Roman tree. ■

Lemma 4. *If $T \in \mathcal{T}$, then T is a Roman tree.*

Proof. Suppose $T \in \mathcal{T}$. We proceed by induction on $\gamma(T)$. If $\gamma(T) = 1$, then T is a star $K_{1,r}$ for $r \geq 1$, and so T is a Roman tree. Suppose, then, that the result is true for every tree in $\mathcal{T}$ with domination number less than m, where $m \geq 2$. Let $T \in \mathcal{T}$ satisfy $\gamma(T) = m$. Then, T can be obtained from a sequence $T_1, \ldots, T_j$ ($j \geq 1$) of trees such that T_1 is a star $K_{1,r}$ for $r \geq 1$, and, if $j \geq 2$, T_{i+1} can be obtained recursively from T_i by one of the three operations $\mathcal{T}_1$, $\mathcal{T}_2$ and $\mathcal{T}_3$. Since $\gamma(T) > 1$, T is not a star, and so $j \geq 2$. Now $\gamma(T_{j-1}) < \gamma(T)$, and so applying the inductive hypothesis to the tree $T_{j-1} \in \mathcal{T}$, T_{j-1} is a Roman tree. By construction, $T = T_j$ is obtained from T_{j-1} by one of the three operations $\mathcal{T}_1$, $\mathcal{T}_2$ and $\mathcal{T}_3$. Hence, by Lemmas 1, 2, and 3 it follows that T is also a Roman tree. ∎

5. Main Result

In this section we provide a constructive characterization of Roman trees. We shall prove:

Theorem 5. *A tree T is a Roman tree if and only if $T \in \mathcal{T}$.*

Proof. The sufficiency follows from Lemma 4. To prove the necessity, we proceed by induction on the domination number $\gamma(T)$ of a Roman tree T. If $\gamma(T) = 1$, then, since the trivial tree K_1 is not a Roman tree, T is a nontrivial star, and so $T \in \mathcal{T}$. Hence, the result is true for the base case when $\gamma(T) = 1$. Suppose the result is true for all Roman trees with domination number less than m, where $m \geq 2$, and let T be a Roman tree with $\gamma(T) = m$. Then, $\text{diam}(T) \geq 3$.

Let f be a $\gamma_R(T)$-function. We may assume that the function f assigns to each strong support vertex the weight 2 and to each leaf adjacent to a strong support vertex the weight 0. Further, we may assume that no adjacent vertices are both assigned the weight 1 under f.

In the proof we shall frequently prune the tree T to a tree T' and then establish that T' is a Roman tree with $\gamma(T') < m$. By the inductive hypothesis, $T' \in \mathcal{T}$. We then show that T can be obtained from T' by operation $\mathcal{T}_1$, $\mathcal{T}_2$ or $\mathcal{T}_3$.

Let T be rooted at the end-vertex r of a longest path P. Let w be the vertex at distance $\text{diam}(T) - 2$ from r on P, and let v be the child of w on P. Since $\text{diam}(T) \geq 3$, $w \neq r$. Let x denote the parent of w.

Claim 1. *If $w \in S(T)$, then $T \in \mathcal{T}$.*

Proof. Let $T' = T - T_v$. Any $\gamma(T')$-set can be extended to a dominating set of T by adding v, and so $\gamma(T) \le \gamma(T') + 1$. On the other hand, let S be a $\gamma(T)$-set. We may assume that S contains every support vertex, and so $\{v, w\} \subseteq S$. Thus, $S - \{v\}$ is a dominating set of T', and so $\gamma(T') \le |S'| = \gamma(T) - 1$. Consequently, $\gamma(T) = \gamma(T') + 1$.

Any $\gamma_R(T')$-function can be extended to an RDF of T by assigning the weight 2 to v and the weight 0 to each child of v, and so $\gamma_R(T) \le \gamma_R(T') + 2$. Therefore, $2\gamma(T) = \gamma_R(T) \le \gamma_R(T') + 2 \le 2\gamma(T') + 2 = 2\gamma(T)$. Hence, we must have equality throughout this inequality chain. In particular, $\gamma_R(T) = \gamma_R(T') + 2$ and $\gamma_R(T') = 2\gamma(T')$. Thus, T' is a Roman tree. By the inductive hypothesis, $T' \in \mathcal{T}$.

Let f' be the restriction of f to T'. Suppose $f(w) \ge 1$. Then, f' is an RDF of T'. If $w(f') > \gamma_R(T')$, then $2\gamma(T') = \gamma_R(T') < w(f') = w(f) - 2 = 2\gamma_R(T) - 2 = 2\gamma(T')$, which is impossible. Hence, f' is a $\gamma_R(T')$-function.

Any $\gamma_R(T' - w)$-function can be extended to an RDF of T by assigning the weight 2 to v and the weight 0 to each neighbor (including w) of v, and so $\gamma_R(T) \le \gamma_R(T' - w) + 2$. Hence, if $\gamma_R(T' - w) < \gamma_R(T')$, then $\gamma_R(T) \le \gamma_R(T' - w) + 2 < \gamma_R(T') + 2 = w(f') + 2 = w(f) = \gamma_R(T)$, which is impossible. Thus we must have $\gamma_R(T' - w) \ge \gamma_R(T')$, and so $w \in V_S(T')$.

Suppose $\deg v = 2$. Let S' be a $\gamma(T')$-set. We may assume that $S(T') \subseteq S'$. In particular, $w \in S'$. Now let $g': V(T') \to \{0, 1, 2\}$ be the function defined by $g'(z) = 2$ if $z \in S'$ and $g'(z) = 0$ otherwise. Then, g' is an RDF of T' of weight $2\gamma(T')$. Since T' is a Roman tree, g' is a $\gamma_R(T')$-function. Let $g: V(T) \to \{0, 1, 2\}$ be the function defined by $g(z) = g'(z)$ if $z \in V(T')$, $g(v) = 0$ and $g(u) = 1$. Then, g is an RDF of T. Thus, $\gamma_R(T) \le w(g) = \gamma_R(T') + 1$, contradicting our earlier observation that $\gamma_R(T) = \gamma_R(T') + 2$. Hence, $\deg v \ge 3$, i.e., v must be a strong support vertex. Thus, T can be obtained from T' by operation $\mathcal{T}_1$, and so $T \in \mathcal{T}$. ■

By Claim 1 we may assume that $w \notin S(T)$, for otherwise $T \in \mathcal{T}$. It follows that every child of w is a support vertex. We show next that at most two children of w are not strong support vertices.

Claim 2. *At most two children of w have degree 2.*

Proof. Suppose that w has three children v_1, v_2 and v_3 each of degree 2. For $i = 1, 2, 3$, let u_i be the child of v_i. Let $V' = \{v_1, v_2, v_3, u_1, u_2, u_3, w\}$. Let $T' = T - V'$.

Any $\gamma(T')$-set can be extended to a dominating set of T by adding the set $\{v_1, v_2, v_3\}$, and so $\gamma(T) \le \gamma(T') + 3$. On the other hand, let S be a $\gamma(T)$-set. We may assume that S contains every support vertex, and so $C(w) \subseteq S$. If $w \in S$, then we can replace w in S with its parent x. Hence, we may assume that $w \notin S$. Thus, $S - \{v_1, v_2, v_3\}$ is a dominating set of T', and so $\gamma(T') \le |S| - 3 = \gamma(T) - 3$. Consequently, $\gamma(T) = \gamma(T') + 3$.

Any $\gamma_R(T')$-function can be extended to an RDF of T by assigning the weight 2 to w and, for each $i = 1, 2, 3$, assigning the weight 0 to v_i and the weight 1 to u_i, and so $\gamma_R(T) \le \gamma_R(T') + 5$. Therefore, $2\gamma(T) = \gamma_R(T) \le \gamma_R(T') + 5 \le 2\gamma(T') + 5 = 2\gamma(T) - 1$, which is impossible. Hence, at most two children of w can have degree 2. ■

By Claim 2, $T_w \in \mathcal{F}_1^*$ or $T_w \in \mathcal{F}_2^*$. In what follows, let $T' = T - T_w$.

Claim 3. $\gamma(T) = \gamma(T') + |C(w)|$ *and* $\gamma_R(T') = 2\gamma(T')$.

Proof. Any $\gamma(T')$-set can be extended to a dominating set of T by adding $C(w)$, and so $\gamma(T) \le \gamma(T') + |C(w)|$. On the other hand, let S be a $\gamma(T)$-set. We may assume that S contains every support vertex, and so $C(w) \subseteq S$. If $w \in S$, then we can replace w in S with its parent x. Hence, we may assume that $w \notin S$. Thus, $S - C(w)$ is a dominating set of T', and so $\gamma(T') \le |S| - |C(w)|$. Consequently, $\gamma(T) = \gamma(T') + |C(w)|$.

Any $\gamma_R(T')$-function can be extended to an RDF of T by assigning the weight 2 to each child of w and the weight 0 to each neighbor of a child of w. Thus, $\gamma_R(T) \le \gamma_R(T') + 2|C(w)|$. Therefore, $2\gamma(T) = \gamma_R(T) \le \gamma_R(T') + 2|C(w)| \le 2\gamma(T') + 2|C(w)| = 2\gamma(T)$. Hence, we must have equality throughout this inequality chain. In particular, $\gamma_R(T') = 2\gamma(T')$. ■

By Claim 3, T' is a Roman tree. Thus, by the inductive hypothesis, $T' \in \mathcal{T}$. Suppose that $T_w \in \mathcal{F}_1^*$. Then, T can be obtained from T' by operation $\mathcal{T}_2$, and so $T \in \mathcal{T}$.

Suppose, finally, that $T_w \in \mathcal{F}_2^*$. Let v_1 and v_2 be the two children of w of degree 2 and let u_1 and u_2 be their respective children. If $|C(w)| = k \ge 3$, then let $C(w) - \{v_1, v_2\} = \{v_3, \ldots, v_k\}$. We may assume that $f(w) = 2$ and that for $i = 1, 2$, $f(v_i) = 0$ and $f(u_i) = 1$. If $|C(w)| = k \ge 3$, then for $j = 3, \ldots, k$, each v_j is a strong support vertex, and so $f(v_j) = 2$ and $f(z) = 0$ for each leaf z adjacent to v_j.

Any $\gamma_R(T' - x)$-function can be extended to an RDF of T by assigning the weight 2 to w and each child of w that is a strong support vertex, the

weight 1 to each of u_1 and u_2, and the weight 0 to all remaining vertices of T_x. Thus, $\gamma_R(T) \leq \gamma_R(T' - x) + 2|C(w)|$.
We show next that $\gamma_R(T' - x) \geq \gamma_R(T')$. Suppose $f(x) = 0$. Let f'_x be the restriction of f' to $T' - x$. Then, $w(f'_x) = w(f') = w(f) - 2|C(w)|$. Since f is an RDF of T and $f(x) = 0$, it follows that f'_x must be an RDF of $T' - x$. If $\gamma_R(T' - x) < w(f'_x)$, then $\gamma_R(T) \leq \gamma_R(T' - x) + 2|C(w)| < w(f'_x) + 2|C(w)| = w(f) = \gamma_R(T)$, which is impossible. Hence, $\gamma_R(T' - x) = w(f'_x)$. Thus, $\gamma_R(T' - x) = w(f') = w(f) - 2|C(w)| = 2\gamma(T) - 2|C(w)| = 2\gamma(T') = \gamma_R(T')$. On the other hand, suppose that $f(x) \geq 1$. Then, f' is an RDF of T'. If $\gamma_R(T' - x) < \gamma_R(T')$, then $\gamma_R(T) \leq \gamma_R(T' - x) + 2|C(w)| < \gamma_R(T') + 2|C(w)| \leq w(f') + 2|C(w)| = w(f) = \gamma_R(T)$, which is impossible. Hence, $\gamma_R(T' - x) \geq \gamma_R(T')$. Therefore, $x \in V_S(T')$. Thus, T can be obtained from T' by operation $\mathcal{T}_3$, and so $T \in \mathcal{T}$. This completes the proof of Theorem 5. ∎

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