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ON THE STRUCTURAL RESULT ON NORMAL PLANE MAPS

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Abstract

We prove the structural result on normal plane maps, which applies to the vertex distance colouring of plane maps. The vertex distance-t chromatic number of a plane graph G with maximum degree $\Delta(G) \le D$, $D \ge 12$ is proved to be upper bounded by $6+\frac{2D+12}{D-2}((D-1)^{(t-1)}-1)$. This improves a recent bound $6+\frac{3D+3}{D-2}((D-1)^{t-1}-1)$, $D \ge 8$ by Jendrol' and Skupień, and the upper bound for distance-2 chromatic number.

Keywords: plane map, distance colouring.

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1. Introduction

The degree of a vertex x, in symbols $d(x)$, is the number of edges incident with x, where each loop incident with x is counted twice. The degree of a face f, in symbols $d(f)$, is the number of edges incident with f, where each cut-edge incident with f is counted twice; thus $d(f)$ is the length of the shortest facial walk which bounds f. Hence the function $d : V \cup F \to \mathbb{N}_0$ is the degree function on the union of the vertex set V and the face set F of a plane graph. Vertices and faces of degree i are called i-vertices and i-faces, respectively. By (a, b)-edge we denote an edge e whose endvertices are of degrees a and b, $a \le b$. The *type* of a k-face f is defined to be the lexicographical minimum of all possible k-sequences $\langle d_1, d_2, \ldots, d_k\rangle$ of degrees of vertices of f as they

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are encountered when traversing the boundary of f in any direction; if, instead of an exact value d_i, only a lower (or upper) bound d'_i is considered, we will write d'_i+ (or d'_i-) at the corresponding position.

A plane map is called a *normal plane map* if interiors of all its faces are homeomorphic to an open disc, and, moreover, degrees of vertices and all faces are not less than three. Note, however, that both loops and multiple edges can appear in a normal plane map. On the other hand, no loops appear in a multigraph.

The classical consequence of the Euler's polyhedral formula is that each plane graph contains a vertex of degree at most 5. Lebesgue [9] proved the existence of small faces of certain types in convex polyhedra. Kotzig [6] proved that each convex polytope contains an edge whose endvertices have their degree–sum at most 13, and at most 11 in the absence of 3-vertices. Borodin [2, 3] and Jendrol' with Skupień [5] generalized Kotzig's result for normal plane maps and used them for proving graph coloring results. Inspired by [5], we will prove the following structural result for normal plane maps which will be later applied for estimating the vertex distance chromatic number.

Definition 1. A $\langle a, b, c, d, e\rangle$-fan is the configuration of three consecutive 3-faces of type $\langle 5, a, b\rangle$, $\langle 5, b, c\rangle$, $\langle 5, c, d\rangle$ and with the 5-vertex in common; the remaining fifth neighbour of this 5-vertex is of degree e.

Definition 2. A $\langle a, b, c, d, e\rangle$-swallow is the configuration of three 3-faces α, β, γ with common 5-vertex, where α is of type $\langle 5, a, b\rangle$, β of type $\langle 5, b, c\rangle$, α and β share the common edge $(5, b)$ and γ is of type $\langle 5, d, e\rangle$ being not adjacent to α and β.

Theorem 1. *Every normal plane map contains one of the following configurations:*

$\mathcal{A}$. *An $\langle a, b, c\rangle$-face with vertices of degrees*

(i) $a = 3 \leq b \leq 10$, *or*

(ii) $a = 4 \leq b \leq 7$ *and* $4 \leq c \leq 11$, *or*

(iii) $a = 4 \leq b \leq 7$ *and a is adjacent with a d-vertex, where* $4 \leq d \leq 11$;

$\mathcal{B}$. *an $\langle a, b, c, d, e\rangle$-fan with vertices of degrees*

(i) $5 \leq b \leq 6$, $5 \leq c \leq 8$, $5 \leq x \leq 8$, *where* $x \in \{a, d\}$, *or*

(ii) $5 \leq b \leq 6$, $5 \leq c \leq 8$, $5 \leq e \leq 8$;

$\mathcal{C}$. *an* $\langle a, b, c, d, e\rangle$*-swallow with vertices of degrees*

(i) $5 \le a \le 6,\ 5 \le b \le 8,\ 5 \le c \le 8,$ *or*

(ii) $5 \le a \le 6,\ 5 \le x \le 8,\ 4 \le y \le 8,$ *where* $x \in \{b, c\}$ *and* $y \in \{d, e\}$, *or*

(iii) $5 \le b \le 6,\ 5 \le x \le 8,\ 4 \le y \le 8,$ *where* $x \in \{a, c\}$ *and* $y \in \{d, e\}$;

$\mathcal{D}$. *a* 4*-face which has either*

(i) *two* 3*-vertices and a third vertex of degree at most* 5, *or*

(ii) *exactly one* 3*-vertex, at least one* 4*-vertex and both remaining vertices of degrees in* $\{4, 5\}$*;*

$\mathcal{E}$. *a* 5*-face with four* 3*-vertices and the fifth vertex of degree at most* 5.

The next section deals with proof of this theorem; the final section describes its application in a vertex distance colouring.

2. Proof of Theorem 1

Proof is by contradiction. Suppose that there exists a counterexample G having none of the configurations described in $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}, \mathcal{E}$. Let V, E, F be sets of vertices, edges and faces of G, respectively. Then clearly

$$2|E| = \sum_{v \in V} d(v) = \sum_{f \in F} d(f).$$

Hence the well-known Euler's formula $|V| - |E| + |F| = 2$ gives

$$\sum_{v \in V} (d(v) - 6) + \sum_{f \in F} (2d(f) - 6) = -12. \tag{1}$$

Consider an initial charge function $g : V \cup F \to \mathbb{Q}$ such that

$$g(v) = d(v) - 6 \text{ for each } v \in V,$$
$$g(f) = 2d(f) - 6 \text{ for each } f \in F.$$

Then (1) is equivalent to

$$\sum_{x \in V \cup F} g(x) = -12.$$

We use five rules in order to transform g into a new charge function $h : V \cup F \to \mathbb{Q}$ by locally redistributing charges so that $\sum_{x\in V\cup F} g(x) = \sum_{x\in V\cup F} h(x)$. For the purpose of the proof, we call an edge e of the graph G to be *weak* and *semi-weak* if e is incident with two 3-faces and exactly one 3-face, respectively. For a d-vertex x, let $e_1, \dots, e_d$ be edges incident with x in a cyclical ordering, and $x_1, \dots, x_d$ be their second endvertices (note that there can be $x_i = x_j$ for $i \neq j$, or $x = x_i$ for some i).

Rule 1. Given a vertex v being incident with a k-face f, assume that $d(v) \leq 5$, $d(f) = k > 3$. Then f sends to v the following charge:
1 if $d(v) = 3$,
$\frac{1}{2}$ if $d(v) = 4$,
$\frac{1}{3}$ if $d(v) = 5$.

Rule 2. Let $e = (w, v)$ be an edge of G with $d(w) \geq 9$ and $d(v) \in \{3, 5\}$. Then the vertex w sends along e to v the charge:
1 if $d(v) = 3$ and e is weak,
$\frac{1}{2}$ if $d(v) = 3$ and e is semi-weak,
$\frac{1}{3}$ if $d(v) = 5$ and e is weak,
$\frac{1}{6}$ if $d(v) = 5$ and e is semi-weak.

Rule 3. Let $e' = (w', v')$ be an edge of G with $d(w') \geq 12$ and $d(v') = 4$. Then the vertex w' sends along e' to v' the charge:
$\frac{2}{3}$ if e' is weak,
$\frac{1}{2}$ if e' is semi-weak.

Rule 4. Let $e'' = (w'', v'')$ be an edge of G with $d(w'') \in \{8, 9, 10, 11\}$ and $d(v'') = 4$. Then the vertex w'' sends along e'' to v'' the charge:
$\frac{1}{2}$ if e'' is weak,
$\frac{1}{4}$ if e'' is semi-weak.

Rule 5. Let $[xyz]$ be a triangular face such that x, y are 7- or 8-vertices and z is a 5-vertex. Then each of x and y sends $\frac{1}{6}$ to z.

Thus, after redistribution of charges,

$$\sum_{x\in V\cup F} h(x) = -12.$$

We are going to show that, for every $x \in V \cup F$, $h(x) \geq 0$, a contradiction. We consider several cases:

Case 1. Let f be a k-face, $k \geq 3$. Then $g(f) = 2k - 6$.

1.1. $k = 3$. Then $h(f) = g(f) = 2 \cdot 3 - 6 = 0$.

1.2. $k = 4$. If f is not incident with a 3-vertex, then f sends a charge to at most four vertices, all of them may be 4-vertices. Therefore, $h(f) \geq g(f) - 4 \cdot \frac{1}{2} = 2 \cdot 4 - 6 - 2 = 0$. Otherwise (since no 4-face satisfying $\mathcal{D}$ can appear in G) the type of f is one of

(i) $\langle 3, 3, 6+, 6+\rangle$; then $h(f) = 2 \cdot 4 - 6 - 2 \cdot 1 = 0$;
(ii) $\langle 3, 4, 4, 6+\rangle$; then $h(f) = 2 \cdot 4 - 6 - 1 - 2 \cdot \frac{1}{2} = 0$;
(iii) $\langle 3, 4, 5, 6+\rangle$; then $h(f) = 2 \cdot 4 - 6 - 1 - \frac{1}{2} - \frac{1}{3} > 0$;
(iv) $\langle 3, 4, 6+, 6+\rangle$; then $h(f) = 2 \cdot 4 - 6 - 1 - \frac{1}{2} > 0$;
(v) $\langle 3, 5+, 5+, 5+\rangle$; then $h(f) \geq 2 \cdot 4 - 6 - 1 - 3 \cdot \frac{1}{3} > 0$.

1.3. $k = 5$. If f is not incident with more than three 3-vertices then $h(f) \geq 2 \cdot 5 - 6 - 3 \cdot 1 - 2 \cdot \frac{1}{2} = 0$. Suppose that f is incident with four 3-vertices; then the fifth incident vertex is of degree greater than 5 (since f does not satisfy $\mathcal{D}$). Hence, $h(f) \geq 2 \cdot 5 - 6 - 4 \cdot 1 = 0$.

1.4. $k \geq 6$. Then $h(f) \geq g(f) - d(f) = 2 \cdot k - 6 - k \geq 0$.

Case 2. Let y be a k-vertex, $k \geq 3$. Then $g(y) = k - 6$.

2.1. $k = 3$. Due to $\mathcal{A}(i)$, each triangle incident with y is of type $\langle 3, 11+, 11+\rangle$. If y is incident with

(i) three 3-faces, then by Rule 2 $h(y) = g(y) + 3 \cdot 1 = 3 - 6 + 3 = 0$;
(ii) two 3-faces, then according to the Rule 1,2 $h(y) = g(y) + 1 + 1 + 2 \cdot \frac{1}{2} = 3 - 6 + 2 + 1 = 0$;
(iii) one 3-face, then $h(y) = g(y) + 2 \cdot 1 + 2 \cdot \frac{1}{2} = 3 - 6 + 2 + 1 = 0$;
(iv) no 3-face, then $h(y) = g(y) + 3 \cdot 1 = 0$.

2.2. $k = 4$. Due to $\mathcal{A}$ (ii) – (iii), each triangle incident with y is of type $\langle 4, 8+, 8+\rangle$ or $\langle 4, k, 12+\rangle$, $4 \leq k \leq 7$.

(i) If y is incident with four non-triangular faces then, by Rule 1, $h(y) = g(y) + 4 \cdot \frac{1}{2} = 0$.

(ii) If y is incident with three non-triangular faces then they send $3 \cdot \frac{1}{2}$ to y. The vertex y is incident with at most one semi-weak $(4, w)$-edge, $w \in \{4, \dots, 7\}$; thus, y is incident with at least one semi-weak $(4, u)$-edge, $u \geq 12$, or y is incident with at least two semi-weak $(4, v)$-edges, $v \geq 8$. Therefore one can see that $h(y) \geq g(y) + 3 \cdot \frac{1}{2} + z \geq 0$ because y gets $z \geq \frac{1}{2}$ along mentioned edges, cf. Rule 3 and Rule 4.

(iii) If y is incident with two non-triangular faces then y is incident with at most one weak or semi-weak $(4, w)$-edge, $w \in \{4, \dots, 7\}$. Then $h(y) \geq g(y) + 2 \cdot \frac{1}{2} + z \geq 0$ because $z = 2 \cdot \frac{1}{2}$, if y is incident with weak $(4, w)$-edge, $z = \frac{1}{2} + \frac{2}{3}$ or $z = 3 \cdot \frac{1}{2}$, if y is incident with semi-weak $(4, w)$-edge or $z \geq 2 \cdot \frac{1}{4} + \frac{1}{2}$, or $z \geq 4 \cdot \frac{1}{4}$ if neither weak nor semi-weak $(4, w)$-edges appear in our configuration.

(iv) If y is incident with one non-triangular face then y is incident with at most one weak or semi-weak $(4, w)$-edge, $w \in \{4, \dots, 7\}$. Then $h(y) \geq g(y) + \frac{1}{2} + 2 \cdot \frac{1}{2} + \frac{2}{3} \geq 0$ or $h(y) \geq g(y) + \frac{1}{2} + 2 \cdot \frac{2}{3} > 0$, or $h(y) \geq g(y) + \frac{1}{2} + 2 \cdot \frac{1}{2} + 2 \cdot \frac{1}{4} = 0$, if neither weak nor semi-weak $(4, w)$-edges appear in our configuration.

(v) Let y be incident with four triangular faces. Then we distinguish two possibilities: if y is incident with one weak $(4, w)$-edge, $w \in \{4, \dots, 7\}$, then $h(y) = g(y) + 3 \cdot \frac{2}{3} = 0$ by Rule 3; otherwise all four edges are $(4, w)$-edges, $w \geq 8$ and $h(y) \geq g(y) + 4 \cdot \frac{1}{2} = 0$ by Rules 4 and 3.

2.3. $k = 5$. Then neither weak nor semi-weak $(3, 5)$-edges appear in our counterexample G.

(i) If y is incident with three or more non-triangular faces then, by Rule 1, $h(y) \geq g(y) + 3 \cdot \frac{1}{3} = 0$.

(ii) If y is incident with two non-triangular faces then they send $2 \cdot \frac{1}{3}$ to y. According to the cases $\mathcal{B}$, $\mathcal{C}$ of Theorem, y is incident with at least one weak $(5, w)$-edge or with two semi-weak $(5, w)$-edges, where $w \geq 9$, or y is incident with at least one triangular face of type $\langle 5, i, j \rangle$, $i, j \in \{7, 8\}$. Then $h(y) \geq g(y) + 2 \cdot \frac{1}{3} + \frac{1}{3} = 0$ by Rule 2 or $h(y) \geq g(y) + 2 \cdot \frac{1}{3} + 2 \cdot \frac{1}{6}$ by Rules 2 and 5, respectively.

(iii) If y is incident with one non-triangular face, then y may be incident only with:

– at least two weak $(5, w)$-edges; or with

– two semi-weak and at least one weak $(5, w)$-edge; or with

– at least one weak $(5,w)$-edge and with at least one triangular face of type $\langle 5,i,j\rangle$; or with

– at least two triangular faces of type $\langle 5,i,j\rangle$.

(In the possibilities above, $w \geq 9$ and $i,j \in \{7,8\}$). Then $h(y) \geq g(y) + \frac{1}{3} + z \geq 0$, because, according to the four cases above, $z = 2\cdot\frac{1}{3}$ or $z = \frac{1}{3} + 2\cdot\frac{1}{6}$ or $z = \frac{1}{3} + 2\cdot\frac{1}{6}$ or $z = 4\cdot\frac{1}{6}$.

(iv) If y is incident only with triangular faces, then y may be incident only with

– at least three weak $(5,w'')$-edges; or with

– at least two weak $(5,w'')$-edges and with one triangular face of type $\langle 5,i,j\rangle$; or with

– at least one weak $(5,w'')$-edge and two triangular faces of type $\langle 5,i,j\rangle$; or with

– at least three triangular faces of type $\langle 5,i,j\rangle$.

(In the possibilities above, $w'' \geq 9$ and $i,j \in \{7,8\}$). In each of these cases, $h(y) \geq g(y) + 3\cdot\frac{1}{3} = 0$.

2.4. $k = 6$. Then $h(y) = g(y) \geq 0$ because y neither gets nor gives anything.

2.5. $k = 7$. Transfer from y is possible only to 5-vertices and only when y and a 5-neighbour of it have the common neighbour of degree $w \in \{7,8\}$; then there are at most four transfers from y by Rule 5 and we have $h(y) \geq g(y) - 4\cdot\frac{1}{6} \geq 0$.

2.6. $k = 8$. Transfer from y is possible only to 4-vertices along edges, which are weak or semi-weak and to 5-vertices in the way described in Rule 5. Observe that for each $i = 1,\ldots,8$, the charge sent from y along two consecutive edges e_i, e_{i+1} (indices are taken modulo 8) is $\leq \frac{1}{2}$; then $h(y) \geq g(y) - 4\cdot\frac{1}{2} = 0$.

2.7. $k \in \{9,10\}$. Transfer from y is possible only to 4-vertices and 5-vertices along edges which are weak or semi-weak. It is easy to check that the maximum charge transferred from y is in the case when y is incident only with weak $(k,5)$- or $(k,4)$-edges. Then, considering that y is incident with p weak $(k,4)$-edges and at most $k-(2p+1)$ weak $(k,5)$-edges, we have $h(y) \geq g(y) - \left[\frac{1}{3}\Big(k-(2p+1)\Big) + \frac{1}{2}p\right] = g(y) + \frac{1}{6}p + \frac{1}{3} - \frac{1}{3}k \geq 0$, because $g(y) = 3$ with $k = 9$ and $p \in \{0,1,2,3,4\}$ or $g(y) = 4$ with $k = 10$ and $p \in \{0,1,2,3,4,5\}$.

2.8. $k = 11$. Observe that if some of x_i is of degree 4 or 5, then total charge sent from y is at most 5; if the only transfers of a charge from y are to adjacent 3-vertices, the total charge sent from y is also at most 5 (because for each two consecutive edges e_i, e_{i+1}, $i = 1, \ldots, 11$, the charge sent along them is at most 1, and there are three consecutive edges such that y sends at most 1 along them). This gives $h(y) \geq g(y) - 5 = 0$.

2.9. $k \geq 12$. In order to show that $h(y) \geq 0$, we use the averaging argument, i.e., we compute how much in average y sends along $e_1, \ldots, e_k$. Firstly, assign to each e_i the charge sent from y to x_i. Consider the following averaging redistribution:

– if e_i is assigned with 1 due to the Rule 1, then it donates $\frac{1}{2}$ to each of e_{i-1}, e_{i+1}

– if e_i is assigned with $\frac{2}{3}$ due to the Rule 3, then it donates $\frac{1}{6}$ to each of e_{i-1}, e_{i+1}.

After this redistribution, every edge e_i preserves at most $\frac{1}{2}$. Hence, $h(y) \geq g(y) - k \cdot \frac{1}{2} \geq 0$. ■

3. Vertex Distance Colouring

Let G be a multigraph. The distance-t chromatic number of G, $t \geq 0$, in symbols $\chi^{(t)}(G)$, is defined to be the minimum number of colours required to colour the vertices of G so that any two vertices whose distance apart is at most t receive distinct colours.

The vertex distance chromatic number was introduced and investigated by Kramer and Kramer, cf. [7, 8], see also Baldi [1], Wegner [12] and Skupień [10]. In [5] Jendrol' and Skupień proved that upper bound for $\chi^{(t)}(G)$ of a planar graph G is

$$\chi^{(t)}(G) \leq 6 + \frac{3D+3}{D-2}\Big((D-1)^{t-1} - 1\Big), \tag{2}$$

where $D = \max\{8, \Delta(G)\}$. Using Theorem 1, this upper bound can be improved as follows:

Theorem 2. *Given a planar graph G, let $D = \max\{12, \Delta(G)\}$. Then the distance-$t$ chromatic number of G is*

$$\chi^{(t)}(G) \leq 6 + \frac{2D+12}{D-2}\Big((D-1)^{(t-1)} - 1\Big). \tag{3}$$

It can be seen that the right hand side of (3), which we denote $\phi_t(D)$, is a polynomial in D of degree $t-1$,

$$\phi_t(D) = 6 + \frac{2D+12}{D-2}\Big((D-1)^{(t-1)} - 1\Big). \tag{4}$$

The proof of this theorem proceeds in the same way as in [5]; thus, the next definitions, theorems and proofs are taken from [5], with respect to improvements in Theorem 1.

Given a plane graph G and its edge e, let $G \circ e$ stand for a plane graph obtained from G by removing e, identifying both endvertices of e, and leaving exactly one edge from each pair of multiple edges which come from a 3-cycle which includes e in G. Thus $G \circ e$ is a result of *specialized contracting* the edge e of G. In particular, if e is incident to a 3-face with a 3-vertex non-incident to e, or e is a common $(3,3)$-edge of two 3-faces, then $G \circ e$ is not a normal plane map because it has a 2-vertex.

We call an (a,b)-edge e to be *light* if $a \leq 2$, or $a = 3$ and $a+b \leq 13$ or else $a+b \leq 11$; moreover, if $a = 3$ and $b = \Delta(G) \in \{9,10\}$ then the edge e is required to be incident to a 3-face.

Proposition 1. *If e is a light edge of a plane map G and $\Delta(G) \leq \widetilde{D}$, where $\widetilde{D} \geq 12$, then $\Delta(G \circ e) \leq \widetilde{D}$.*

The function ϕ_t has the following utilization.

Lemma 1. *Each planar graph G has a vertex v of degree at most 5 such that $\phi_t(\Delta(G))$ is the upper bound on the number of vertices at distance t or less from v.*

Proof. If $\delta(G) \leq 2$, let v be a vertex of degree 1 or 2; otherwise, let v be a vertex of the smallest degree in a configuration whose existence in a plane embedding of G is ensured by Theorem 1. In a routine manner we can get an upper bound, B_i, on the number of vertices at distance i from v for $i = 0, 1, \ldots, t$. The largest sum of these numbers is $\phi_t(\Delta(G))$ and is obtained in subcases of $\mathcal{A}(ii)$ of Theorem 1. Then $\phi_t(\Delta(G)) = \sum_{i=0}^{t} B_i$ where $B_0 = 1, B_2 = 5$, and else $B_i = (2\Delta + 12)(\Delta - 1)^{i-2}$ where $\Delta = \Delta(G)$. ■

Proof of Theorem 2. By contradiction. Suppose that there exists a graph G with $\chi^{(t)}(G) > \phi_t$. We can assume that G is connected and has the least possible number of vertices, say n. Let $D = \max\{12, \Delta(G)\}$; then

$n > \phi_t(D)$. Since G is a simple plane graph, it has a light edge $e = vw$ incident to a vertex v of degree at most 5, v being specified in the proof of Lemma 1. Contract the edge e in order to obtain $G' = G \circ e$. Let w^* be the image in G' of endvertices v, w of e. Then Theorem holds for G' because G' has less vertices than G. Hence, G' has a distance-t colouring with $\phi_t(D')$ (or less) colours, where $D' = \max\{12, \Delta(G')\}$. This is also a distance-t colouring of vertices in $G - v$ if colour of w^* is assoociated with the vertex w in G. Moreover, $D' \leq D$. Hence, because $\phi_t(D)$ is increasing, $G - v$ has a distance-t colouring with $\phi_t(D)$ colours. Since, by Lemma 1, G has at most $\phi_t(D) - 1$ vertices whose distances from v are from 1 to t, at least one colour exists which can be associated with v to get a distance-t colouring of $V(G)$ with $\phi_t(D)$ colours, a contradiction. ∎

As for the lower bound of $\chi^{(t)}(G)$, in [4] it is proved that for any t and sufficiently large Δ there exists a plane graph G such that $\chi^{(t)}(G) \geq a\Delta^{\lfloor t/2 \rfloor} - \mathcal{O}(\Delta^{\lfloor t/2 \rfloor} - 1), a$ is $\frac{9}{2}$ for t odd and $\frac{3}{2}$ for t even. Hence, the upper bound proved here is expected to be much smaller.

For $t = 2$, Wegner [12] conjectures that $\chi^{(2)}(G) \leq \Delta(G) + 5$ for plane G with $4 \leq \Delta(G) \leq 7$ and $\chi^{(2)}(G) \leq \lfloor \frac{3\Delta(G)}{2} \rfloor + 1$ if $\Delta(G) \geq 8$. van den Heuvel and McGuinness [11] proved that for plane G, $\chi^{(2)}(G) \leq 2\Delta(G) + 25$ holds. From Theorem 2 it follows that for plane G with $\Delta(G) \geq 12$, $\chi^{(2)}(G) \leq 2\Delta(G) + 17$.

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