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CONNECTED PARTITION DIMENSIONS OF GRAPHS

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Abstract

For a vertex v of a connected graph G and a subset S of $V(G)$, the distance between v and S is $d(v,S) = \min\{d(v,x)|x \in S\}$. For an ordered k-partition $\Pi = \{S_1, S_2, \cdots, S_k\}$ of $V(G)$, the representation of v with respect to Π is the k-vector $r(v|\Pi) = (d(v,S_1), d(v,S_2), \cdots, d(v,S_k))$. The k-partition Π is a resolving partition if the k-vectors $r(v|\Pi)$, $v \in V(G)$, are distinct. The minimum k for which there is a resolving k-partition of $V(G)$ is the partition dimension $\mathrm{pd}(G)$ of G. A resolving partition $\Pi = \{S_1, S_2, \cdots, S_k\}$ of $V(G)$ is connected if each subgraph $\langle S_i \rangle$ induced by S_i $(1 \leq i \leq k)$ is connected in G. The minimum k for which there is a connected resolving k-partition of $V(G)$ is the connected partition dimension $\mathrm{cpd}(G)$ of G. Thus $2 \leq \mathrm{pd}(G) \leq \mathrm{cpd}(G) \leq n$ for every connected graph G of order $n \geq 2$. The connected partition dimensions of several classes of well-known graphs are determined. It is shown that for every pair a, b of integers with $3 \leq a \leq b \leq 2a-1$, there is a connected graph G having $\mathrm{pd}(G) = a$ and $\mathrm{cpd}(G) = b$. Connected graphs of order $n \geq 3$ having connected partition dimension 2, n, or $n-1$ are characterized.

Keywords: distance, resolving partition, connected resolving partition.

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1. Introduction

The *distance* $d(u, v)$ between two vertices u and v in a connected graph G is the length of a shortest $u - v$ path in G. The *diameter* of G is the largest distance between two vertices in G and is denoted by $\operatorname{diam} G$. For a set S of vertices of G and a vertex v of G, the *distance* $d(v, S)$ *between* v *and* S is defined as

$$d(v, S) = \min\{d(v, x) \mid x \in S\}.$$

For an ordered k-partition $\Pi = \{S_1, S_2, \cdots, S_k\}$ of $V(G)$ and a vertex v of G, the *representation of* v *with respect to* Π is defined as the k-vector

$$r(v|\Pi) = (d(v, S_1), d(v, S_2), \cdots, d(v, S_k)).$$

The partition Π is called *a resolving partition* for G if the distinct vertices of G have distinct representations with respect to Π. The minimum k for which there is a resolving k-partition of $V(G)$ is the *partition dimension* $\mathrm{pd}(G)$ of G. A resolving partition of $V(G)$ containing $\mathrm{pd}(G)$ elements is called *a minimum resolving partition*.

As an illustration of these concepts, we consider the graph G in Figure 1.

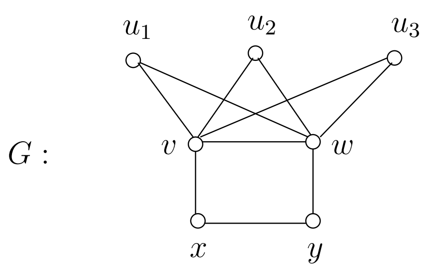

Figure 1: A graph G

Let $\Pi = \{S_1, S_2, S_3\}$, where $S_1 = \{u_1, v\}$, $S_2 = \{u_2, w\}$, and $S_3 = \{u_3, x, y\}$. Then

$$r(u_1|\Pi) = (0, 1, 2), \;\; r(u_2|\Pi) = (1, 0, 2), \;\; r(u_3|\Pi) = (1, 1, 0),$$
$$r(v|\Pi) = (0, 1, 1), \; r(w|\Pi) = (1, 0, 1), \; r(x|\Pi) = (1, 2, 0), \; r(y|\Pi) = (2, 1, 0).$$

So Π is a resolving partition for G. Since there is no resolving 2-partition in G, it follows that $\mathrm{pd}(G) = 3$.

The example just presented also illustrates an important point. Let $\Pi = \{S_1, S_2, \cdots, S_k\}$ be a resolving partition of $V(G)$. If $u \in S_i$ and $v \in S_j$, where $i \neq j$ and $i, j \in \{1, 2, \cdots, k\}$, then $r(u \mid \Pi) \neq r(v \mid \Pi)$ since $d(u, S_i) = 0$ and $d(u, S_j) \neq 0$. Thus, when determining whether a given partition Π of vertices of a graph G is a resolving partition for G, we need only verify that the vertices of G belonging to same element in Π have distinct representations with respected to Π.

The following lemma appeared in [4] will be useful to us.

Lemma 1.1. *Let* Π *be a resolving partition of* $V(G)$ *and* $u, v \in V(G)$. *If* $d(u, w) = d(v, w)$ *for all* $w \in V(G) - \{u, v\}$, *then* u *and* v *belong to distinct elements of* Π.

A resolving partition $\Pi = \{S_1, S_2, \cdots, S_k\}$ of $V(G)$ is *connected* if each subgraph $\langle S_i \rangle$ induced by S_i $(1 \leq i \leq k)$ is connected in G. The minimum k for which there is a connected resolving k-partition of $V(G)$ is the *connected partition dimension* $\mathrm{cpd}(G)$ of G. A connected resolving partition of $V(G)$ containing $\mathrm{cpd}(G)$ elements is called a *cr-partition of* $V(G)$. Certainly, every connected resolving partition of a connected graph is a resolving partition. In general, however, the converse is not true. Thus if G is a connected graph of order $n \geq 2$, then

$$2 \leq \mathrm{pd}(G) \leq \mathrm{cpd}(G) \leq n. \tag{1}$$

As an example, we again consider the graph G of Figure 1. In the resolving partition Π described above, the subgraph $\langle S_3 \rangle$ induced by S_3 is disconnected in G and so Π is not connected. On the other hand, let $\Pi' = \{S'_1, S'_2, S'_3, S'_4\}$, where $S'_1 = \{u_1, v\}$, $S'_2 = \{u_2, w\}$, $S'_3 = \{u_3\}$, and $S'_4 = \{x, y\}$. Then Π' is a connected resolving partition of $V(G)$. By a case-by-case analysis, one can show that Π' is a *cr*-partition of G and so $\mathrm{cpd}(G) = 4$. Thus $\mathrm{pd}(G) < \mathrm{cpd}(G)$ for the graph G of Figure 1. The following observation is useful.

Observation 1.2. *Let* G *be a connected graph. Then* $\mathrm{pd}(G) = \mathrm{cpd}(G)$ *if and only if* G *contains a minimum resolving partition that is connected.*

The concept of resolvability in graphs has previously appeared in the literature. In [8] and later in [9], Slater introduced and studied these ideas with different terminology. Slater described the usefulness of these ideas when working with U.S. sonar and coast guard Loran (Long range aids to navigation) stations. Harary and Melter [5] discovered these concepts independently as well. These concepts were rediscovered by Johnson [6, 7] of the Pharmacia Company while attempting to develop a capability of large datasets of chemical graphs. A basic problem in chemistry is to provide mathematical representations for a set of chemical compounds in a way that gives distinct representations to distinct compounds. Thus, a graph-theoretic interpretation of this problem is to provide representations for the vertices of a graph in such a way that distinct vertices have distinct representations. The resolving partition and partition dimension of a graph were introduced and studied in [3, 4]. We refer to the book [1] for graph theory notation and terminology not described here.

2. Some Basic Results on Connected Partition Dimensions of Graphs

We have seen that if G is a connected graph of order $n \geq 2$, then $2 \leq \mathrm{cpd}(G) \leq n$. We now present improved upper and lower bounds for the connected partition dimension of a connected graph in terms of its order and diameter. For integers n and d with $n > d \geq 1$, we define $f(n,d)$ as the least positive integer k for which $kd^{k-1} \geq n$. Thus $f(n,1) = n$ for all $n \geq 2$.

Theorem 2.1. *If G is a connected graph of order $n \geq 3$ and diameter d, then*

$$f(n,d) \leq \mathrm{cpd}(G) \leq n - d + 1.$$

Proof. If $d = 1$, then $G = K_n$ and $\mathrm{cpd}(K_n) = n$ by Lemma 1.1. On the other hand, $f(n,1) = n$ and $n - d + 1 = n$. So the result is true for $d = 1$. Thus we may assume that $d \geq 2$. We first establish the upper bound. Let u and v be vertices of G for which $d(u,v) = d$ and let $u = v_1, v_2, \cdots, v_{d+1} = v$ be a $u-v$ path of length d. Assume that $V(G) = \{v_1, v_2, \ldots, v_d, \cdots, v_n\}$. Then the partition $\Pi = \{S_1, S_2, \cdots, S_{n-d+1}\}$ of $V(G)$, where $S_1 = \{v_1, v_2, \ldots, v_d\}$ and $S_i = \{v_{i+d-1}\}$ for $2 \leq i \leq n-d+1$, is a connected resolving $(n-d+1)$-partition of $V(G)$. Therefore, $\mathrm{cpd}(G) \leq n - d + 1$.

Next we verify the lower bound. Suppose that $\mathrm{cpd}(G) = k$ and that Π is a connected resolving k-partition of $V(G)$. Since (1) each representation of a vertex with respect to Π is a k-vector whose coordinates are nonnegative integers not exceeding d with exactly one coordinate zero and (2) all n representations are distinct, it follows that $kd^{k-1} \geq n$. Thus $f(n,d) \leq k = \text{cpd}(G)$. ∎

Note that the upper and lower bounds given in Theorem 2.1 can be attained. Consider the graphs G_1 and G_2 of Figure 2. It can be verified that $\text{cpd}(G_i) = 3$ for $i = 1, 2$. A *cr*-partition in each of G_1 and G_2 is also shown in Figure 2. The graph G_1 has order $n = 5$ and $\text{diam}\, G_1 = 3$. Thus $\text{cpd}(G_1) = 3 = n - \text{diam}\, G_1 + 1$, attaining the upper bound. On the other hand, The graph G_2 has order $n = 9$ and $\text{diam}\, G_2 = 4$. Since $f(9,4) = 3$, it follows that $\text{cpd}(G_2) = f(9,4)$, attaining the lower bound.

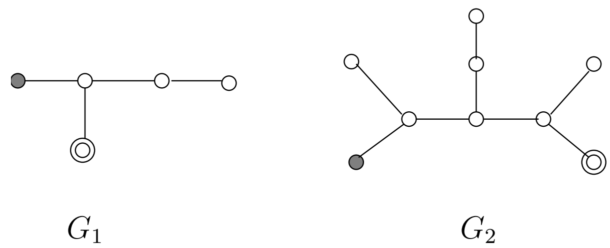

Figure 2: The graphs G_1 and G_2

For each integer $n \geq 2$, it was shown in [4] that the path P_n of order n is the only connected graph of order n having partition dimension 2 and the complete graph K_n is the only connected graph of order n having partition dimension n. We show that this is also true for the connected partition dimension of a connected graph.

Proposition 2.2. *Let G be a connected graph of order $n \geq 2$. Then*

(a) $\text{cpd}(G) = 2$ *if and only if* $G = P_n$,

(b) $\text{cpd}(G) = n$ *if and only if* $G = K_n$.

Proof. We first verify (a). Let $P_n : v_1, v_2, \cdots, v_n$, where $n \geq 2$, and let $\Pi = \{S_1, S_2\}$ be the partition of $V(P_n)$ with $S_1 = \{v_1\}$ and $S_2 = \{v_2, v_3, \cdots, v_n\}$. Then $\langle S_1 \rangle = K_1$ and $\langle S_2 \rangle = P_{n-1}$ are connected in P_n. Since $r(v_1|\Pi) = (0,1)$ and $r(v_i|\Pi) = (i-1, 0)$ for $2 \leq i \leq n$, it follows that Π is a *cr*-partition of P_n and so $\text{cpd}(P_n) = 2$ by (1). For the converse, if G is a connected graph of order $n \geq 2$ with $\text{cpd}(G) = 2$. Then $\text{pd}(G) = 2$ by (1) and so $G = P_n$, which establishes (a).

Next we verify (b). We have seen that $\mathrm{cpd}(K_n) = n$. On the other hand, if G is not a complete graph, then $\mathrm{diam}\, G \geq 2$. It then follows from Theorem 2.1 that $\mathrm{cpd}(G) \leq n - 1$. ■

It was shown in [4] that $\mathrm{pd}(K_{r,s}) = r + 1$ if $r = s$ and $\mathrm{pd}(K_{r,s}) = \max\{r, s\}$ if $r \neq s$. We now show that this is also true for the connected partition dimension of $K_{r,s}$ for all positive integers r, s.

Proposition 2.3. *For positive integers r, s,*

$$\mathrm{cpd}(K_{r,s}) = \begin{cases} r+1 & \text{if } r = s, \\ \max\{r, s\} & \text{if } r \neq s. \end{cases}$$

Proof. Let $G = K_{r,s}$ with partite sets $V_1 = \{u_1, u_2, \cdots, u_r\}$ and $V_2 = \{v_1, v_2, \cdots, v_s\}$. By Observation 1.2, it suffices to show that G contains a minimum resolving partition that is connected. For $r = s$, let $\Pi = \{S_1, S_2, \cdots, S_{r+1}\}$, where $S_i = \{u_i, v_i\}$ $(1 \leq i \leq r-1)$, $S_r = \{u_r\}$, and $S_{r+1} = \{v_r\}$. Since Π is a connected resolving $(r+1)$-partition of $V(G)$, it follows that $\mathrm{cpd}(K_{r,s}) = r + 1$ if $r = s$. For $r \neq s$, assume, without loss of generality, that $r > s$. Let $\Pi = \{S_1, S_2, \cdots, S_r\}$, where $S_i = \{u_i, v_i\}$ $(1 \leq i \leq s)$ and $S_i = \{u_i\}$ $(s+1 \leq i \leq r)$. Since Π is a connected resolving r-partition of $V(G)$, it follows that $\mathrm{cpd}(K_{r,s}) = r = \max\{r, s\}$. ■

Thus, if G is a path, a complete graph, or a complete bipartite graph, then $\mathrm{pd}(G) = \mathrm{cpd}(G)$. This observation yields the following.

Corollary 2.4. *For each integer $k \geq 2$, there is a connected graph G with*

$$\mathrm{pd}(G) = \mathrm{cpd}(G) = k.$$

Note that every graph G encountered thus far has the property that either $\mathrm{pd}(G) = \mathrm{cpd}(G)$ or $\mathrm{cpd}(G) - \mathrm{pd}(G) \leq 1$. This might lead one to believe that $\mathrm{cpd}(G)$ and $\mathrm{pd}(G)$ are close for every connected graph G. However, this is not the case. In fact, as we will see in the next section, the difference $\mathrm{cpd}(G) - \mathrm{pd}(G)$ can be arbitrarily large.

3. Connected Partition Dimensions of Trees That Are Not Paths

Although the partition dimensions of some special types of trees that are not paths have been studied in [3], there is no general formula for the partition dimension of a tree that is not a path. In this section we present a formula for the connected resolving partition dimension of a tree that is not a path. First, we need some additional definitions.

A vertex of degree at least 3 in a connected graph G is called a *major vertex* of G. An end-vertex u of G is said to be a *terminal vertex of a major vertex* v of G if $d(u, v) < d(u, w)$ for every other major vertex w of G. The *terminal degree* $ter(v)$ of a major vertex v is the number of terminal vertices of v. A major vertex v of G is an *exterior major vertex* of G if it has positive terminal degree. Let $\sigma(G)$ denote the sum of the terminal degrees of the major vertices of G and let $ex(G)$ denote the number of exterior major vertices of G. If G is a tree that is not path, then $ex(G)$ is the number of end-vertices of G. For example, the tree T of Figure 3 has four major vertices, namely, v_1, v_2, v_3, v_4. The terminal vertices of v_1 are u_1 and u_2, the terminal vertices of v_3 are u_3, u_4, and u_5, and the terminal vertices of v_4 are u_6 and u_7. The major vertex v_2 has no terminal vertex and so v_2 is not an exterior major vertex of T. Therefore, $\sigma(T) = 7$ and $ex(T) = 3$.

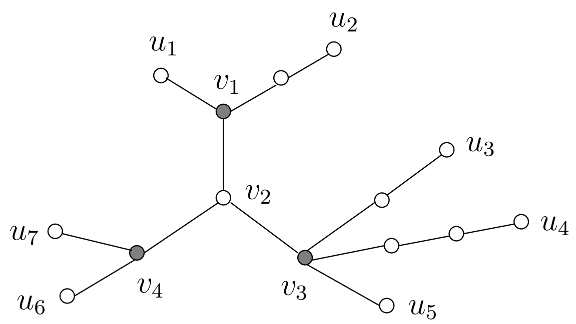

Figure 3: A tree with its exterior major vertices

We first present a lemma that provides a lower bound for the connected partition dimension of a connected graph G in terms of $\sigma(G)$ and $ex(G)$.

Lemma 3.1. *If G is a connected graph, then*

$$\operatorname{cpd}(G) \geq \sigma(G) - ex(G) + 1.$$

Proof. Let $\Pi = \{S_1, S_2, \cdots, S_k\}$ be a connected resolving partition of G. Suppose that G contains p exterior major vertices $v_1, v_2, \cdots, v_p$. For each i

with $1 \leq i \leq p$, let $u_{i1}, u_{i2}, \cdots, u_{ik_i}$ be the terminal vertices of v_i. For each i with $1 \leq i \leq p$, let P_{ij} be the $v_i - u_{ij}$ path in G for all $1 \leq j \leq k_i$ and let x_{ij} be a vertex in P_{ij} that is adjacent to v_i. Then let Q_{ij} be the $x_{ij} - u_{ij}$ subpath of P_{ij} for all $1 \leq i \leq p$ and $1 \leq j \leq k_i$.

Without loss of generality, assume that $v_1 \in S_1$. We claim that at least one vertex, say a_{1j}, from the path Q_{1j} $(1 \leq j \leq k_1)$ such that all vertices a_{1j} $(1 \leq j \leq k_1)$ belong to distinct elements in Π and $a_{1j} \notin S_1$ for all $1 \leq j \leq k_1$ with at most one exception. Assume, to the contrary, that this is not the case. We may assume that $V(Q_{11})$ and $V(Q_{12})$ are contained in the same element of Π. Since $d(x_{11}, v) = d(x_{12}, v)$ for all $v \in V(G) - ((V(Q_{11}) \cup V(Q_{12}))$, it follows that $r(x_{11} \mid \Pi) = r(x_{12} \mid \Pi)$, which is a contradiction. Thus assume, without loss of generality, that $a_{1j} \in S_j$ for $2 \leq j \leq k_1$. Since Π is a connected partition of $V(G)$ and $v_1 \in S_1$, no vertex in $V(G) - \left(\bigcup_{j=2}^{k_1} V(Q_{1j})\right)$ belongs to S_j for all $2 \leq j \leq k_1$; for otherwise, the subgraph $\langle S_j \rangle$ cannot be connected in G. On the other hand, the vertex a_{11} is either in S_1 or in S_t for some integer t with $k_1 + 1 \leq t \leq k$. In either case, $k \geq k_1 = (k_1 - 1) + 1$.

Next, we consider the exterior major vertex v_2. Since $v_2 \notin S_j$ for all j with $2 \leq j \leq k_1$, we assume that $v_2 \in S_\ell$, where $\ell = 1$ or $k_1 + 1 \leq \ell \leq k$. Similarly, at least one vertex, say a_{2j}, from the path Q_{2j} $(1 \leq j \leq k_2)$ such that all vertices a_{2j} $(1 \leq j \leq k_2)$ belong to distinct elements in Π and $a_{2j} \notin S_\ell$ for all $1 \leq j \leq k_2$ with at most one exception. Thus, we may assume that $a_{2j} \in S_{j+k_1-1}$ for $2 \leq j \leq k_2$ and $j + k_1 - 1 \neq \ell$. Then no vertex in $V(G) - \left(\bigcup_{j=2}^{k_2} V(Q_{2j})\right)$ belongs to S_{j+k_1-1} for all j with $2 \leq j \leq k_2$. Note that either $S_1 = S_\ell$, or $S_1 \neq S_\ell$. In either case, all elements S_i, where $1 \leq i \leq k_1 + k_2 - 1$, are distinct elements in Π. Thus $k \geq k_1 + k_2 - 1 = (k_1 - 1) + (k_2 - 1) + 1$.

Continuing this procedure to the remaining exterior major vertices of G, we obtain

$$k \geq \left(\sum_{i=1}^{p} (k_i - 1) \right) + 1 = \sigma(G) - ex(G) + 1.$$

Therefore, $\mathrm{cpd}(G) \geq \sigma(G) - ex(G) + 1$. ∎

In order to determine the connected partition dimension of a tree that is not a path, we will apply a lemma appeared in [2]. First, some additional definitions are needed. For an ordered set $W = \{w_1, w_2, \cdots, w_k\}$ of vertices in a connected graph G and a vertex v of G, the k-vector

$$r(v|W) = (d(v, w_1), d(v, w_2), \cdots, d(v, w_k))$$

is referred to as the (*metric*) *representation of* v *with respect to* W. The set W is called a *resolving set* for G if the vertices of G have distinct representations with respect to W. For more information on this topic, see [2, 5, 8, 9]. The following lemma [2] is useful.

Lemma 3.2. *Let T be a tree that is not a path, having order $n \geq 4$ and p exterior major vertices $v_1, v_2, \cdots, v_p$. For $1 \leq i \leq p$, let $u_{i1}, u_{i2}, \cdots, u_{ik_i}$ be the terminal vertices of v_i and let P_{ij} be the $v_i - u_{ij}$ path $(1 \leq j \leq k_i)$. Suppose that W is a set of vertices of T. Then W is a resolving set of T if and only if W contains at least one vertex from each of the paths $P_{ij} - v_i$ $(1 \leq j \leq k_i$ and $1 \leq i \leq p)$ with at most one exception for each i with $1 \leq i \leq p$.*

We are prepared to present a formula for the connected partition dimension of a tree that is not a path.

Theorem 3.3. *If T is a tree of order $n \geq 4$ that is not a path, then*

$$\mathrm{cpd}(T) = \sigma(T) - ex(T) + 1.$$

Proof. By Lemma 3.1, $\mathrm{cpd}(T) \geq \sigma(T) - ex(T) + 1$. Thus it remains to show that $\mathrm{cpd}(T) \leq \sigma(T) - ex(T) + 1$. Let $k = \sigma(T) - ex(T) + 1$. Suppose that T contains p exterior major vertices $v_1, v_2, \cdots, v_p$. For each i with $1 \leq i \leq p$, let $u_{i1}, u_{i2}, \cdots, u_{ik_i}$ be the terminal vertices of v_i. For each i with $1 \leq i \leq p$, let P_{ij} be the $v_i - u_{ij}$ path in T for all $1 \leq j \leq k_i$ and let x_{ij} be a vertex in P_{ij} that is adjacent to v_i. Then let Q_{ij} be the $x_{ij} - u_{ij}$ subpath of P_{ij} for all $1 \leq i \leq p$ and $1 \leq j \leq k_i$.

Let $U = \{v_1, u_{11}, u_{21}, \cdots, u_{p1}\}$ and let T_1 be the subtree of T of smallest size such that T_1 contains U. Let $S_0 = V(T_1)$ and $S_{ij} = V(Q_{ij})$ for all $1 \leq i \leq p$ and $2 \leq j \leq k_i$. Define a k-partition Π of $V(T)$ by

$$\Pi = \{S_0, S_{12}, S_{13}, \cdots, S_{1k_1}, S_{22}, S_{23}, \cdots, S_{2k_2}, \cdots, S_{p2}, S_{p3}, \cdots, S_{pk_p}\}.$$

Then Π is connected. We now show that Π is a resolving partition of $V(T)$. Note that it suffices to show that the vertices of T belonging to same element of Π have distinct representations with respect to Π. Let $x, y \in V(T)$. We consider two cases.

Case 1. $x, y \in S_0$.
Then $d(x, S_{ij}) = d(x, x_{ij})$ and $d(y, S_{ij}) = d(y, x_{ij})$ for all pairs i, j with $1 \leq i \leq p$ and $2 \leq j \leq k_i$. Let

$$B = \{x_{ij} : 1 \leq i \leq p \text{ and } 2 \leq j \leq k_i\}.$$

By Lemma 3.2, the set B is a resolving set of T and so $r(x \mid B) \neq r(y \mid B)$. Observe that the first coordinate in each of $r(x \mid \Pi)$ and $r(y \mid \Pi)$ is 0, the remaining $k-1$ coordinates of $r(x \mid \Pi)$ are exactly those of $r(x \mid B)$, and the remaining $k-1$ coordinates of $r(y \mid \Pi)$ are exactly those of $r(y \mid B)$. Since $r(x \mid B) \neq r(y \mid B)$, it follows that $r(x \mid \Pi) \neq r(y \mid \Pi)$.

Case 2. $x, y \in S_{ij}$, where $1 \leq i \leq p$ and $2 \leq j \leq k_i$.
Then $d(x, S_0) = d(x, v_i)$ and $d(y, S_0) = d(y, v_i)$. Since x and y are two distinct vertices in the $x_{ij} - u_{ij}$ path Q_{ij}, it follows that $d(x, v_i) \neq d(y, v_i)$ and so $d(x, S_0) \neq d(y, S_0)$. Therefore, $r(x \mid \Pi) \neq r(y \mid \Pi)$.

Therefore, Π is a connected resolving k-partition of $V(T)$ and so $\text{cpd}(T) \leq k = \sigma(T) - ex(T) + 1$. ■

To illustrate Theorem 3.3, we consider the tree of Figure 2, which is redrawn in Figure 4. We have seen that $\sigma(T) = 7$ and $ex(T) = 3$. The subtree T_1 of T that contains $U = \{v_1, u_{11}, u_{21}, u_{31}\}$ and the four subpath $Q_{12}, Q_{22}, Q_{32}, Q_{33}$ are shown in Figure 4, where T_1 is drawn in bold. By Theorem 3.3 the 5-partition $\Pi = \{S_0, S_{12}, S_{22}, S_{32}, S_{33}\}$ of $V(T)$ is a *cr*-partition and so $\text{cpd}(T) = 5 = \sigma(T) - ex(T) + 1$.

By Theorem 3.3, we are now able to show that every pair a, b of integers with $3 \leq a < b \leq 2a - 1$ is realizable as the partition dimension and the connected partition dimension of some connected graph.

Theorem 3.4. *For every pair a, b of integers with $3 \leq a < b \leq 2a-1$, there is a connected graph G such that* $\text{pd}(G) = a$ *and* $\text{cpd}(G) = b$.

Proof. Let G be a double star with central vertices u and v and $N(u) = \{x_1, x_2, \cdots, x_a\}$ and $N(v) = \{y_1, y_2, \cdots, y_{b-a+1}\}$. Then $\text{cpd}(G) = b$ by Theorem 3.3. Thus it remains to show that $\text{pd}(G) = a$. By Lemma 1.1, the vertices of $N(u)$ must belong to distinct elements in a resolving partition of G and so $\text{pd}(G) \geq a$. On the other hand, if $b = a+1$, let $\Pi = \{S_1, S_2, \cdots, S_a\}$, where $S_i = \{x_i\}$ for $1 \leq i \leq a-3$, $S_{a-2} = \{x_{a-2}, y_2\}$, $S_{a-1} = \{u, x_{a-1}, y_1\}$, and $S_a = \{v, x_a\}$; while if $b > a+1$, let $\Pi = \{S_1, S_2, \cdots, S_a\}$, where $S_1 = \{u, x_1, y_1\}$, $S_2 = \{v, x_2, y_2\}$, $S_i = \{x_i, y_i\}$ for $3 \leq i \leq b-a+1$,

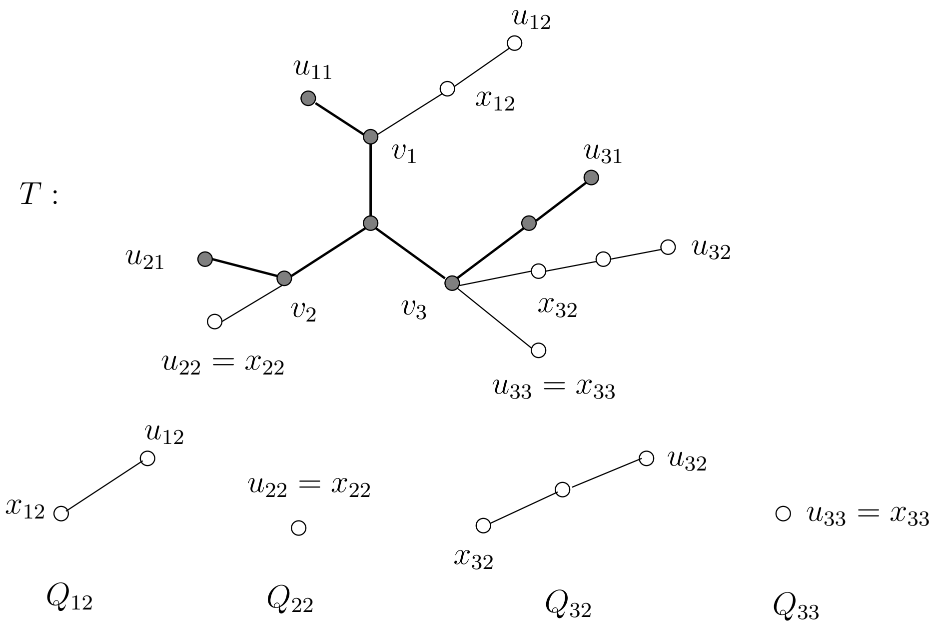

Figure 4: Illustrating Theorem 3.3

and $S_j = \{x_j\}$ for $b-a+2 \le j \le a$. In either case, Π is a resolving partition of $V(G)$ and so $\text{pd}(G) = a$. ■

On the other hand, the following problem is still open.

Problem 3.5. *For which pairs a, b of integers with $a \ge 3$ and $b \ge 2a$, does there exists a connected graph G such that* $\text{pd}(G) = a$ *and* $\text{cpd}(G) = b$*?*

As a consequence of Theorem 3.4, we see that, for some connected graph G, the difference $\text{cpd}(G) - \text{pd}(G)$ can be arbitrarily large. In fact, we have the following result.

Corollary 3.6. *For each positive integer N, there is an infinite class of connected graphs G such that*

$$\text{cpd}(G) - \text{pd}(G) \ge N.$$

Proof. For each integer a with $a \ge \max\{N+1, 3\}$, let G_a be the double star with central vertices u and v such that $\deg u = \deg v = a$. Then $\text{cpd}(G_a) = 2a - 1$ by Theorem 3.3 and $\text{pd}(G_a) = a$ by the proof of Theorem 3.4. Therefore, $\text{cpd}(G_a) - \text{pd}(G_a) = a - 1 \ge N$, as desired. ■

4. Graphs With Connected Partition Dimension $n-1$

We have seen that the complete graph K_n of order $n \geq 2$ is the only connected graph of order n with connected partition dimension n. Thus, if G is a connected graph of order $n \geq 3$ that is not a complete graph, then $\text{cpd}(G) \leq n-1$. It was shown in [4] that the graphs $K_{1,n-1}$, $K_n - e$, $K_1 + (K_1 \bigcup K_{n-2})$ are the only connected graphs of order $n \geq 3$ with partition dimension $n-1$. Applying the same technique used in [4], we now show that those graphs are also the only connected graphs of order $n \geq 3$ with connected partition dimension $n-1$. In order to do this, we first present a lemma which is an immediate consequence of Theorem 2.1.

Lemma 4.1. *If G is a connected graph of order $n \geq 3$ and* $\text{cpd}(G) = n-1$, *then* $\text{diam}\, G = 2$.

Theorem 4.2. *Let G be a connected graph of order $n \geq 3$. Then* $\text{cpd}(G) = n-1$ *if and only if G is one of the graphs $K_{1,n-1}$, $K_n - e$, $K_1 + (K_1 \bigcup K_{n-2})$.*

Proof. It is routine to verify that the graphs mentioned in the theorem have connected partition dimension $n-1$. For the converse, assume that G is a connected graph of order $n \geq 3$ with connected partition dimension $n-1$. By Lemma 4.1, it follows that the diameter of G is 2. Suppose first that G is bipartite. Since the diameter of G is 2, it follows that $G = K_{r,s}$ for some integers r and s with $n = r + s \geq 3$. By Proposition 2.3, it follows that $G = K_{1,n-1}$.

We now suppose that G is not bipartite. Let Y be the vertex set of a maximum clique of G. We show that $|Y| \geq 3$. Since G is not bipartite, G contains an odd cycle. Let $C_{2\ell+1}$ be the smallest odd cycle in G. Since the diameter of G is 2, it follows that $C_{2\ell+1}$ is C_3 or C_5. Suppose first that $C_{2\ell+1} = C_5 : v_1, v_2, v_3, v_4, v_5, v_1$. Let $\Pi = \{S_1, S_2, \cdots, S_{n-2}\}$, where $S_1 = \{v_1, v_2, v_3\}$, $S_2 = \{v_4\}$, $S_3 = \{v_5\}$, and S_i $(4 \leq i \leq n-2)$ contains a single vertex of $V(G) - \{v_1, v_2, v_3, v_4, v_5\}$. Then each $\langle S_i \rangle$ is connected for all $1 \leq i \leq n-2$. Since $r(v_1|\Pi) = (0, 2, 1, \cdots), r(v_2|\Pi) = (0, 2, 2, \cdots)$, and $r(v_3|\Pi) = (0, 1, 2, \cdots)$, it follows that Π is a connected resolving $(n-2)$-partition of $V(G)$, contradicting $\text{cpd}(G) = n-1$. Therefore, $C_{2\ell+1} = C_3$. Since G contains K_3 as a subgraph, it follows that $|Y| \geq 3$.

Let $U = V(G) - Y$. Since G is not complete, $|U| \geq 1$. Assume first that $|U| = 1$. Then $G = K_s + (K_1 \cup K_t)$ for some integers s and t. Since

G is connected and G is not complete, $s \geq 1$ and $t \geq 1$. Let $V(K_s) = \{u_1, u_2, \cdots, u_s\}$, $V(K_t) = \{v_1, v_2, \cdots, v_t\}$, and $V(K_1) = \{w\}$. We consider two cases.

Case 1. $s \geq t$.
Let $\Pi = \{S_1, S_2, \cdots, S_{s+1}\}$, where $S_i = \{u_i, v_i\}$ $(1 \leq i \leq t)$, $S_i = \{u_i\}$ $(t+1 \leq i \leq s)$, and $S_{s+1} = \{w\}$. Since $d(u, w) = 1$ for $u \in V(K_s)$ and $d(v, w) = 2$ for $v \in V(K_t)$, it follows that Π is a connected resolving $(s+1)$-partition of $V(G)$. Hence $\text{cpd}(G) \leq s + 1$. By Lemma 1.1, $\text{cpd}(G) \geq s$. However, $\text{cpd}(G) \neq s$, for otherwise $s = n - 1$ and $G = K_n$. Therefore, $\text{cpd}(G) = s + 1$. Since $\text{cpd}(G) = n - 1$, it follows that $s = n - 2$ and $t = 1$. Therefore,

$$G = K_{n-2} + (K_1 \cup K_1) = K_n - e\,.$$

Case 2. $s < t$.
Then let $\Pi = \{S_1, S_2, \cdots, S_{t+1}\}$, where $S_i = \{u_i, v_i\}$ $(1 \leq i \leq s)$, $S_i = \{v_i\}$ $(s+1 \leq i \leq t)$, and $S_{t+1} = \{w\}$, is a connected resolving partition of $V(G)$. Thus $\text{cpd}(G) \leq t+1$. By Lemma 1.1, $\text{cpd}(G) \geq t$. However, $\text{cpd}(G) \neq t$, for otherwise $t = n - 1$ and $s = 0$, implying that G is disconnected. Therefore, $\text{cpd}(G) = t + 1$. Since $\text{cpd}(G) = n - 1$, we have $t = n - 2$ and $s = 1$. Therefore,

$$G = K_1 + (K_1 \cup K_{n-2})\,.$$

Next we assume that $|U| \geq 2$. We first claim that U is an independent set of vertices. Suppose, to the contrary, that this is not the case. Then U contains two adjacent vertices u and w. Because of the defining property of Y, there exist $v \in Y$ such that $uv \notin E(G)$ and $v' \in Y$ such that $wv' \notin E(G)$, where v and v' are not necessarily distinct. We also consider these two cases.

Case 1. There exists a vertex $v \in Y$ such that $uv, wv \notin E(G)$.
We now consider two subcases.

Subcase 1.1. There exists a vertex $x \in Y$ that is adjacent to exactly one of u and w, say u.
Since $|Y| \geq 3$, there exist a vertex $y \in Y$ that is distinct from v and x. Thus G contains the subgraph shown in Figure 5, where dashed lines indicate that the given edge is not present.

Let Π=$\{S_1,\ S_2,\ \cdots,\ S_{n-2}\}$, there S_1=$\{u,\ w\}$, S_2=$\{v,\ x\}$, S_3=$\{y\}$, and each of remaining sets S_i $(4 \leq i \leq n-2)$ contains exactly one vertex from $V(G) - \{u, w, y, x, v\}$. Then $\langle S_i \rangle$ is connected for all $1 \leq i \leq n-2$. Since

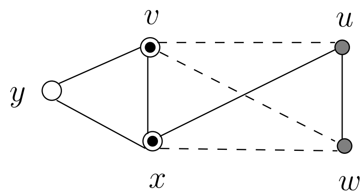

Figure 5: The subgraph of G in Subcase 1.1

$r(u|\Pi) = (0, 1, \cdots)$, $r(v|\Pi) = (2, 0, \cdots)$, $r(w|\Pi) = (0, 2, \cdots)$, and $r(x|\Pi) = (1, 0, \cdots)$, it follows that Π is a connected resolving $(n-2)$-partition of $V(G)$, contradicting the fact that $\text{cpd}(G) = n-1$. Thus this subcase cannot occur.

Subcase 1.2. Every vertex of Y is adjacent to either both u and w or to neither u nor w.
If u and w are adjacent to every vertex in $Y - \{v\}$, then the vertices of $(Y - \{v\}) \cup \{u, w\}$ are pairwise adjacent, contradicting the defining property of Y. Thus, there exists a vertex $y \in Y$ such that y is distinct from v, and y is adjacent to neither u nor w. Since the diameter of G is 2, there is a vertex x of G that is adjacent to both u and v and a vertex z of G such that z is adjacent to both y and w. Since x and z are not necessary distinct and they do not necessary belong to Y, we consider two cases.

Subcase 1.2.1. $x = z$.
Then G contains the subgraph shown in Figure 6. Let $\Pi = \{S_1, S_2, \cdots, S_{n-2}\}$, where $S_1 = \{x, y, w\}$, $S_2 = \{u\}, S_3 = \{v\}$, and each of the remaining sets S_i $(4 \le i \le n-2)$ contains only one vertex from $V(G) - \{u, w, y, x, v\}$. Then $\langle S_i \rangle$ is connected for all $1 \le i \le n-2$. Since $r(x|\Pi)$=$(0, 1, 1, \cdots)$, $r(y|\Pi) = (0, 2, 1, \cdots)$, and $r(w|\Pi) = (0, 1, 2, \cdots)$, it follows that Π is a connected resolving $(n-2)$-partition of $V(G)$, contradicting the fact that $\text{cpd}(G) = n-1$.

Subcase 1.2.2. $x \neq z$.
Then G contains the subgraph shown in Figure 7. Let $\Pi = \{S_1, S_2, \cdots, S_{n-2}\}$, where $S_1 = \{u\}$, $S_2 = \{w\}$, $S_3 = \{v, x\}$, $S_4 = \{y, z\}$, and each of the remaining sets S_i $(5 \le i \le n-2)$ contains only one vertex from $V(G) - \{v, u, w, x, y, z\}$. Then $\langle S_i \rangle$ is connected for all $1 \le i \le n-2$. Since $r(v|\Pi) = (2, 2, 0, \cdots)$, $r(x|\Pi) = (1, 2, 0, \cdots)$, $r(y|\Pi) = (2, 2, 1, 0, \cdots)$, and $r(z|\Pi) = (*, 1, *, 0, \cdots)$, where $*$ is either 1 or 2, it follows that Π is a connected resolving $(n-2)$-partition of $V(G)$, contradicting the fact that

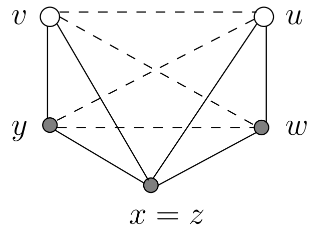

Figure 6: The subgraph of G in Subcase 1.2.1

$\mathrm{cpd}(G) = n - 1$. Thus Subcase 1.2 and, in fact, Case 1 cannot occur.

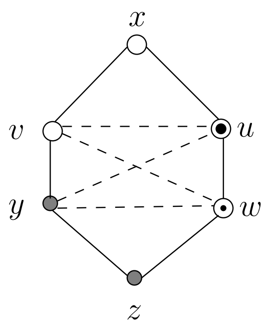

Figure 7: The subgraph of G in Subcase 1.2.2

Case 2. There exist distinct vertices v and v' in Y such that $uv, wv' \notin E(G)$. For each vertex y_0 of Y, y_0 is adjacent to at least one of u and w, for otherwise, we have the conditions of Case 1.

Necessarily, then $vw, v'u \in E(G)$. Since $|Y| \geq 3$, there exists a vertex y in Y distinct from v and v'. Also, at least one of the edges yu and yw must be present in G, say yu. Thus G contains the subgraph shown in Figure 8.

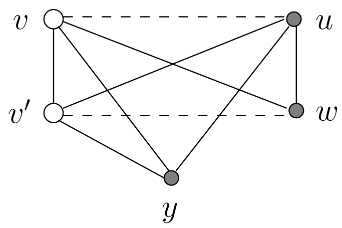

Figure 8: The subgraph of G in Case 2

Let $\Pi=\{S_1,\ S_2,\ \cdots,\ S_{n-2}\}$, where $S_1 = \{u, w, y\}$, $S_2 = \{v\}, S_3 = \{v'\}$, and each of the remaining sets S_i $(4 \le i \le n-2)$ contains only one vertex from $V(G) - \{u, w, y, v, v'\}$. Since $r(u|\Pi) = (0, 2, 1, \cdots)$, $r(w|\Pi) = (0, 1, 2, \cdots)$, and $r(y|\Pi) = (0, 1, 1, \cdots)$, it follows that Π is a connected resolving $(n-2)$-partition of $V(G)$, contradicting the fact that $\text{cpd}(G) = n-1$. Therefore, U is an independent set.

Next we claim the $N(u) = N(w)$ for all $u, w \in U$. It suffices to show that if $uv \in E(G)$, then $vw \in E(G)$. Suppose that $uv \in E(G)$ for some vertex v of G. Necessarily $v \in Y$. Assume, to the contrary, that $wv \notin E(G)$. Since Y is the vertex set of a maximum clique, there exists $y \in Y$ such that $uy \notin E(G)$. Since G is connected and U is independent, w is adjacent to some vertex of Y. we consider two cases.

Case 1. w is adjacent only to y.

Since w and y are not adjacent to u, it follows that $d(w, u) = 3$, which contradicts the fact that the diameter of G is 2.

Case 2. There exists a vertex x in Y distinct from y such that $wx \in E(G)$.

Thus G contains the subgraph shown in Figure 9. Let $\Pi = \{S_1, S_2, \cdots, S_{n-2}\}$, where $S_1 = \{w, x\}$, $S_2 = \{u, v\}$, $S_3 = \{y\}$, and each of the remaining sets S_i $(4 \le i \le n-2)$ contains only one vertex of $V(G) - \{u, w, x, v, y\}$. Then $\langle S_i \rangle$ is connected for all $1 \le i \le n-2$. Since $r(u|\Pi) = (*, 0, 2, \cdots)$, where $*$ is either 1 or 2, $r(v|\Pi) = (1, 0, 1, \cdots)$, $r(w|\Pi) = (0, 2, 1, \cdots)$, and $r(x|\Pi) = (0, 1, 1, \cdots)$, it follows that Π is a connected resolving $(n-2)$-partition of $V(G)$, contradicting the fact that $\text{cpd}(G) = n-1$.

Therefore, $V(G) = Y \cup U$, where $\langle Y \rangle$ is complete, U is independent, $|Y| \ge 3$, $|U| \ge 2$, and $N(u) = N(w)$ for all $u, w, \in U$.

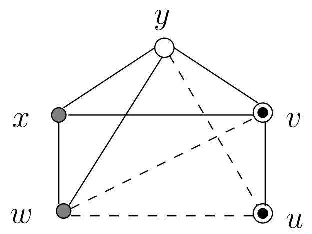

Figure 9: The subgraph of G in Subcase 2.2

Next we show that for each $u \in U$, there exists at most one vertex of Y not contained in $N(u)$. Suppose, to the contrary, that there are two vertices $x, y \in Y$ not in $N(u)$. Let w be a vertex of U that is distinct from u. Thus $wx, wy \notin E(G)$. Since G is connected, there exists $z \in Y$ such that $z \in N(u) = N(w)$. Thus G contains the subgraph shown in Figure 10.

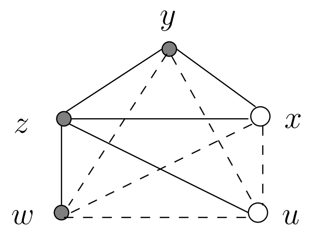

Figure 10: The subgraph of G

Let $\Pi = \{S_1, S_2, \cdots, S_{n-2}\}$, where $S_1 = \{y, z, w\}$, $S_2 = \{u\}$, $S_3 = \{x\}$, and each of the remaining sets S_i ($4 \le i \le n-2$) contains only one vertex of $V(G) - \{y, z, w, u, x\}$. Since $r(y|\Pi) = (0, 2, 1, \cdots)$, $r(z|\Pi) = (0, 1, 1, \cdots)$, and $r(w|\Pi) = (0, 2, 2, \cdots)$, it follows that Π is a connected resolving $(n-2)$-partition of $V(G)$, contradicting the fact that $\operatorname{cpd}(G) = n - 1$.

Now either $N(u) = Y$ or $N(u) = Y - \{v\}$ for some $v \in Y$. If $N(u) = Y$, then $G = K_s + \overline{K}_t$ for $s = |Y| \ge 3$ and $t = |U| \ge 2$. If $N(u) = Y - \{v\}$, then $G = K_s + (K_1 \cup \overline{K}_t)$, where $V(K_1) = \{v\}$, $s = |Y| - 1 \ge 2$, and $t = |U| \ge 2$. However, $K_s + (K_1 \cup \overline{K}_t) = K_s + \overline{K}_{t+1}$. In either case, $G = K_s + \overline{K}_t$, where $t \ge 3$ and so $s \le n - 3$. Let $V(K_s) = \{u_1, u_2, \cdots, u_s\}$ and $V(\overline{K}_t) = \{v_1, v_2, \cdots, v_t\}$. We consider three cases.

Case 1. $s = t$.

Let $\Pi = \{S_1, S_2, \cdots, S_{s+1}\}$, where $S_i = \{u_i, v_i\}$ $(1 \le i \le s-1)$, $S_s = \{u_s\}$, and $S_{s+1} = \{v_s\}$. Since $d(u, v_s) = 1$ $(u \in V(K_s))$ and $d(v, v_s) = 2$ $(v \in V(K_t))$, it follows that Π is a connected resolving $(s+1)$-partition of $V(G)$. Hence $\mathrm{cpd}(G) \le s + 1 \le n - 3 + 1 = n - 2$, which is a contradiction, and this case cannot occur.

Case 2. $s > t$.
Let $\Pi = \{S_1, S_2, \cdots, S_{s+1}\}$, where $S_i = \{u_i, v_i\}$ $(1 \le i \le t-1)$, $S_i = \{u_i\}$ $(t+1 \le i \le s)$, and $S_{s+1} = \{v_t\}$. Since $d(u, v_t) = 1$ $(u \in V(K_s))$ and $d(v, v_t) = 2$ $(v \in V(K_t))$, it follows that Π is a connected resolving $(s+1)$-partition of $V(G)$. Hence $\mathrm{cpd}(G) \le s + 1 \le n - 3 + 1 = n - 2$, which is a contradiction, and this case cannot occur.

Case 3. $s < t$.
Let $\Pi = \{S_1, S_2, \cdots, S_t\}$, where $S_i = \{u_i, v_i\}$ $(1 \le i \le s)$ and $S_i = \{v_i\}$ $(s+1 \le i \le t)$. Since Π is a connected resolving t-partition of $V(G)$, it follows that $\mathrm{cpd}(G) \le t \le n - 2$, which is a contradiction, and this case cannot occur. ■

5. Topics for Study

If G is a connected graph with $V(G) = \{v_1, v_2, \cdots, v_n\}$, then the ordered partition $\Pi = \{S_1, S_2, \cdots, S_n\}$, where $S_i = \{v_i\}$ for $1 \le i \le n$, into singleton subsets of $V(G)$ is always a resolving partition of $V(G)$. Since $\langle S_i \rangle$ is trivially connected for each i $(1 \le i \le n)$, it follows that Π is a connected resolving partition of $V(G)$ as well, and, consequently, $\mathrm{cpd}(G)$ is defined.

This suggests a variety of concepts to study. If P is any graphical property possessed by a trivial subgraph of a connected graph G, then the ordered partition Π of $V(G)$ described above is said to satisfy property P and the P-partition dimension $\mathrm{pd}_P(G)$ is defined. Among the various properties P, in addition to the property of being connected, are:

(1) the property of being acyclic,
(2) the property of being a path,
(3) the property of being a star,
(4) the property of linear forest (every component is a path),
(5) the property of being a galaxy (every component is a star), and
(6) the property of being planar.

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